A skein theoretic model for the double affine Hecke algebras

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Dreams

Dreams

In the broad light of day mathematicians check their equations and their proofs, leaving no stone unturned in their search for rigour.

But, at night, under the full moon, they dream, they float among the stars and wonder at the mystery of the heavens: they are inspired.

Without dreams there is no art, no mathematics, no life.

Michael Atiyah

Introduction

Peter Samuelson and I worked some years ago on the skein-based algebra of closed curves in the thickened torus.

Peter related this to a special case of the elliptic Hall algebra, following a paper of Schiffman and Vasserot, which connects the elliptic Hall algebra with quotients of the double affine Hecke algebras of Cherednik.

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More recently Peter and I had a look at the models of the double affine Hecke algebras by Burella et al, and came up with a variant of these based this time on *braids* in the thickened torus.

We are hopeful that we can enhance these by including *closed curves* in the torus so as to get a skein based model for the complete elliptic Hall algebra which is consistent with our own earlier work and also the account of Schiffman and Vasserot.

Main sections

- **Torus braids with a base string**
- **Double affine Hecke algebras**
- **The elliptic Hall algebra**

Braids in the torus

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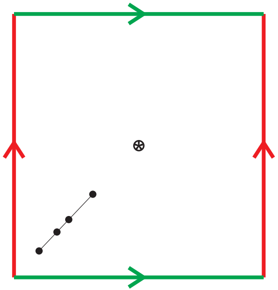
Regard T^2 as given by identifying opposite pairs of sides in the unit square $[0,1] \times [0,1]$.

Take the base point $*$ to be the centre of the square. Fix n endpoints for n-string braids in $T^2 \times I - \{*\} \times I$.

Braids in the torus

This is a plan view from above.



Braids in the torus

A braid should be thought of as n strings in $T^2 \times I$ which run monotonically upwards without meeting each other, along with a fixed base string.

We can imagine n distinct points starting from our choice of endpoints and moving around the torus as time progresses, along with a base point $*$ which does not move.

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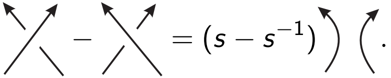
We can imagine n distinct points starting from our choice of endpoints and moving around the torus as time progresses, along with a base point $*$ which does not move.

In general we want to use a ribbon (a framed curve), rather than a string, but that can be done implicitly here in a consistent way.

Making an algebra

The based skein $H_n(T^2, *)$ is defined to be $\mathbf{Z}[s^{\pm 1}, c^{\pm 1}]$-linear combinations of braids, up to equivalence, subject to the local *skein relation*



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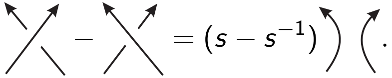
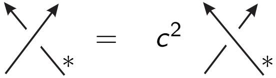


In addition a braid string is allowed to cross through the base string $*$ at the expense of multiplying by a parameter c according to the local relation



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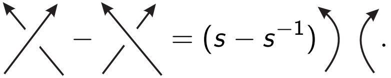

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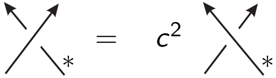

Composition of braids induces a product on $H_n(T^2, *)$, making it into an algebra over $\mathbf{Z}[s^{\pm 1}, c^{\pm 1}]$.

Related algebras

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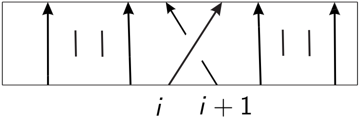

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The skein relation, applied at this crossing point, gives

$$\sigma_i - \sigma_i^{-1} = (s - s^{-1})\mathrm{Id}$$

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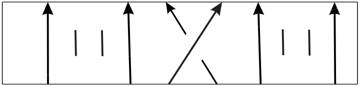

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$i \qquad i+1$

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which becomes the quadratic relations

$$(\sigma_i - s)(\sigma_i + s^{-1}) = 0$$

Related algebras

They also satisfy Artin's braid relations.

These are

- Non-adjacent generators σ_i, σ_j commute.
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

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In some contexts the parameter q or t is used in place of s^2.

