Convex Integration for the Gradient Flow of Polyconvex Functionals

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BIRS Workshop on Convex Integration
Banff, Alberta, Canada

August 11–16, 2019

Plan of the Talk

I discuss how the convex integration approaches in [Kim & Y. '15-'18] on the *Perona-Malik and forward-backward equations* can be generalized to study some general *diffusion systems*, including the gradient flow of some polyconvex functionals; this may be viewed as parallel to the study on critical points for polyconvex functionals of [Székelyhidi '04], but focusing on the aspects of **nonuniqueness and instability (flexibility)** of the IBVP.

1. **Introduction and Main Results**
    - Gradient flow as nonhomogeneous PDI
    - Convex integration: T_N-configurations and the building blocks
2. **Condition (OC) and Existence for Diffusion System**
    - General existence for diffusion system by Baire's category
    - Construction and the density of subsolution sets $\mathcal{U}$ and $\mathcal{U}_\epsilon$
3. **Compatibility of (OC) with Polyconvexity**
    - τ_5-configuration supported by a polyconvex function on $\mathbb{M}^{2\times 2}$
    - Perturbations, the polyconvex functions F and open sets Σ

Let $\mathbb{M}^{m\times n}$ be the space of $m \times n$ matrices and $F\colon \mathbb{M}^{m\times n} \to \mathbb{R}$ be smooth. Consider the energy

$$\mathcal{E}(\mathbf{u}) = \int_\Omega F(D\mathbf{u})dx, \quad \mathbf{u}\colon \Omega \to \mathbb{R}^m; \tag{1}$$

here $\Omega \subset \mathbb{R}^n$ is bounded open and $D\mathbf{u}$ is the Jacobian matrix of $\mathbf{u}$.

- **Minimization** of $\mathcal{E}$ over a Sobolev space is closely related to the notion of *Morrey's quasiconvexity*. We say that F is **strongly quasiconvex** if for some $\nu > 0$

$$\int_\Omega (F(A + D\phi) - F(A))dx \geq \frac{\nu}{2} \int_\Omega |D\phi|^2 dx \tag{2}$$

  holds for all $A \in \mathbb{M}^{m\times n}, \phi \in C_c^\infty(\Omega; \mathbb{R}^m)$; ($\nu = 0$ is Morrey's quasiconvexity.) In this case, F may not be convex if $m, n \geq 2$.
- If F is C^1, then (2) implies that the *strong rank-one monotonicity*:

$$\langle DF(A + p\otimes\alpha) - DF(A),\, p\otimes\alpha\rangle \geq \nu|p|^2|\alpha|^2 \tag{3}$$

  for all $A \in \mathbb{M}^{m\times n}, p \in \mathbb{R}^m$, and $\alpha \in \mathbb{R}^n$, where $\langle A, B\rangle$ stands for the inner product of $\mathbb{M}^{m\times n}$ and $p\otimes\alpha$ for the matrix $(p_i\alpha_k)$.

- In addition, if F is C^2, condition (3) is equivalent to the uniform strong *Legendre-Hadamard* condition:

$$\sum_{i,j=1}^{m}\sum_{k,l=1}^{n}\frac{\partial^2 F(A)}{\partial a_{ik}\partial a_{jl}}p_i p_j \alpha_k \alpha_l \geq \nu |p|^2 |\alpha|^2 \quad \forall\, p \in \mathbb{R}^m,\ \alpha \in \mathbb{R}^n. \tag{4}$$

- Minimizers of $\mathcal{E}$ in a Dirichlet class satisfy the *Euler-Lagrange equations*:

$$\mathrm{div} DF(D\mathbf{u}) = 0 \quad \text{in } \Omega. \tag{5}$$

We say (5) is **strongly elliptic** if (4) holds for some $\nu > 0$. The well-known results of [Evans '86] and [Müller & Šverák '03; Székelyhidi '04] show that, *unlike for a convex F,* a Lipschitz weak solution $\mathbf{u}$ of *elliptic system* (5) may not be a minimizer of $\mathcal{E}$.

