Convex Integration in PDEs, Geometry, and Variational Calculus

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There is vast ongoing interest in the so-called technique of convex integration in several areas of mathematics, as demonstrated by diverse recent contributions. Notably, the seemingly unrelated fields of materials science, fluid dynamics and symplectic geometry enjoyed significant advances through this method, to mention a few: flexibility of the energy-minimizing solutions in the description of shape memory alloys, the resolution of Onsager's conjecture formulated in the realm of statistical mechanics, or the classification of overtwisted contact structures in all dimensions; all based on Gromov's h-principle and further appropriate extensions of Nash and Kuiper's iterative convex integration scheme developed for the classical isometric embedding problem in Riemannian geometry.

This 5-day workshop brought together experts from geometry, variational calculus and mathematical fluid dynamics to share existing knowledge as well as to open up new perspectives and collaborations across different mathematical subfields.

1 Overview of the Field

The first example of convex integration, albeit *avant la lettre*, was given by John Nash in 1954 [34]. Nash considered a classical problem in differential geometry, namely the isometric embedding problem. He showed that the $(n-1)$-dimensional unit sphere admits a C^1-isometric embedding into a ball $B_\epsilon \subset \mathbb{R}^n$ with arbitrarily small radius $\epsilon > 0$. The ingenious idea leading to this counterintuitive result was to shrink the sphere so that it fits inside B_ϵ and, consequently, all distances decrease by the factor of ϵ, and then to successively add "wrinkles" in order to increase the intrinsic distances back to the Euclidean ones.

Considerably later, Gromov [24] vastly extended Nash's idea and suggested the term *convex integration* for the iterative procedure in the isometric embedding problem. He developed very general ideas how to apply such approach to solve differential inclusions; roughly speaking, the method works well with systems of very non-linear partial differential equations, such that the convex envelope of each small domain of the submanifold representing the equation in the jet space has a non-empty interior. Consequently, Gromov used convex integration in his studies of the symplectic rigidity and Eliashberg started using systematically this kind of ideas to classify the overtwisted contact forms in [19].

The 1990s and early 2000s saw a further application of ideas of a similar flavour in the calculus of variations, the theory of partial differential equations, and in materials science. Two landmark papers from this period are the one by Dacorogna-Marcellini [12] and the work of Müller-Šverák [33], together with other parallel developments concerning counterexamples to elliptic regularity, existence of mappings with incompatible gradients and fluid-like behaviour of shape memory alloys.

A renewed interest in convex integration techniques was sparked around ten years ago, when De Lellis–Székelyhidi showed that the method could be adapted for constructing weak solutions to the incompressible Euler equations of fluid dynamics [16]. Such applicability of convex integration to time-dependent partial differential equations of mathematical physics came highly unexpected (Gromov himself had previously expressed his disbelief in such a possibility), as it violates the supposedly deterministic nature of classical mechanics. There is an ongoing debate in the fluid dynamics community how to make sense of the solutions of De Lellis-Székelyhidi in terms of turbulence theory.

The resulting ideas and techniques have had important implications on the solvability of the Cauchy problem for the incompressible Euler equations [37], the non-uniqueness of entropy solutions in compressible fluid dynamics [6, 7], and the proof of Onsager's Conjecture of turbulence theory [28, 4].

There is a vast ongoing interest in the technique of convex integration in several fields of mathematics, as demonstrated by a number of diverse recent contributions. To mention a few: Conti et al. [11] improved the original Nash embedding result, Kim-Yan [29] treated an equation related to image processing, Lewicka-Pakzad [30] showed flexibility of the Monge-Ampère equations in appropriate Hölder continuity regimes, and Rüland et al. [35] examined higher regularity in the theory of elasticity.

2 Recent Developments and Open Problems

As explained above, convex integration is used in various fields of mathematics, including (symplectic and contact) geometry, elliptic partial differential equations, calculus of variations, materials science, and mathematical fluid dynamics. Researchers employing convex integration, however, are usually experts only in one of these fields. Accordingly, convex integration is mostly discussed at meetings that focus solely on one of these mathematical subfields and there seems to be little interaction in particular between geometers and analysts working with convex integration techniques.

