Determinantal Hypersurfaces, Joint Spectra , and Representations of Coxeter Groups

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Based on joint works with Z. Cuckovic, T. Peebles, A. Tchernev, and J.Weyman

Let $A_1, ..., A_n$ be $k \times k$ matrices. The set

$$\sigma(A_1, ..., A_n) = \left\{[x_1, ..., x_n] \in \mathbb{CP}^{n-1} : \det(x_1 A_1 + ... + x_n A_n) = 0\right\}$$

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If $A_1, ..., A_n$ are operators acting on a Hilbert space X, the **projective joint spectrum** of $A_1, ..., A_n$ introduced by Yang (2008) is

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We will concentrate on the case when $A_n = I$ and denote by

$$\sigma_p(A_1, ..., A_{n-1}) = \sigma(A_1, ..., A_{n-1}, I) \cap \{x_n \neq 0\} \text{ (so that } x_n = -1).$$

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Theorem (S., Tchernev)

Let $A_1, ..., A_n$ be bounded operators on a Hilbert space X with A_1 normal, and let $\lambda \neq 0$ be an isolated spectral point of $A_!$ of finite multiplicity. Then, there is a neignbourhood $\mathcal{O} \subset \mathbb{CP}^n$ of $[1/\lambda, 0, ..., 0, -1]$ such that $\sigma_p(A_1, ..., A_n) \cap \mathcal{O}$ is an analytic set of pure codimension one.
The same is true without the assumption of normality if λ is a simple isolated spectral point.

Q.1

Given a hypersurface $\Gamma \subset \mathbb{CP}^n$ when are there matrices $A_1, ..., A_{n+1}$ such that

$$\Gamma = \sigma(A_1, ..., A_{n+1})?$$

In the case when the answer is affirtmative, it is said that Γ has a determinatal representation.

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Q.2

Given that $\Gamma \subset \mathbb{CP}^n$ has a determinantal representation, what does its geometry say about the relations between the matrices in the tuple?

Motzkin and Taussky (1952): Two self-adjoint matrices commute $\iff$ $\sigma(A_1, A_2, I)$ is a union of projective lines.
Chagouel, S., Zhu (2015) extended this result to tuples of compact self-adjoint operators in a Hilbert space, and tuples of normal matrices.

If $A_1, ..., A_n$ have a common invariant subspace of dimension k, then $\sigma_p(A_1, ..., A_n)$ contains an algebraic hypersurface of order k. Simple examples show that the converse is not true. For example, if

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 7 & 1 \\ 1 & 1 & 1/2 \end{bmatrix},$$

then

$$\sigma_p(A_1, A_2) = \{(x, y) \in \mathbb{C}^2 : (x+y-1)(5xy+5y^2-15y-10x+2) = 0\}.$$

There are a line and a quadratic in the joint spectrum, but no common eigenvectors and no common two-dimensional invariant subspaces.

Q. 2′

Find a necessary and sufficient conditions for an appearance of an algebraic hypersurface of order k in $\sigma_p(A_1, ..., A_n)$ to indicate that there is a k-dimensional common invariant subspace.

It turned out that the case $n = 2, \ k = 1$ is the most important here.

Theorem (S., Tchernev)

Let $A_1, \dots, A_n$ be self-adjoint, $\lambda \neq 0$ be an isolated point of $\sigma(A_1)$, and there exists $\rho > 0$ such that, up to multiplicity,

$$\Delta_\rho(1/\lambda, 0, \dots, 0) \cap \{\lambda x_1 + a_2 x_2 + \cdots + a_n x_n = 1\} \\ = \Delta_\rho(1/\lambda, 0, \dots, 0) \cap \sigma_p(A_1, \dots, A_n)$$

where $\Delta_\rho(w) = \{z \in \mathbb{C}^n : |z_j - w_j| < \rho\}$.

The following are equivalent:

(1) The eigensubspace of A_1 corresponding to eigenvalue λ is an eigensubspace for each of the operators $A_2, \dots, A_n$;

(2) There exist an $\epsilon \in \mathbb{R}$, $\epsilon \neq 1$, and $\rho' > 0$ such that $A_1(\epsilon, \lambda)$ is invertible and, up to multiplicity,

$$\Delta_{\rho'}(\lambda, 0, \dots, 0) \cap \{(1/\lambda)x_1 + a_2 x_2 + \cdots + a_n x_n = 1\} \\ = \Delta_{\rho'}(\lambda, 0, \dots, 0) \cap \sigma_p\big(A_1(\epsilon, \lambda)^{-1}, A_2(\epsilon, a_2), \dots, A_n(\epsilon, a_n)\big),$$

where $A(\epsilon, b) = (1 + \epsilon)A - b\epsilon I$.