Related algebras

The extensions to braids in either an annulus, or the whole torus, give models for the affine, or (we believe) double affine Hecke algebras, $\dot{H}_n, \ddot{H}_n$

Pictorial views

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For example, σ_i appears in plan view as

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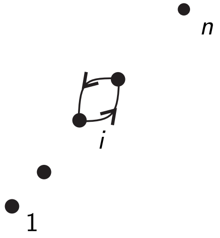

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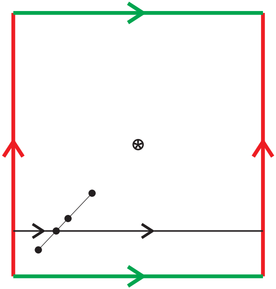
$x_i \quad =$  ,

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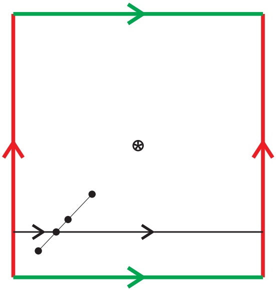
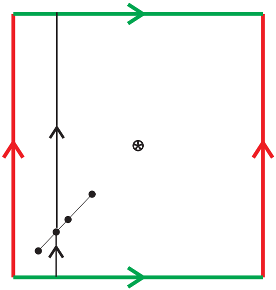
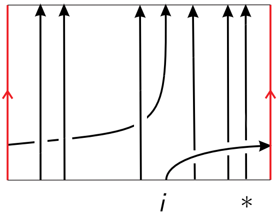
$x_i =$  , $y_i =$ 

View from the side

Seen from the side they look like

$x_i \quad =$ 

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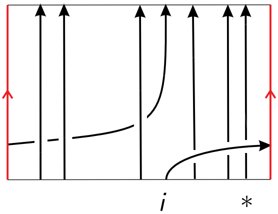

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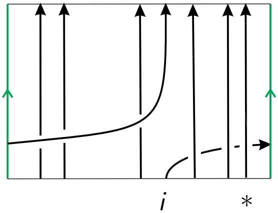

Torus braid relations

We can see that

$$\begin{aligned} \sigma_i^{-1} x_i \sigma_i^{-1} &= x_{i+1} \\ \sigma_i y_i \sigma_i &= y_{i+1}. \end{aligned}$$

Visualising braid composition

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The braids $\{x_i\}$ commute among themselves, since their paths in the plan view are disjoint.

The same is true for the braids $\{y_i\}$, and equally the braids σ_i commute with x_j and y_j when $j \neq i, i+1$.

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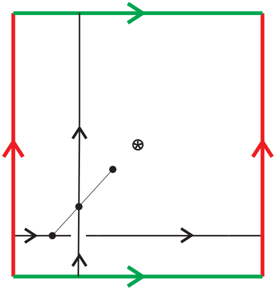
$x_1 y_2 =$ 

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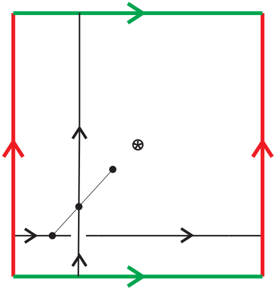
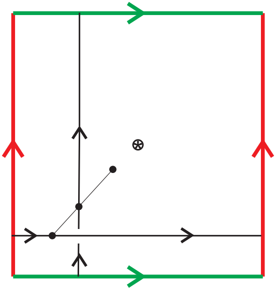
$x_1 y_2 =$  $y_2 x_1 =$ 

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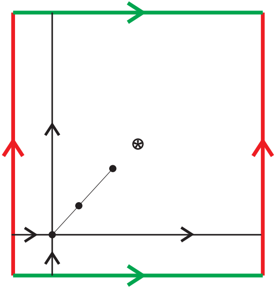
$x_1 y_1 \quad =$ 