It has therefore been our main goal to bring together a diverse poll of experts in order to acquire a broader horizon on these techniques and to make new connections between different fields. To this end, we had survey talks and specialised talks alike, presenting cutting-edge research of very recent years, or even work in progress. Ample time was left for discussion and collaboration.

To our knowledge, the proposed workshop has been unique in focusing on convex integration across different mathematical disciplines. We managed to bring together a group of mathematicians several of whom had never met before. Let us mention a few open problems related to convex integration that enjoyed lively discussion at the workshop:

1. Rigidity vs. flexibility of isometric embeddings and solutions to Monge-Ampere equation – what is the threshold Hölder exponent?

2. Higher regularity of convex integration solutions for realistic models of nonlinear elasticity.

3. Dissipative Hölder-continuous Euler flows satisfying the local energy inequality.

4. Fluid equations of Navier-Stokes type, cf. [9].

5. Admissibility criteria for systems of fluid dynamics of inviscid fluids, designed to eliminate the "wild" solutions. One of them could be the maximal dissipation principle proposed by Dafermos or the viscosity criterion.

6. The structure of the set of initial data that give rise to "wild" solutions for the compressible (barotropic) and full Euler system; are such initial data dense in the phase space?

3 Presentation Highlights

3.1 Incompressible Fluids

OVERVIEW TALK

Roman Shvydkoy: *Mechanisms for energy balance restoration in the Onsager (super-)critical flows*

In this talk, Roman gave an overview of the current state of the Onsager conjecture, its relevance in laws of turbulence, and a series of new results on flows that have Onsager-critical or even supercritical regularity. In a number of recent works it was observed that some solutions to the Euler equation still conserve energy despite being critical or supercritical. Such energy law restoration comes as a result of several additional mechanisms that are not taken into account in the classical commutator estimate approach of proving the law. In this talk Roman highlighted those mechanisms, which include the use of incompressibility in a geometric way, such as in the vortex-sheet case, transport structure of the equation that comes into light in the pressureless case, singularity set organized on a smooth structure, e.g. the case of a point singularity and Hamiltonian structure that comes into place in this situation or as in the case of Caffarelli-Kohn-Nirenberg theorem for suitable solutions, and finally the vanishing viscosity limit in 2D with supercritical condition on vorticity was discussed. Many of these cases set certain restrictions on the application of the convex integration method in constructing critical dissipative solutions to the incompressible systems.

This overview calls for a more systematic study of these mechanisms and their possible generalization to less structured flows.

RESEARCH TALKS

Alexey Cheskidov: *Wild solutions to the 3D Navier-Stokes equations*

Alexey showed that there exists a nontrivial finite energy periodic stationary weak solution to the 3D Navier-Stokes equations (NSE) with zero force. This provides the first proof of a non-uniqueness for the stationary Naiver-Stokes equations. Moreover, the result gives an alternative proof of a non-uniqueness for the evolutionary Naiver-Stokes equations, recently obtained by Buckmaster and Vicol. Indeed, a nontrivial stationary solution can be used as an initial value for the evolutionary problem. Lerays theorem implies the existence of a Leray-Hopf solution starting from this initial data, which cannot coincide with the constructed stationary solution. This solution exhibit what is called the *anomalous energy influx*, the backward energy cascade that precisely balances the energy dissipation at each scale. The stationary solution does not lose any energy even though its enstrophy is positive (in fact, infinite).

The construction relies on a convex integration scheme utilizing new stationary building blocks designed specifically for the NSE. In order to increase concentration and decreases the intermittency dimension, Alexey and his collaborator Xiaoyutao Luo design *viscous eddies*, compactly supported approximate stationary solutions of the NSE that are divergence-free up to the leading term. The constructed viscous eddies are also used to prove the existence of weak solutions of the NSE with energy profiles discontinuous on a dense set of positive Lebesgue measure.