Corollary

Let A_1 be a unitary involution ($A_1^2 = I$) with 1 being a spectral point of A_1 of finite multiplicity, and let $A_2, ..., A_n$ be self-adjoint. If $\sigma_p(A_1, ..., A_n)$ contains a part of a hyperplane passing through $(1, 0..., 0)$ that lies in a neighborhood of $(1, 0, ..., 0)$, then $A_1, ..., A_n$ have a common eigenvector.

Remark: If the multiplicity is infinite, it is no longer true.

Algebraic curves in the spectrum

Let A_1 and A_2 be two self-adjoint operators on X and suppose that $\lambda \neq 0$ is an isolated spectral point of A_1 of finite multiplicity. Suppose that for some neighborhood $\mathcal{O}$ of a point $(1/\lambda, 0)$ the part of the joint spectrum $\sigma_p(A_1, A_2)$ which is in $\mathcal{O}$ is an an algebraic curve

$$\sigma_p(A_1, A_2) \cap \mathcal{O} = \{(x_1, x_2) \in \mathcal{O} : \mathcal{P}(x_1, x_2) = 0\},$$

$$\mathcal{P}(x_1, x_2) = \sum_{j=0}^{k} R_j(x_1, x_2),$$

$R_j(x_1, x_2)$ is a homogeneous polynomial of degree j, $R_0 = -1$.

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We assume that $(1/\mathrm{lambda}, 0)$ is not a singular point of $\sigma(A_1, A_2)$ and that the line $\{x_2 = 0\}$ is not tangent to $\sigma_p(A_1, A_2)$ at $(1/\lambda, 0)$, so that $\forall x = (x_1, x_2) \in \mathcal{O}$, $\{\tau x : \tau \in \mathbb{C}\} \cap \sigma_p(A_1, A_2) \neq \emptyset$.

Let $x = (x_1, x_2) \in \mathcal{O}$. Write

$$A(x) = x_1 A_1 + x_2 A_2.$$

We have

$$tx = (tx_1, tx_2) \in \sigma_p(A_1, A_2) \iff \sum_{j=0}^{k} t^j R_j(x_1, x_2) = 0, \qquad (1)$$

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and μ satisfies

$$\mu^k - \sum_{j=1}^{k} R_{k-j}(x_1, x_2)\mu^j = 0.$$

If $\mathcal{O}$ is small enough, the last equation has a root $\mu(x)$ close to 1 which is an eigenvalue of $A(x)$.

If $\xi(x)$ is an eigenvector of $A(x)$ with eigenvalue $\mu(x)$, then

$$\left(A(x)^k - \sum_{j=1}^{k} R_{k-j}(x)A(x)^j\right)\xi = 0,$$

$$\left(A(x)^k - \sum_{j=1}^{k} R_{k-j}(x)A(x)^j\right)P(x)\eta = 0, \ \forall \eta \in X,$$

$P(x)$ is the orthogonal projection X onto the eigenspace of $A(x)$ with eigenvalue $\mu(x)$.

$$\Longrightarrow \left(A(x)^k - \sum_{j=1}^{k} R_{k-j}(x)A(x)^j\right)P(x) = 0.$$

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$$A(x)^m P(x) = \frac{1}{2\pi i} \int_\gamma z^m (zI - A(x))^{-1} dz$$

Therefore,

$$\frac{1}{2\pi i} \int_\gamma \left(z^k - \sum_{j=1}^{k} R_{k-j}(x) z^j \right) (zI - A(x))^{-1} dz = 0.$$

Let $x = (1/\lambda, y)$, with y being small. Then

$$A(x) = (1/\lambda)A_1 + yA_2,$$

$$(zI - A(x))^{-1} = (zI - (1/\lambda)A_1)^{-1}(I - yA_2(zI - (1/\lambda)A_1)^{-1})^{-1}$$

$$= (zI - (1/\lambda)A_1)^{-1} \sum_{j=0}^{\infty} y^j \left[A_2(zI - (1/\lambda)A_1)^{-1}\right]^j,$$

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$$= (zI - (1/\lambda)A_1)^{-1} \sum_{j=0}^{\infty} y^j \left[A_2(zI - (1/\lambda)A_1)^{-1}\right]^j,$$

$$\Rightarrow \sum_{j=0}^{\infty} y^j \frac{1}{2\pi i} \int_\gamma (z^k - \sum_{j=1}^{k} R_{k-j}(1/\lambda, y)z^j)(zI - (1/\lambda)A_1)^{-1} S^j dz,$$

where $S = [A_2(zI - (1/\lambda)A_1]$.