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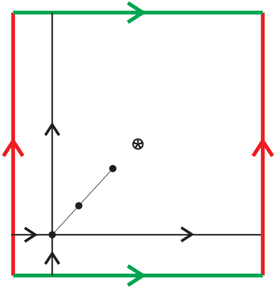
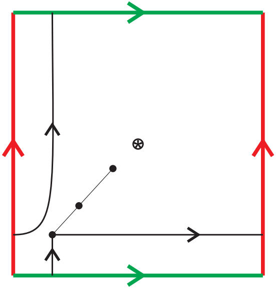
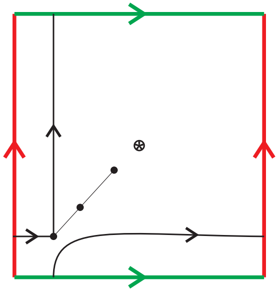
$x_1 y_1 =$  $=$ 

Visualising braid composition

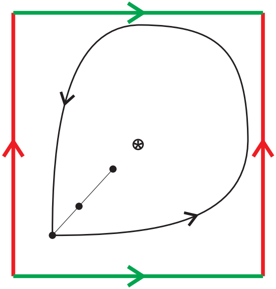
while $y_1 x_1$ diverts to

$y_1 x_1 \quad =$ 

Visualising braid composition

With further smoothing we get the plan view of the commutator

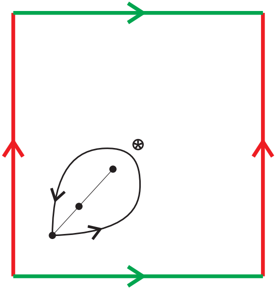
$$x_1 y_1 x_1^{-1} y_1^{-1} \quad =$$



Visualising braid composition

Moving this braid across the base string leads to

$$x_1 y_1 x_1^{-1} y_1^{-1} \quad = \quad c^2$$



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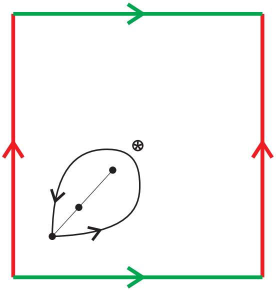
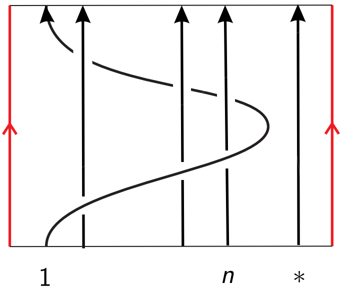
$x_1 y_1 x_1^{-1} y_1^{-1} \;=\; c^2$  $:= c^2 \beta_n$

Visualising braid composition

Seen from the side

β_n =



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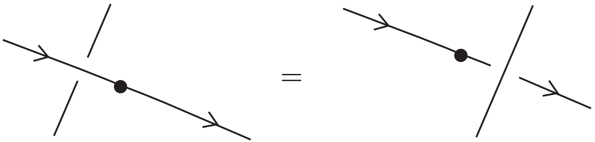

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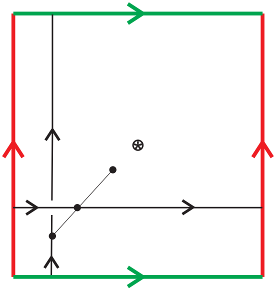

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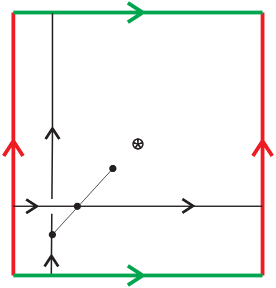
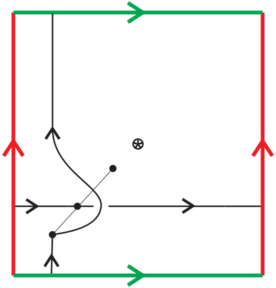
$y_1x_2 =$  $=$ 