Mimi Dai: *Flexibility and rigidity for some incompressible flows*

Mimi discussed some recent trends in the study of incompressible flows. The emphasis is on the topic of flexibility and rigidity of solutions to the governing PDE model which do not have high enough regularity. For the three dimensional magnetohydrodynamics with Hall effect, Mimi showed that weak solutions in Leray-Hopf class are not unique, by tailoring a convex integration scheme to combine with classical theory. It is worth to mention that the uniqueness of weak solutions in Leray-Hopf class for the three dimensional Navier-Stokes equation (NSE) remains an open problem. On the rigidity side of the 3D NSE with external force, she showed that two weak solutions coincide in the energy space for large time provided that they coincide on the low frequency part below a certain time-dependent wavenumber. This wavenumber is called determining wavenumber. She further proved that the time average of the determining wavenumber is bounded by Kolmogorov's dissipation wavenumber which separates the inertial range from the dissipation range. A stronger version of the result shows the uniqueness of trajectory on the global attractor under the constraint on the low modes part. Moreover, the work built a bridge to connect mathematical theory with phenomenological law of physics.

Sara Daneri: *On non-uniqueness below Onsager's critical exponent*

In this talk, Sara considered the following initial value problem for the Euler equations on the three-dimensional torus $\mathbb{T}^3$

$$\begin{cases} \partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0 & \text{in } (0,T) \times \mathbb{T}^3 \\ \operatorname{div} v = 0 & \text{in } (0,T) \times \mathbb{T}^3 \\ v(\cdot, 0) = v_0 & \text{on } \mathbb{T}^3 \end{cases} \tag{1}$$

In (1), $v : [0,T) \times \mathbb{T}^3 \to \mathbb{R}^3$ is the velocity field of the fluid, $p : [0,T) \times \mathbb{T}^3 \to \mathbb{R}^3$ the pressure field and $v_0 : \mathbb{T}^3 \to \mathbb{R}^3$ is a given divergence free velocity field, the prescribed initial datum for the Cauchy problem.

While for initial data in $C^{1,\alpha}$ one has short time existence and uniqueness of classical solutions, a completely different picture appears for weak solutions with lower regularity. In the seminal paper [16], De Lellis and Székelyhidi showed the existence of infinitely many bounded solutions of the Euler equations with compact support in space and time, in any dimension greater than or equal to two. A feature of these non-physical solutions is that the total kinetic energy of the fluid, namely the map

$$[0,T) \ni t \mapsto \int_{\mathbb{T}^3} |v(t,x)|^2 \, dx. \tag{2}$$

increases at time $t = 0$. Therefore, in [17] they considered solutions satisfying an additional *admissibility condition*, which among the possible formulations takes the form

$$\int_{\mathbb{R}^3} |v(t,x)|^2 \, dx \le \int_{\mathbb{R}^3} |v_0|^2 \, dx, \qquad \forall \, t \ge 0, \tag{3}$$

and asked themselves if in this class one can prevent non-uniqueness. The answer turns out to be negative: there are initial data v_0 in L^∞, called by the authors *wild initial data*, which give rise to infinitely many bounded and admissible weak solutions of (1). Moreover, in [36] they were shown to be dense in the solenoidal fields in L^2. Notice that, due to the weak-strong uniqueness result by Brenier, De Lellis and Székelyhdidi [3], not any initial datum can be a wild initial datum, since whenever a classical solution exists, this is the unique solution in the class of weak admissible solutions with the same initial datum.

Therefore a natural question arises, namely whether there exists a regularity threshold above which solutions are unique, for all initial data, and below which non-uniqueness may happen.

The aim of this talk was to show, by reviewing some already published results [13, 15] and announcing a result obtained in collaboration with E. Runa and L. Székelyhidi [14], that such a threshold must be bigger than Hölder continuity in space of any order $\beta < 1/3$ (with Hölder constant uniformly bounded in time).