A rearrangement of terms gives

$$\sum_{j=0}^{\infty} \frac{y^j}{2\pi i} \int_\gamma \Psi_j(z)dz = 0,$$

where $\Psi_j(z)$ are operator-valued meromorphic functions of z obtained from the equation above.

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Thus,

$$\mathrm{Rez}(\Psi_j)|_{z=1} = 0, \; j = 0, 1, ... \tag{2}$$

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Remark It is possible to show that conditions of the last relation imply that all Ψ_j are holomorphic and that these conditions are necessary and sufficient for the curve $\mathcal{P}(x_1, x_2) = 0\}$ to be in the spectrum.

For this talk we will need relations (2) only for $j = 1, 2$.

Recall that we denoted by P the projection onto the λ-eigenspace of A_1. Now we introduce the following operator $T(A_1)$.

1). In the case of matrices, let $\lambda = \lambda_1, \lambda_2, ..., \lambda_s$ be distinct eigenvalues of A_1 and $P = P_1, P_2, ..., P_s$ be the corresponding projections. Then

$$T(A_1) = T = \sum_{j=2}^{s} \frac{\lambda}{\lambda_j - \lambda} P_j.$$

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2). For general self-adjoint operators

$$T = \int_{\sigma(A_1)\setminus\{\lambda\}} \frac{\lambda}{z - \lambda} dE(z),$$

where

$$A_1 = \int_{\sigma(A_1)} z dE(z)$$

is the spectral resolution of A_1.

Theorem (S., Tchernev)

Suppose that A_1 and A_2 are self-adjoint, that $\lambda \neq 0$ is an isolated spectral point of A_1 of finite multiplicity such that

- $(1/\lambda, 0)$ belongs to only one component of $\sigma_p(A_1, A_2)$ and in a neighborhood of $(1/\lambda, 0)$ the proper joint spectrum $\sigma_p(A_1, A_2)$ is given by $\mathcal{P}(x_1, x_2) = 0$;
- $\frac{\partial \mathcal{R}}{\partial x_1}\Big|_{(1/\lambda,0)} \neq 0$, so that locally $\{\mathcal{P} = 0\}$ defines x_1 as an implicit function of x_2, $x_1 = x_1(x_2)$, $x_1(0) = 1/\lambda$.

Then

$$PA_2P = -x_1'(0)P \tag{3}$$

$$PA_2TA_2P = -\frac{x_1''(0)}{2}P. \tag{4}$$

This result is used to prove Theorem about common eigenvalues for tuples.

Another application of this result is to the case when the unit circle is in the spectrum.

Theorem (Cuckovic, S., Tchernev)

Let A_1, A_2 be self-adjoint operators on an N-dimensional Hilbert space X, and suppose that A_1 is invertible and that $\|A_2\| = 1$.

Further suppose that the "complex unit circle" $\{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 1\}$ is a reduced component of both $\sigma_p(A_1, A_2)$ and $\sigma_p(A_1^{-1}, A_2)$, of multiplicity n, and that the points $(\pm 1, 0)$ do not belong to any other component of either $\sigma_p(A_1, A_2)$ or $\sigma_p(A_1^{-1}, A_2)$, and that the points $(0, \pm 1)$ do not belong to any other component of $\sigma_p(A_1, A_2)$.

Theorem (Continued)

Then:

1. A_1 and A_2 have a common $2n$-dimensional invariant subspace L;
2. The pair of restrictions $A_1|_L$ and $A_2|_L$ is unitary equivalent to the following pair of $2n \times 2n$ involutions C_1 and C_2, each block-diagonal with n equal 2×2 blocks along the diagonal:

$$C_1 = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & -1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & -1 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

3. The group generated by C_1 and C_2 represents the Coxeter group B_2.

Corollary

If in the previous Theorem A_1 is an involution and the ”circle” is in the spectrum with $(\pm 1, 0)$, $(0, \pm 1)$ not being singular points of the spectrum, then the conclusions of the above Theorem hold.