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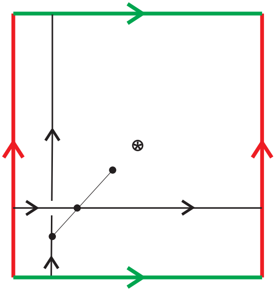
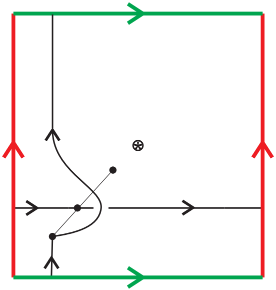
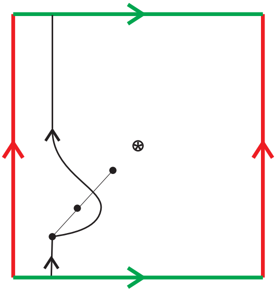
$y_1 x_2 =$  $=$ 

$= \quad x_2 \alpha_2$

Braid relations

where

$\alpha_2 =$  $= \sigma_1^2 y_1,$

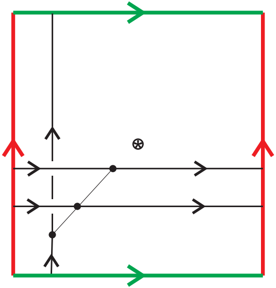
and thus

$$x_2 y_1^{-1} = y_1^{-1} x_2 \sigma_1^2$$

Further relations

A similar argument, moving one path across braid points $2 \dots n$, shows that

$y_1 x_2 x_3 \cdots x_n =$



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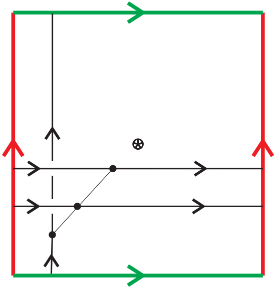
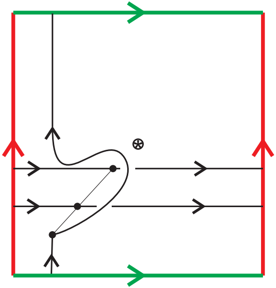
$y_1 x_2 x_3 \cdots x_n =$  $=$ 

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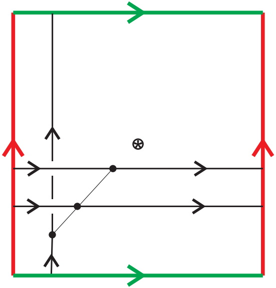
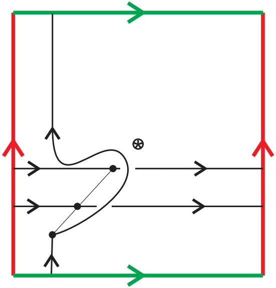
$y_1 x_2 x_3 \cdots x_n =$  $=$ 

$= \quad x_2 x_3 \cdots x_n \alpha_n$

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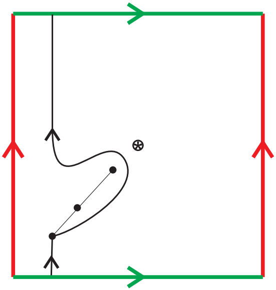
$\alpha_n =$  $= \beta_n y_1$

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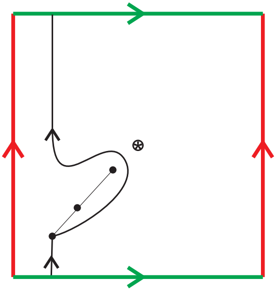

$= \beta_n y_1$

Now

$$\beta_n = c^{-2} x_1 y_1 x_1^{-1} y_1^{-1}$$

Hence

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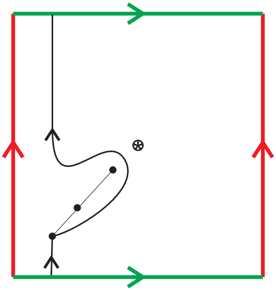
$\alpha_n =$  $= \beta_n y_1$

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The case $n = 1$

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This is the simplest case $\ddot{H}_1$ of a double affine Hecke algebra.