3.2 Compressible Fluids

OVERVIEW TALK

Eduard Feireisl: *Convex integration and compressible Euler system*

This was a survey of recent results obtained via the method of convex integration for the Euler system describing the motion of a compressible inviscid fluid. The system written in terms of the variables $\varrho = \varrho(t,x)$ - the mass density, $\mathbf{m} = \varrho \mathbf{u}$ - the momentum, $S = \varrho s$ - the total entropy, reads:

$$\partial_t \varrho + \operatorname{div} \mathbf{m} = 0,$$

$$\partial_t \mathbf{m} + \operatorname{div} \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla p(\varrho, S) = 0,$$

$$\partial_t \left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e(\varrho, S) \right) + \operatorname{div} \left[\left(\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + \varrho e(\varrho, S) \right) \frac{\mathbf{m}}{\varrho} \right] = 0,$$

supplemented in the context of weak solutions by the entropy inequality

$$\partial_t S + \operatorname{div} \left(S \frac{\mathbf{m}}{\varrho} \right) \ge 0.$$

The method of convex integration can be used to show the existence of infinitely many weak solutions for a rather vast class of initial data satisfying also the entropy inequality, see [8], [23]. There are more results available for the isentropic system, where $S = \varrho\overline{s}$, $\overline{s}$ suitable constant. The approach via convex integration is based on the original results of DeLellis and Székelyhidi proved in the context of incompressible Euler system [17] and a generalization of the so–called Oscillatory Lemma in [20].

RESEARCH TALKS

Martina Hofmanová: *Ill-posedness in fluid dynamics what can we do about it?*

Martina discussed several puzzling results related to solvability and ill-posedness of the isentropic Euler system. It is nowadays well understood that the multidimensional isentropic Euler system is desperately ill–posed. Indeed, the method of convex integration can be used to construct infinitely many wild solutions as well as rather surprising approximation results. On the other hand, Martina and collaborators propose a new concept of dissipative solution to the compressible Euler system based on a careful analysis of possible oscillations and/or concentrations in the associated generating sequence. Unlike the conventional measure–valued solutions or rather their expected values, the dissipative solutions comply with a natural compatibility condition they are classical solutions as long as they enjoy certain degree of smoothness. Moreover, ideas from Markov selections allow to select a semiflow of physically reasonable solutions and to exclude oscillation defects in certain cases. The solution semiflow enjoys the standard semigroup property and the solutions coincide with the strong solutions as long as the latter exist. Moreover, they minimize the energy (maximize the energy dissipation) among all dissipative solutions. Finally, Martina shows that any weakly converging sequence of solutions to the isentropic NavierStokes system on the full physical space R^d, $d = 2, 3$, in the vanishing viscosity limit either (i) converges strongly in the energy norm, or (ii) the limit is not a weak solution of the associated Euler system. The same result holds for any sequence of approximate solutions in the spirit of DiPerna and Majda. This is in sharp contrast to the incompressible case, where (oscillatory) approximate solutions may converge weakly to solutions of the Euler system.

Simon Markfelder: *On some non-uniqueness results for the compressible Euler equations based on convex integration*

Simon considered the compressible Euler equations in multiple space dimensions. These equations describe the time evolution of a compressible inviscid fluid. He gave an overview over some results showing existence of infinitely many solutions. All these results are based on the convex integration method, which was used by De Lellis and Székelyhidi in the context of the incompressible Euler equations. After considering general initial data, Simon and collaborators focussed on a special type of two-dimensional initial data, namely those which are constant in each of the two half spaces (Riemann data).

3.3 Conservation Laws

RESEARCH TALKS

Sam Krupa: *Uniqueness of solutions versus convex integration for conservation laws in one space dimension*

For hyperbolic systems of conservation laws in one space dimension, the best theory of well-posedness is restricted to solutions with small total variation (Bressan et al. 2000). Looking to expand on this theory, Sam pushes in new directions. One key difficulty is that for many systems of conservation laws, only one nontrivial entropy exists. In 2017, in joint work with A. Vasseur, Sam proved uniqueness for the solutions to the scalar conservation laws which verify only a single entropy condition. Their result was the first result in this direction which worked directly on the conservation law. Further, their method was based on the theory of shifts and a-contraction developed by Vasseur and his team. These theories are general theories and apply also to the systems case, leading one to hope the framework they built for the scalar conservation laws will apply to systems. In this talk, Sam reviewed the current progress on using the theory of shifts and a-contraction to push forward the theory of well-posedness for systems of conservation laws in one space dimension. On the other hand, he also briefly discussed the potential for non-uniqueness of solutions via convex integration. This is joint work with A. Vasseur and L. Székelyhidi.