Unitary Matrices

Lemma

Let A_1 and A_2 be bounded self-adjoint involutions on a Hilbert space X that is $A_1^2 = A_2^2 = I$. Then:

1) The set $\sigma_p(A_1, A_2)$ is the union of all the "complex ellipses" $\mathcal{E}_\alpha = \{x^2 + \alpha xy + y^2 = 1\}$ with $\alpha \in \sigma(A_1A_2 + A_2A_1)$.

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2) When $\sigma(A_1A_2 + A_2A_1)$ is a finite set then each connected component of $\sigma_p(A_1, A_2) \setminus \{(\pm 1, 0)\ (0, \pm 1)\}$ is either $L \setminus \{(\pm 1, 0)\ (0, \pm 1)\}$ with L one of the lines $x \pm y = \pm 1$, or $\mathcal{E}_\alpha \setminus \{(\pm 1, 0)\ (0, \pm 1)\}$ for some $\alpha \in \sigma(A_1A_2 + A_2A_1)$.

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3) When X is finite dimensional each reduced component of $\sigma_p(A_1, A_2)$ is either a line of the form $x \pm y = \pm 1$, or a “complex ellipse” $\mathcal{E}_\alpha$ with $\alpha \in \sigma(A_1A_2 + A_2A_1) \setminus \{-2, 2\}$.

Proof If $(x, y) \in \sigma_p(A_1, A_2)$, then

$$\begin{aligned}(xA_1 + yA_2)^2 - I &= (xA_1 + yA_2 - I)(xA_1 + yA_2 + I) \\ &= (x^2 + y^2 - 1)I + xy(A_1A_2 + A_2A_1).\end{aligned}$$

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If $(x, y) \neq (\pm 1, 0)$ or $(0, \pm 1)$, then

$$\frac{1 - x^2 - y^2}{xy} \in \sigma(A_1A_2 + A_2A_1).$$

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Since $\| A_j \| = 1$,

$$\alpha = \left| \frac{1 - x^2 - y^2}{xy} \right| \leq 2,$$

and in the case of finite dimension 1) follows. In infinite dimensional case it is derived from the conclusion that $\sigma_p(A_1, A_2) \cup (-\sigma_p(A_1, A_2))$ contains the "ellipse".

The following result is derived from the previous two:

Theorem

Let A_1 and A_2 be unitary self-adjoint linear operators on a finite-dimensional Hilbert space X. Then:

1) Every reduced component of $\sigma_p(A_1, A_2)$ is either a line $\{x \pm y = \pm 1\}$ or an "ellipse" $\{x^2 + 2xy\cos(2\pi\theta) + y^2 = 1\}$ for some $0 < \theta < 1/2$.

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2) If a line $\{x \pm y = \pm 1\}$ is a reduced component of multiplicity r of the joint spectrum $\sigma_p(A_1, A_2)$ then A_1 and A_2 have a corresponding common eigenspace of dimension r.

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2) If a line $\{x \pm y = \pm 1\}$ is a reduced component of multiplicity r of the joint spectrum $\sigma_p(A_1, A_2)$ then A_1 and A_2 have a corresponding common eigenspace of dimension r.
3) If an "ellipse" $\{x^2 + 2xy\cos(2\pi\theta) + y^2 = 1\}$ with $0 < \theta < 1/2$ is a reduced component of the proper joint spectrum $\sigma_p(A_1, A_2)$ of multiplicity r, then A_1 and A_2 have a correponding common invariant subspace of dimension $2r$ that is a direct sum of r two-dimensional common invariant subspaces.

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3) is proved by successive scaling and using the above CST result.

Proposition

Let A_1 and A_2 be as in the previous Theorem, and let $m \geq 2$ be an integer. The following are equivalent:

(1) $(A_1 A_2)^m = I$,

(2) $\sigma(A_1 A_2 + A_2 A_1) \subseteq \{ \mathcal{E}_\alpha : \alpha = 2\cos(2\pi k/m) \mid k = 0, \ldots, m-1 \}$.