Double affine Hecke algebras

The *double affine Hecke algebra* $\ddot{H}_n$ of Cherednik is an algebra over $\mathbb{Z}[s^{\pm 1}, q^{\pm 1}]$ generated by

$$\{T_i\}, 1 \le i \le n-1, \{X_j\}, \{Y_j\}, 1 \le j \le n$$

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with relations

$$\begin{aligned} (T_i + s)(T_i - s^{-1}) &= 0 \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1} \\ [T_i, T_j] &= 0, |i - j| > 1 \end{aligned}$$

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X_1^{-1} Y_2 &= Y_2 X_1^{-1} T_1^{-2} \\
Y_1 X_1 \cdots X_n &= q X_1 \cdots X_n Y_1
\end{aligned}$$

Comparison with the skein algebra

These equations are all satisfied in our skein algebra $H_n(T^2, *)$ taking $X_i = x_i, Y_i = y_i, T_i = \sigma_i^{-1}$ and $q = c^{-2}$.

Comparison with the skein algebra

These equations are all satisfied in our skein algebra $H_n(T^2, *)$ taking $X_i = x_i$, $Y_i = y_i$, $T_i = \sigma_i^{-1}$ and $q = c^{-2}$.

We believe that this is an isomorphism of algebras.

Going further

There is scope for going further, and incorporating closed curves in the skein algebra. The reason for trying this is to model the *elliptic Hall algebra*, which is shown by Schiffman and Vasserot to relate to a limit of quotients of the algebras $\ddot{H}_n$.

Going further

There is scope for going further, and incorporating closed curves in the skein algebra. The reason for trying this is to model the *elliptic Hall algebra*, which is shown by Schiffman and Vasserot to relate to a limit of quotients of the algebras $\ddot{H}_n$.

Key elements in their model are the power sums

$$X_1^m + \cdots + X_n^m, Y_1^m + \cdots + Y_n^m$$

and commutators of them.

Skein theory appearance of power sums

An important connection comes from the skein of the annulus with n boundary points.

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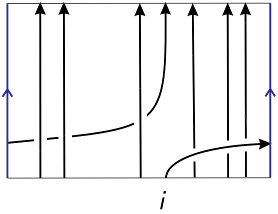
$Z_i \quad =$ 

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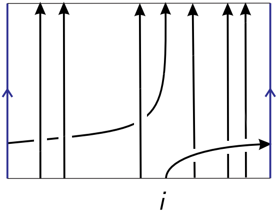
$Z_i \quad =$ 

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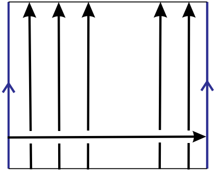

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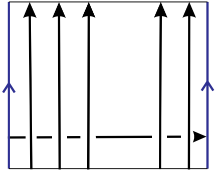

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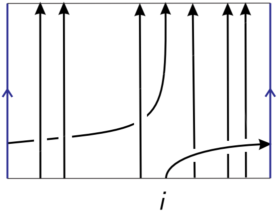
$Z_i \quad =$ 

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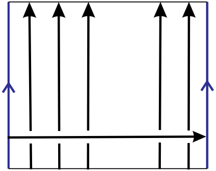

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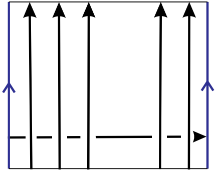
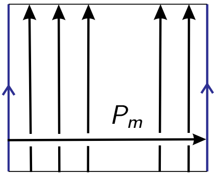


$= \quad (s - s^{-1}) \sum Z_i$

Skein theory appearance of power sums

For each m there is also a well-established element P_m in the skein of the annulus which satisfies the relation



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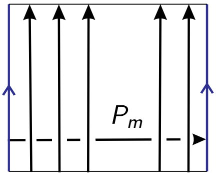
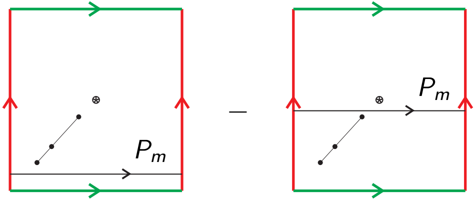


$$= (s^m - s^{-m}) \sum Z_i^m$$

Realising power sums in the torus

Placing the annulus in the direction of the $(1,0)$ curve in the torus, and including the closed curve decorated by P_m leads to



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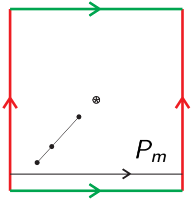

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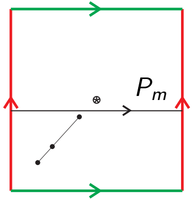
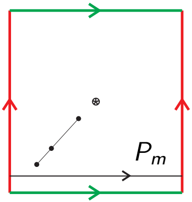


$$= \quad (s^m - s^{-m}) \sum_i X_i^m$$

This is the view from above.