Emil Wiedemann: *Convex integration vs. the chain rule*

Emil provided counterexamples to the chain rule for divergence-free vectorfields, based on a Gromov-type convex integration scheme. This was based on joint work with G. Crippa, N. Gusev, and S. Spirito.

Agnieszka Świerczewska-Gwiazda: *Measure-valued–strong uniqueness for general conservation laws and some convex integration for nonlocal Euler system*

In the last years measure-valued solutions started to be considered as a relevant notion of solutions if they satisfy the so-called measure-valued–strong uniqueness principle. This means that they coincide with a strong solution emanating from the same initial data if this strong solution exists. Following a result of Yann Brenier, Camillo De Lellis and Laszlo Székelyhidi Jr. [3] for the incompressible Euler equation, this property has been examined for many systems of mathematical physics, including incompressible and compressible Euler system, compressible Navier-Stokes system, polyconvex elastodynamics et al., [18, 21]. One observes also some results concerning general hyperbolic systems, [25]. Agnieszka's goal has been to provide a unified framework for general systems, that would cover the most interesting cases of systems. Additionally she introduced a new concept of dissipative measure valued solution to general hyperbolic system.

In the second part of the talk she considered several modifications of the Euler system of fluid dynamics including its pressureless variant driven by non-local interaction repulsive-attractive and alignment forces. These models arise in the study of self-organisation in collective behavior modeling of animals and crowds. Agnieszka discussed how to adapt the method of convex integration to show the existence of infinitely many global-in-time weak solutions for any bounded initial data. Then she considered the class of *dissipative* solutions satisfying, in addition, the associated global energy balance (inequality). She identified a large set of initial data for which the problem admits infinitely many dissipative weak solutions. Finally, she established a weak-strong uniqueness principle for the pressure driven Euler system with non-local interaction terms as well as for the pressureless system with Newtonian interaction, see [5]. Agnieszka directed her attention also to dissipative measure-valued solutions to the pressureless Euler system with non-local terms, and showed that the property of measure-valued - strong uniqueness holds.

Piotr Gwiazda: *Onsager's conjecture for general conservation laws*

A common feature of systems of conservation laws of continuum physics is that they are endowed with natural companion laws which are in such case most often related to the second law of thermodynamics. This observation easily generalizes to any symmetrizable system of conservation laws. They are endowed with nontrivial companion conservation laws, which are immediately satisfied by classical solutions. Not surprisingly, weak solutions may fail to satisfy companion laws, which are then often relaxed from equality to inequality and overtake a role of a physical admissibility condition for weak solutions. Piotr discussed what is the critical regularity of weak solutions to a general system of conservation laws to satisfy an associated companion law as an equality. An archetypal example of such result was derived for the incompressible Euler system by Constantin et al. [1] in the context of the seminal Onsager conjecture. This general result can serve as a simple criterion to numerous systems of mathematical physics to prescribe the regularity of solutions needed for an appropriate companion law to be satisfied. The talk was based on common results with C. Bardos, E. Feireisl, P. Gwiazda, E. S. Titi, and E. Wiedemann [10, 22, 26, 2].