- We study a parabolic companion of (5), known as **the** (L^2) **gradient flow** of energy $\mathcal{E}$. To be more specific, given $T > 0$ and $\mathbf{u}_0 : \bar{\Omega} \to \mathbb{R}^m$, we study the initial-boundary value problem (IBVP):

$$\begin{cases} \mathbf{u}_t = \mathrm{div}\, DF(D\mathbf{u}) & \text{in } \Omega_T = \Omega \times (0,T), \\ \mathbf{u}(x,t) = \mathbf{u}_0(x) & (x \in \partial\Omega,\ 0 < t < T), \\ \mathbf{u}(x,0) = \mathbf{u}_0(x) & (x \in \Omega). \end{cases} \tag{6}$$

- If F is convex, then **monotone operator theory** applies to (6); in particular, (6) has a unique weak solution. However, there is no general theory on the solvability of IBVP (6) under condition (3). For general gradient problems (see [Ambrosio et al '05]), one may use a time-discretization approximation based on the **implicit Euler scheme** to produce the so-called *generalized minimizing movements* and *Young measure solutions* for (6).
- The existence of true weak solutions remains essentially open for general nonconvex F's, including the **strongly polyconvex** functions

$$F(A) = \epsilon|A|^2 + G(A, \det A) \quad (\epsilon > 0,\ G(A,\delta) \text{ smooth convex}) \quad (7)$$

  on $\mathbb{M}^{2\times 2}$ considered in [Székelyhidi '04], which satisfy (2) with $\nu = 2\epsilon$.

  The similar open question remains open for **elastodynamics problems**, despite many existing works; see [Kim & Koh '19].
- Our main result is concerning the **nonuniqueness and instability (or flexibility)** of Lipschitz weak solutions of (6) for certain strongly polyconvex functions F of the form (7).

The main result

Theorem (A) (Y. '19)

There exist smooth strongly polyconvex functions $F\colon \mathbb{M}^{2\times 2} \to \mathbb{R}$ and smooth functions $\mathbf{u}_0$ such that the IBVP (6) *possesses a sequence of Lipschitz weak solutions that converges weakly* to a function which is not a Lipschitz weak solution itself.*

- We stress that the polyconvex functions and anomalous solutions for system (5) constructed in [Székelyhidi '04] would not give an example for our theorem. One must study the full parabolic problem, not just the stationary elliptic problem.
- In the theorem we may choose $\mathbf{u}_0(x) = Ax$ for some $A \in \mathbb{M}^{2\times 2}$. In this case, the Lipschitz weak solutions in the given sequence are (eventually) distinct and not a classical solution by *quasiconvexity*; this proves the **nonuniqueness** of the IBVP. However, we will not address the further irregularity of these weak solutions: e.g., whether they can be nowhere C^1 in x, but $C^{1,\alpha}$ in t.)

The main approach

Consider general nonlinear diffusion system in divergence form:

$$\mathbf{u}_t = \operatorname{div} \sigma(D\mathbf{u}) \quad \text{in } \Omega_T, \tag{8}$$

where $\sigma = (\sigma^i_k(A)) \colon \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a given diffusion flux.
If there exist functions $\mathbf{v}^1, \dots, \mathbf{v}^m \colon \Omega_T \to \mathbb{R}^n$ such that

$$u^i = \operatorname{div} \mathbf{v}^i, \quad \mathbf{v}^i_t = \sigma^i(D\mathbf{u}) \quad a.e.\ (x,t) \in \Omega_T, \tag{9}$$

then $\mathbf{u} = (u^1, \dots, u^m)$ is a weak solution of (8). We generalize the framework of [Zhang '06; Kim & Y. '15–'18] to setup (9) as a (space-time) **partial differential inclusion** (PDI), by introducing the function

$$\mathbf{w} = [\mathbf{u}, (\mathbf{v}^i)] \colon \Omega_T \to \mathbb{R}^m \times (\mathbb{R}^n)^m$$

with space-time Jacobian matrix $\nabla \mathbf{w} = \begin{bmatrix} D\mathbf{u} & \mathbf{u}_t \\ (D\mathbf{v}^i) & (\mathbf{v}^i_t) \end{bmatrix} \in \mathbb{M}^{(m+nm)\times(n+1)}$;

here $\mathbb{M}^{(m+nm)\times(n+1)}$ is the space of matrices $X = \begin{bmatrix} A & a \\ (B^i) & (b^i) \end{bmatrix}$ with

$$A \in \mathbb{M}^{m\times n},\ a \in \mathbb{R}^m,\ \ B^i \in \mathbb{M}^{n\times n},\ b^i \in \mathbb{R}^n\ \ (i = 1, \dots, m).$$

- For $z \in \mathbb{R}^m$, define the matrix set $\mathcal{K}(z) \subset \mathbb{M}^{(m+nm)\times(n+1)}$ by

$$\mathcal{K}(z) = \left\{ \begin{bmatrix} A & a \\ (B^i) & (\sigma^i(A)) \end{bmatrix} : \operatorname{tr}(B^i) = z^i \ (i = 1, \dots, m) \right\}. \quad (10)$$