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Proof (1) $\Longrightarrow$ (2). For each $n \geq 0$ set

$$R_n = (1/2)\Big[(A_1 A_2)^n + (A_2 A_1)^n\Big].$$

Then

$$\begin{aligned} R_0 &= I, \\ R_1 &= (1/2)(A_1 A_2 + A_2 A_1), \quad \text{and} \\ R_n &= 2R_1 R_{n-1} - R_{n-2} \qquad \text{for } n \geq 2. \end{aligned}$$

It follows by induction that for each $n \geq 0$ we have

$$R_n = T_n(R_1),$$

where $T_n(z)$ are Tchebyshev's polynomials of the first kind defined by

$$\begin{aligned} T_0(z) &= 1, \\ T_1(z) &= z, \quad \text{and} \\ T_n(z) &= 2zT_{n-1}(z) - T_{n-2}(z) \qquad \text{for } n \geq 2. \end{aligned}$$

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Now, suppose $(A_1A_2)^m = I$. Thus $(A_2A_1)^m = I$ as well, hence $R_m = T_m(R_1) = I$. Since $\sigma(R_m) = T_m\big(\sigma(R_1)\big)$, we must have $T_m(\alpha) = 1$ for each $\alpha \in \sigma(R_1)$. Therefore $\sigma(R_1) \subseteq \{\cos(2\pi k/m) \mid k = 0, \ldots, m-1\}$, which implies (2) as desired.

Application to representations of Coxeter groups

Definiton For $N \times N$ matrices $A_1, ..., A_n$ the proper joint spectrum in the divisor form, $\sigma_p^d(A_1, ..., A_n)$ is defined as the zero-divisor of the polynomial $\det(x_1 A_1 + ... + x_n A_n - I)$.

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The multiplicity ascribed to a point $(x_1, ..., x_n) \in \sigma_p^d(A_1, ..., A_n)$ is equal to the rank of the projection

$$\frac{1}{2\pi i} \int_\gamma (zI - \sum_{j=1}^n x_j A_j)^{-1} dz,$$

(γ is asmall contour around 1).

Recall that a **Coxeter group** is a finitely generated group with generators $g_1, ..., g_n$ satisfying the relations

$$g_j^2 = 1,\ j = 1, ..., n;\ (g_i g_j)^{m_{ij}} = 1,\ 2 \leq m_{ij} \leq \infty \text{ for } i \neq j.$$

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A Coxeter group is defined by the **Coxeter matrix**

$$M = (m_{ij}),\ m_{ii} = 1,$$

that is symmetric (obviously $m_{ij} = m_{ji}$)

A traditional way of presentation of a Coxeter group is through its **Coxeter diagram**, which is a graph constructed by the following rules:

- the vertices of the graph are the generator subscripts;
- vertices i and j form an edge if and only if $m_{ij} \geq 3$;
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In particular, two generators commute if and only if they are not connected by an edge. The disjoint union of Coxeter diagrams yields a direct product of Coxeter groups, and a Coxeter group is connected if its diagram is a connected graph.

The finite connected Coxeter groups consist of the one-parameter families A_n, B_n, D_n, and $I(n)$, and the six exceptional groups E_6, E_7, E_8, F_4, H_3, and H_4. They were classified by Coxeter.

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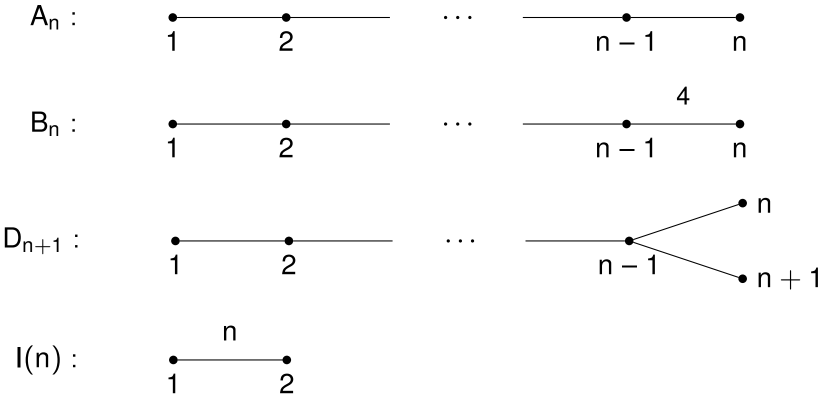
The Coxeter diagrams for the groups A_n, B_n, D_{n+1}, and $I(n)$ that we study here are as follows:



I(n) is called **Dihedral group**.