3.4 Differential Geometry and Equations of Elliptic and Parabolic Type

RESEARCH TALKS

Dominik Inauen: *Rigidity and Flexibilty of Isometric Embeddings*

The problem of embedding abstract Riemannian manifolds isometrically (i.e. preserving the lengths) into Euclidean space stems from the conceptually fundamental question of whether abstract Riemannian manifolds and submanifolds of Euclidean space are the same. As it turns out, such embeddings have a drastically different behaviour at low regularity (i.e. C^1) than at high regularity (i.e. C^2). For example, by the famous Nash–Kuiper theorem it is possible to find C^1 isometric embeddings of the standard 2-sphere into arbitrarily small balls in $\mathbb{R}^3$, and yet, in the C^2 category there is (up to translation and rotation) just one isometric embedding, namely the standard inclusion. Analoguous to the Onsager conjecture, one might ask if there is a sharp regularity threshold in the Hölder scale which distinguishes these flexible and rigid behaviours. In his talk, Dominik reviewed some known results and argued why the Hlder Exponent 1/2 can be seen as a critical exponent in the problem.

Seonghak Kim: *Fine phase mixtures in 1-D hyperbolic-elliptic problem*

In this talk, Seonghak presented fine phase mixtures of weak solutions to the initial-boundary value problem for a class of hyperbolic-elliptic equations in one space dimension. Such solutions are constructed through a carefully modified method of convex integration to capture fine scale oscillations of spatial derivatives of solutions. Seonghak also included a numerical simulation via FEM to assert that his solutions are indeed reasonable candidates from infinitely many solutions to the problem.

Young-Heon Kim: *The Monge problem in Brownian stopping optimal transport*

Young-Heon discussed recent progress in an optimal Brownian stopping problem, called the optimal Skorokhod embedding problem, which is an active research area especially in relation to mathematical finance. Given two probability measures with appropriate order, the problem considers the stopping time under which the Brownian motion carries one probability measure to the other, while minimizing the transportation cost. Young-Heon focussed on the cost given by the distance between the initial and the final point. A strong duality result of this optimization problem is obtained, which enables one to prove that the optimal stopping time is given by the first hitting time to a barrier determined by the optimal dual solutions.

The main part of this talk was based on joint work of Young-Heon Kim with Nassif Ghoussoub (UBC) and Aaron Palmer (UBC).

Marta Lewicka: *Convex integration for the Monge-Ampère equation*

In this talk, Marta discussed the dichotomy of rigidity vs. flexibility for the $\mathcal{C}^{1,\alpha}$ solutions to the Monge-Ampère equation in two dimensions:

$$\mathcal{D}et\nabla^2 v := -\frac{1}{2}\text{curl curl }(\nabla v \otimes \nabla v) = f \qquad \text{in } \Omega \subset \mathbb{R}^2. \tag{4}$$

The reported results appeared in [30]. Fistly, they showed that below the regularity threshold $\alpha < 1/7$, the very weak $\mathcal{C}^{1,\alpha}(\bar{\Omega})$ solutions to (4) are dense in the set of all continuous functions. This flexibility statement is a consequence of the convex integration h-principle, that is a method proposed by Gromov for solving certain partial differential relations and that, as Marta showed, turns out to be applicable to the Monge-Ampère equation. Here, she directly adapted the iteration method of Nash and Kuiper, in order to construct the oscillatory solutions to (4).

Secondly, Marta proved that the same class of very weak solutions fails the above flexibility in the regularity regime $\alpha > 2/3$: any $\mathcal{C}^{1,\alpha}(\Omega)$ solution to (4) with positive Hessian determinant f is necessarily convex, while when $f = 0$ the graph of u is necessarily developable. In these examples, convexity and developability are the two global characteristics displaying the solutions' rigidity. Their results and techniques are parallel with those concerning the low co-dimension isometric immersions, the Onsager conjecture for the Euler equation, the Perona-Malik equation, the active scalar equation, and also should be compared with results on the regularity of Sobolev solutions to the Monge-Ampère equation whose study is important in the variational description of shape formation [31, 32].

Baisheng Yan: *Convex integration for the gradient flow of polyconvex functionals*

In this talk, Baisheng discussed non-uniqueness and instability for a class of nonlinear diffusion equations, including the gradient flows of some nonconvex energy functionals, under the framework of partial differential inclusions by convex integration and Baire's category methods. The existence of infinitely many Lipschitz weak solutions to the initial-boundary value problem is proved if the diffusion flux function satisfies a structural condition called Condition (OC). For parabolic systems, this condition proves to be compatible with strong polyconvexity. As a result of such compatibility by brutal constructions, instability for the gradient flows of certain strongly polyconvex functionals is established in the sense that the initial-boundary value problem for the gradient flow possesses a weakly* convergent sequence of Lipschitz weak solutions whose limit is not a weak solution even for smooth initial-boundary data.