Realising power sums in the torus

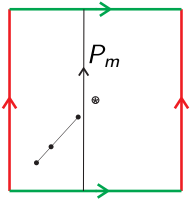
By moving the closed curve in the second diagram round the torus past the base string we can then write

$$(s^m - s^{-m}) \sum_i X_i^m \quad = \quad (1 - c^{2m})$$ 

Realising power sums in the torus

Similarly

$$(s^m - s^{-m}) \sum_i Y_i^m \quad = \quad (1 - c^{2m})$$



The elliptic Hall algebra

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We could then use the skein algebras $H_n(T^2, *)$, enhanced with suitable closed curves, as models for the full elliptic Hall algebra.

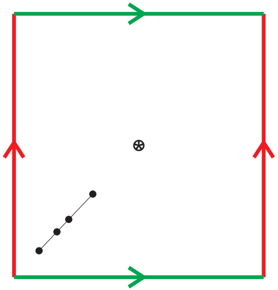
Enhanced skein algebra

Here is a speculative, and maybe overoptimistic approach.

All the same it does give a nice interpretation in the DAHA setting, which is independent of n to a large extent, provided that our skein construction really gives us isomorphisms with $\ddot{H}_n$.

Construction of elements $W_{\mathbf{x}}$

Fix a disc D in T^2 which includes the braid points and the base point. A suitable choice for our purposes is a neighbourhood of the diagonal in the square.



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Any oriented embedded curve in the complement of this disc is determined up to isotopy by a primitive element $\mathbf{y} \in \mathbb{Z}^2$, representing the homology class of the curve.

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Any oriented embedded curve in the complement of this disc is determined up to isotopy by a primitive element $\mathbf{y} \in \mathbb{Z}^2$, representing the homology class of the curve.

For each primitive $\mathbf{y}$ define an element $W_{\mathbf{y}}$ of the skein $H_n(T^2, *)$ by the oriented curve corresponding to $\mathbf{y}$, along with vertical braid strings and base string.

The closed curve is taken to be framed by its neighbourhood in T^2.

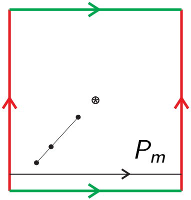
Construction of elements $W_{\mathbf{x}}$

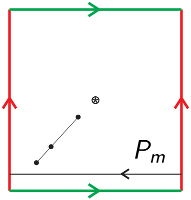
For any other non-zero $\mathbf{x} \in \mathbb{Z}^2$ write $\mathbf{x} = m\mathbf{y}$ with $m > 0$ and $\mathbf{y}$ primitive, and define $W_{\mathbf{x}}$ to be $W_{\mathbf{y}}$ with the closed curve decorated by the element P_m.

Write $d(\mathbf{x}) = m$ to denote the multiple m.

Views

We then have plan views of $W_{(\pm m,0)}$ and $W_{(0,\pm m)}$ as

$W_{(m,0)}$ = 

, $W_{(-m,0)}$ = 

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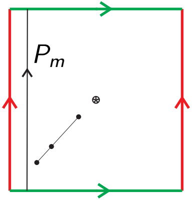
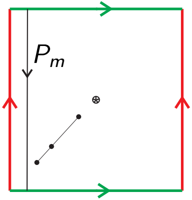
$W_{(0,m)}$ = 

, $W_{(0,-m)}$ = 

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Relating power sums to the elements $W_{\mathbf{x}}$