Then (9) is equivalent to the **nonhomogeneous** PDI for $\mathbf{w}$

$$\nabla \mathbf{w}(x,t) \in \mathcal{K}(\mathbf{u}(x,t)) \quad a.e. \ (x,t) \in \Omega_T. \quad (11)$$

- The celebrated works [Müller & Šverák '03; Székelyhidi '04] mentioned above rely on studying the elliptic system (5) in 2-D as a **homogeneous** PDI for $U = (\mathbf{u}, \tilde{\mathbf{u}}) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^{2m}$,

$$DU = \begin{pmatrix} D\mathbf{u} \\ D\tilde{\mathbf{u}} \end{pmatrix} \in K_F = \left\{ \begin{pmatrix} A \\ DF(A)J \end{pmatrix} : A \in \mathbb{M}^{m\times 2} \right\}, \quad (12)$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\tilde{\mathbf{u}}$ is a stream function of $DF(D\mathbf{u})$.

- Under (3), the set K_F has *no rank-1 connections*; however, its *rank-1 convex hull* K_F^{rc} is sufficiently large to contain many special *T_4 or T_5 configurations* to build the so-called in-approximations; in this way, Gromov's **convex integration** is adapted to constructing *Lipschitz but nowhere-C^1 weak solutions* for certain strongly quasiconvex or polyconvex functions F on $\mathbb{M}^{2\times 2}$.

The convex integration and Baire's category methods

- There are primarily two approaches for studying PDIs. One is a generalization of Gromov's **convex integration method** by Müller & Šverák; the other is the **Baire category method** developed by Dacorogna & Marcellini based on early ideas for ordinary differential inclusions. Both methods rely on intermittent approximations by certain relaxed (often open) relations.
- In addition to many important earlier applications to phase-transition and ferromagnetics problems, the method of convex integration has recently found remarkable success in many important PDE problems, e.g.: Incompressible Euler equations ([De Lellis & Székelyhidi '09, '13; et al '15]); Active scalar equations ([Shvydkoy '11]); Porous medium equations ([Cordoba, Faraco & Gancedo '11]); Perona-Malik and forward-backward parabolic equations ([Zhang '06; Kim & Y. '15–'18]); 2-D Monge-Ampère equations ([Lewicka & Pakzad '17]); Onsager's conjecture ([Isett '18]); Navier-Stokes equation ([Buckmaster & Vicol '19]), etc.

The main building blocks

The key building blocks for convex integration of PDIs are the rank-1 convex hulls of matrix sets. We need the following generalization of Tartar's famous *T_4-configurations*.

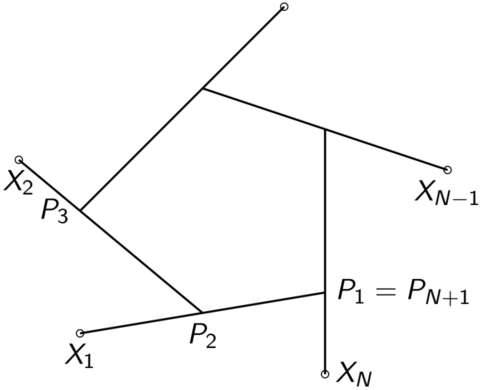
Definition: Let $N \geq 2$ and $\{X_1, X_2, \ldots, X_N\} \subset \mathbb{M}^{p\times q}$. The N-tuple $(X_1, X_2, \ldots, X_N)$ is called a **T_N-configuration** if $\exists$ P, $C_1, \ldots, C_N$ in $\mathbb{M}^{p\times q}$ and $\kappa_1, \ldots, \kappa_N$ in $\mathbb{R}$, with $\operatorname{rank}(C_j) = 1$, $\sum_{j=1}^N C_j = 0$ and $\kappa_j > 1$, such that

$$\begin{cases} X_1 = P + \kappa_1 C_1, \\ X_2 = P + C_1 + \kappa_2 C_2, \\ \quad\vdots \\ X_N = P + C_1 + \cdots + C_{N-1} + \kappa_N C_N. \end{cases} \tag{13}$$

Let $P_1 = P$, $P_j = P + C_1 + \cdots + C_{j-1}$ for $j = 2, 3, \ldots, N$, and define

$$T(X_1, \ldots, X_N) = \cup_{j=1}^N \{(1-\lambda)X_j + \lambda P_j \colon 0 < \lambda \leq 1\}. \tag{14}$$

Remark: We do not require that $\{X_1, X_2, \ldots, X_N\}$ contain no rank-1 connections; this allows for $N = 2$ and rank-1 connections.