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Known:

Every linear representation of a finite group is equivalent to a unitary representation.

Corollary

Two linear representations of the Dihedral group $I(n)$, ρ_1 and ρ_2, are equivalent if and only if

$$\sigma_p^d(\rho_1(g_1), \rho_1(g_2)) = \sigma_p^d(\rho_2(g_1), \rho_2(g_2)),$$

where g_1, g_2 are the Coxeter generators of $I(n)$.

Another Corollary to the above theorem is the follow result.

Theorem (Cuckovic, S, Tchernev)

Let $U_1, ..., U_n$ be $k \times k$ self-adjoint unitary matrices, and let G be the subgroup of GL_k generated by these matrices. Suppose that for $i \neq j$ the joint spectra

$$\sigma_p(U_i, U_j) = \cup_{s=1}^{r_{ij}} \mathcal{E}_{\alpha_s^{ij}},\ \alpha_s^{ij} = 2\pi \frac{l_s^{ij}}{p_s^{ij}},$$

where l_s^{ij}, p_s^{ij} are mutually prime ($p_S^{ij} = 1$ if $l_s^{ij} = 0$). Denote by

$$m_{ij} = \begin{cases} 2 & \text{if } l_s^{ij} = 0\ \forall s \\ \text{the least common multiple of } \{p_s^{ij}\} & \text{if } \exists l_s^{ij} \neq 0. \end{cases}$$

Then G is isomorphic to a quotient group of the Coxeter group with the Coxeter matrix (m_{ij}).

We saw that the joint spectrum in the divisor form of the Coxeter generators determines a representation of a Dihedral group up to an equivalence.

Q. Are there any other finitely generated groups with the same property: there is a group of generators such that the joint spectrum in the divisor form of these generators determine a representation up to an equivalence?

Theorem (Cuckovic, S., Tchernev)

Suppose G is a finite Coxeter group of type either A, or B, or D, and let $g_1, ..., g_n$ be the Coxeter generators of G. If for two finite dimensional linear representations ρ_1 and ρ_2 of G we have

$$\sigma_p^d(\rho_1(g_1), ..., \rho_1(g_n)) = \sigma_p^d(\rho_2(g_1), ..., \rho_2(g_n)),$$

then the representations ρ_1 and ρ_2 are equivalent.

Comments for the proof.

Write $A_i = \rho_1(g_i), B_i = \rho_2(g_i),\ i = 1, ..., n$. Fix $x \in \mathbb{C}^n$. Then for $\lambda \in \mathbb{C},\ \lambda x \in \sigma_p(A_1, ..., A_n) \Longleftrightarrow \frac{1}{\lambda} \in \sigma(A(x)),\ A(x) = \sum x_j A_j$.

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Thus,

$$\sigma_p^d(A_1, ...A_n) = \sigma_p^d(B_1, ..., B_n) \Rightarrow \sigma(A(x)) = \sigma(B(x)) \qquad (5)$$

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Definition

The character, χ_ρ, of a representation $\rho : G \to GL_K$ is the function

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The above relation shows that if $\sigma_p^d(A_1, ..., A_n) = \sigma_p^d(B_1, ..., B_n)$, then

$$\chi_{\rho_1}(g_j) = \chi_{\rho_2}(g_j),\ j = 1, ..., n. \tag{6}$$

Known:

If for two linear representations ρ_1 and ρ_2 of a finite group G

$$\chi_{\rho_1}(g) = \chi_{\rho_2}(g), \quad \forall g \in G, \tag{7}$$

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Relation (6) means that (7) holds for words of length one.

To prove (7) for all words we remark that (5) implies that $\forall k \in \mathbb{N},\ x \in \mathbb{C}^n$

$$\sigma(A(x)^k) = \sigma(B(x)^k) \Longrightarrow \text{Trace}(A(x)^k) = \text{Trace}(B(x)^k) \quad (8)$$

$$A(x)^k = \sum_{j_1 + ..j_n = k} x_1^{j_1} ... x_n^{j_n} \left(\sum A_{r_1} ... A_{r_k}\right)$$

where the last sum is taken over all $(r_1, ..., r_n)$ with $r_1 + ... + r_n = k$ and $(r_1, ..., r_n)$ contains j_1 $A_1 - s$; j_2 $A_2 - s$; , ..., j_n $A_n - s$. The same is true for $B(x)^k$.