Our equations above show that

$$\begin{aligned}
(1-c^{2m})W_{(m,0)} &= (s^m - s^{-m})\sum x_i^m,\\
(c^{-2m}-1)W_{(-m,0)} &= (s^m - s^{-m})\sum x_i^{-m}\\
(c^{-2m}-1)W_{(0,m)} &= (s^m - s^{-m})\sum y_i^m\\
(1-c^{2m})W_{(0,-m)} &= (s^m - s^{-m})\sum y_i^{-m}.
\end{aligned}$$

Relation with generators of the elliptic Hall algebra

Our best hope for a compatible skein version of $u_{\mathbf{x}}$ is to take

$$u_{\mathbf{x}} = \frac{1}{s^m - s^{-m}} W_{\mathbf{x}}$$

with $m = d(\mathbf{x}) > 0$.

The existing algebraic approach

For non-zero $\mathbf{x} \in \mathbb{Z}^2$ Schiffman and Vasserot define elements $P_{\mathbf{x}}$ in the *spherical algebra* $S\ddot{H}_n$.

Here $S\ddot{H}_n$ is defined as $e\ddot{H}_n e$, where $e \in H_n$ is the *symmetrizer* idempotent in the finite Hecke algebra H_n.

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In their comparisons with the elliptic Hall algebra they make the assignment

$$u_\mathbf{x} \to \frac{1}{q^m - 1} P_\mathbf{x}$$

with $m = d(\mathbf{x})$.

Comparison

Using the identification of $H_n(T^2,*)$ with $\ddot{H}_n$, where $q=c^{-2}, s^2=t$, we can compare our elements $W_{\mathbf{x}}$ to $P_{\mathbf{x}} \in S\ddot{H}_n$.

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The symmetrizer idempotent e has a well-established representation as a linear combination of n-braids in the disc D^2.

Theorem

For every non-zero $\mathbf{x} \in \mathbb{Z}^2$ we have

$$(q^m-1)eW_{\mathbf{x}}e = (s^m - s^{-m})P_{\mathbf{x}},$$

where $m = d(\mathbf{x}) > 0$.

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Our calculations above establish this when $\mathbf{x} = (\pm m, 0), (0, \pm m)$.

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The general definition of $P_{\mathbf{x}}$ uses automorphisms of $\ddot{H}_n$ induced by Dehn twists on the torus starting from $P_{0,m}$, and these same automorphisms also carry $W_{0,m}$ to $W_{\mathbf{x}}$.

Hopes and fears

Peter Samuelson and I are encouraged in this approach by our earlier result in Duke Mathematical Journal.

We show there that the elliptic Hall algebra with $q = 1$ is isomorphic to the skein algebra $H(T^2)$ of closed curves in T^2, with no base point, and no braid points.

Hopes and fears

Our earlier work gives a nice commutativity relation for the elements $W_{\mathbf{x}}$ when there are no braid points or base point.

Theorem (Samuelson and Morton)

$$[W_{\mathbf{x}}, W_{\mathbf{y}}] = (s^d - s^{-d})W_{\mathbf{x}+\mathbf{y}}$$

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This relation between $W_{\mathbf{x}}$ and $W_{\mathbf{y}}$ also works in our model for $\ddot{H}_n$ when $\mathbf{x} = (1, 0)$ and $\mathbf{y} = (0, m)$.

This corresponds to the commutator relation between $u_{\mathbf{x}}$ and $u_{\mathbf{y}}$ in the elliptic Hall algebra in this case.

Hopes and fears

It would be nice to work directly with elements such as $W_{\mathbf{x}}$ in $\ddot{H}_n$ without passing to the spherical versions.

It seems though that some involvement of the spherical algebra is needed to cover the full relations from the Hall algebra.

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It would be nice to work directly with elements such as $W_{\mathbf{x}}$ in $\ddot{H}_n$ without passing to the spherical versions.

It seems though that some involvement of the spherical algebra is needed to cover the full relations from the Hall algebra.

In the case $\mathbf{x} = (-1, 0)$ and $\mathbf{y} = (1, m)$ the commutator relation in the elliptic Hall algebra is a bit more complicated, and we have not been able to get a direct proof of the corresponding relation in our skein version of $S\ddot{H}_n$.

The future

We will continue to dream.

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