3.5 Materials Science

OVERVIEW TALK
Angkana Rüland: *Convex integration in materials science*

In this talk Angkana reviewed a number of applications of the method of convex integration in the calculus of variations and the theory of elasticity. She explained both the results and their applications. She covered:

- examples of irregular solutions to elliptic systems found by Mller-√Sverák as an application of convex integration within the quasiconvex theory of elasticity,
- the dichotomy between rigidity and flexibility in the modelling of shape-memory alloys as a non-quasi-convex model problem,
- the m-matrix problem and its resolution by Chlebík-Kirchheim-Preiss,
- the role of constraints, focusing particularly on the characterisation of gradient Young measures coming from sequences of gradients with a pointwise determinant bound at low integrability,
- elastic plates, folding and crumpling.

  The method of convex integration appears here in a variety of different guises and roles.

RESEARCH TALK
Christian Zillinger: *Convex integration arising in the modelling of shape-memory alloys: some remarks on scaling and numerical implementations*

Christian studied convex integration solutions in the context of the modelling of shape-memory alloys. In a first part, he related the maximal regularity of convex integration solutions to the presence of lower bounds in variational models with surface energy. Hence, variational models with surface energy could be viewed as a selection mechanism allowing for or excluding convex integration solutions. Secondly, he presented the first numerical implementations of convex integration schemes for the model problem of the geometrically linearised two-dimensional hexagonal-to-rhombic phase transformation. This was based on joint work with Angkana Rüland and Jamie Taylor.

4 Scientific Progress Made and Outcome of the Meeting

As indicated in the introduction, the main goal of this workshop has been the sharing of information and the exposition of various convex integration techniques used across different fields. Groups of mathematicians working in geometry and materials science often use closely related ideas but have little idea of progress made by the other group. The speakers from each group made an effort to address such a diverse audience, and all other speakers welcomed the opportunity to present their findings to colleagues they have not discussed with before. The schedule provided ample time for the unstructured mathematical discussions. Several new collaborations have been established: Hofmanova and Feireisl begun a project concerning the stochastic perturbations of the Euler system; Lewicka and Wiedemann started discussions on the possibility of extending the rigidity-flexibility results to the case with boundary conditions.

This mini-workshop had 20 participants, ranging from professors to graduate students. Among them were 6 female mathematicians. All participants gave talk. The participants list was international (USA, Czech Republic, Germany, Korea, Poland, Switzerland), with one participant from Canada.

Participants

Alexey Cheskidov, University of Illinois at Chicago (USA)
Mimi Dai, University of Illinois at Chicago (USA)
Sara Daneri, Universitat Erlangen (Germany)
Eduard Feireisl, Academy of Sciences (Czech Republic)
Piotr Gwiazda, Polish Academy of Sciences (Poland)
Martina Hofmanova, Bielefield University (Germany)
Dominik Inauen, University of Zurich (Switzerland)

Seonghak Kim, Kyungpook National University (Korea)
Young-Heon Kim, University of British Columbia (Canada)
Ondrej Kreml, Academy of Sciences (Czech Republic)
Sam Krupa, University of Texas at Austin (USA)
Marta Lewicka, University of Pittsburgh (USA)
Simon Markfelder, University of Wurzburgh (Germany)
Angkana Ruland, Max Planck Institut (Germany)
Roman Shvydkoy, University of Illinois at Chicago (USA)
Agnieszka Swierczewska-Gwiazda, University of Warsaw (Poland)
Daniel Weser, University of Texas at Austin (USA)
Emil Wiedemann, Universitat Ulm (Germany)
Baisheng Yan, Michigan State University (USA)
Christian Zillinger, University of Southern California (USA)

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