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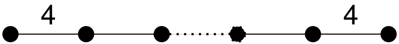
Now (5) implies

$$\sum \text{Trace}(A_{r_1}...A_{r_k}) = \sum \text{Trace}(B_{r_1}...B_{r_k})$$

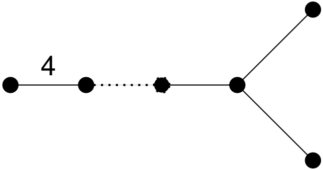
$$\sum \chi_{\rho_1}(g_{r_1}...g_{r_n}) = \sum \chi_{\rho_2}(g_{r_1}...g_{r_n}).$$

Characters of representations of affine Coxeter groups

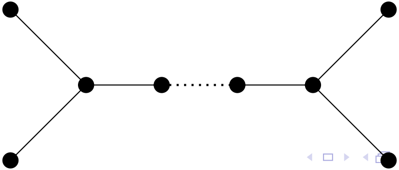
$\tilde{C}_n$



$\tilde{B}_n$



$\tilde{D}_n$



Let us denote by $c_1, ..., c_{n+1}$ Coxeter generators of $\tilde{C}_n$, so that

$$c_1^2 = c_2^2 = \cdots = c_n^2 = c_{n+1}^2 = 1, \; c_j c_k = c_k c_j \; \text{ if } |j - k| \geq 2,$$
$$(c_1 c_2)^4 = (c_{n+1} c_n)^4 = 1, (c_j c_k)^3 = 1, \text{for } \; 2 \leq j, k \leq n.$$

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Write

$$t_j = c_j c_{j+1} ... c_n c_{n+1} c_n ... c_j,\ j = 2, ..., n+1,$$
$$r_1 = c_1 c_2 \cdots c_n c_{n+1} c_n \cdots c_2$$
$$r_2 = c_2 c_1 c_2 \cdots c_n c_{n+1} c_n \cdots c_3$$
$$\vdots$$
$$\vdots$$
$$r_{n-2} = c_{n-2} c_{n-3} \cdots c_2 c_1 c_2 \cdots c_{n+1} c_n c_{n-1}$$
$$r_{n-1} = c_{n-1} c_{n-2} \cdots c_2 c_1 c_2 c_3 \cdots c_{n+1} c_n$$
$$r_n = c_n c_{n-1} \cdots c_2 c_1 c_2 \cdots c_{n+1}$$

Proposition

$N := < r_1, r_2, \ldots, r_n >$ is an abelian normal subgroup of $\tilde{C}_n$ and $\tilde{C}_n = B_n \rtimes N$

Theorem (Peebles, S., Tchernev Weyman)

Let ρ_1, ρ_2 be two finite dimensional linear representations of $\tilde{C}_n$. If

$$\sigma_p^d(\rho_1(c_2), \rho_1(c_3), \ldots, \rho_1(c_n), \rho_1(c_{n+1}), \rho(t_2), ..., \rho(t_n), \rho_1(r_1), \ldots, \rho_1(r_n), \rho_1(r_1^{-1}), \ldots, \rho_1(r_n^{-1}))$$
$$= \sigma_p^d(\rho_2(c_2), \rho_2(c_3), \ldots, \rho_2(c_n), \rho_2(c_{n+1}), \rho_2(t_2), ...\rho_2(t_n) \rho_2(r_1), \ldots, \rho_2(r_n), \rho_2(r_1^{-1}), \ldots, \rho_2(r_n^{-1})),$$

then $\chi_{\rho_1} = \chi_{\rho_2}$.

Some open questions

Q.1 Does the joint spectrum $\sigma_{\mathrm{p}}^{\mathrm{d}}$ of other than Coxeter sets of generators determine a representation up to an equaivalence?

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Q.3 Is a representation of a non-special finite Coxeter group is irreducible if and only if the joint spectrum of the Coxeter generators is irreducible?

Q.4 We saw that an appearance of a "complex ellipse" in the joint spectrum of two matrices indicates the existence of a two-dimensional invariant subspace. Are there other surfaces $\{P(x_1, ..., x_n) = 0\}$ such that if they appear in the joint spectrum of tuple of n matrices, these matrices have common invariant subspace of dimension equal to the degree of P?

THANK YOU!