To study the space-time PDI (11), due to the linear constraints in $\mathcal{K}(z)$, we focus on the **admissible** T_N**-configurations** in $\mathbb{M}^{(m+nm)\times(n+1)}$ whose determining rank-1 matrices are of the form

$$C = \begin{bmatrix} p \otimes \alpha & sp \\ (\beta^i \otimes \alpha) & (s\beta^i) \end{bmatrix}; \;\; p \in \mathbb{R}^m,\; s \in \mathbb{R},\; \alpha \neq 0,\; \beta^i \in \mathbb{R}^n,\; \beta^i \cdot \alpha = 0.$$

Theorem (**Convex Integration Building Blocks**)

(i) *Let* $Y \in T(X_1, \dots, X_N)$, *where* $(X_1, \dots, X_N)$ *is an admissible* T_N*-configuration in* $\mathbb{M}^{(m+nm)\times(n+1)}$. *Then, for all bounded open* $G \subset \mathbb{R}^{n+1}$ *and* $\epsilon > 0$, $\exists\, \omega = [\varphi, (\psi^i)] \in C_c^\infty(\mathbb{R}^{n+1}; \mathbb{R}^m \times (\mathbb{R}^n)^m)$ *with*

- (a) $\operatorname{supp} \omega \subset\subset G$, $div\, \psi^i = 0$ *in* $\mathbb{R}^{n+1}$ *for all* $i = 1, \dots, m$, *and* $\int_{\mathbb{R}^n} \varphi(x,t)\, dx = 0$ *for all* $t \in \mathbb{R}$;
- (b) $\|\omega\|_{L^\infty(\mathbb{R}^{n+1})} < \epsilon$ *and* $Y + \nabla\omega \in [\overline{T(X_1, \dots, X_N)}]_\epsilon$ *on* $\mathbb{R}^{n+1}$;
- (c) *there exist an open set* $V \subset\subset G$ *such that*

$$|V| \geq (1-\epsilon)|G|, \quad Y + \nabla\omega \in \{X_1, X_2, \dots, X_N\} \ \text{ in } V.$$

(ii) **[Kim & Y. '15]** *Let* $\phi \in W_0^{1,\infty}(Q_0)$ *satisfy* $\int_{\tilde{Q}_0} \phi(x,t)\, dx = 0$ *for all* $t \in (0,1)$. *Let* $\tilde{\phi} = (\mathcal{L}_{\bar{y},l}\phi)(y) = l\phi(\frac{y-\bar{y}}{l})$ *for* $y \in Q_{\bar{y},l}$. *Then there exists* $\tilde{g} = \mathcal{R}_{\bar{y},l}\phi$ *in* $W_0^{1,\infty}(Q_{\bar{y},l}; \mathbb{R}^n)$ *such that* $div\, \tilde{g} = \tilde{\phi}$ *a.e. in* $Q_{\bar{y},l}$ *and*

$$\|\tilde{g}_t\|_{L^\infty(Q_{\bar{y},l})} \leq C_n l \|\tilde{\phi}_t\|_{L^\infty(Q_{\bar{y},l})}. \tag{15}$$

Moreover, if in addition $\phi \in C^1(Q_0)$ *then* $\tilde{g} = \mathcal{R}_{\bar{y},l}\phi \in C^1(Q_{\bar{y},l}; \mathbb{R}^n)$.

Definition: An N-tuple $(\xi_1, \xi_2, \ldots, \xi_N)$ with $\xi_j \in \mathbb{M}^{m\times n} \times (\mathbb{R}^n)^m$ is called a τ_N-**configuration** provided that there exist ρ, $\gamma_1, \ldots, \gamma_N$ in $\mathbb{M}^{m\times n} \times (\mathbb{R}^n)^m$ and $\kappa_1 > 1, \ldots, \kappa_N > 1$ such that

$$\begin{cases} \xi_1 = \rho + \kappa_1\gamma_1, \\ \xi_2 = \rho + \gamma_1 + \kappa_2\gamma_2, \\ \quad\vdots \\ \xi_N = \rho + \gamma_1 + \cdots + \gamma_{N-1} + \kappa_N\gamma_N, \end{cases} \tag{16}$$

where $\gamma_j = [p_j \otimes \alpha_j, (s_j\beta_j^i)]$, with $s_j \in \mathbb{R}$, α_j, $\beta_j^i \in \mathbb{R}^n$, $\alpha_j \neq 0$ and $p_j \in \mathbb{R}^m$ satisfying

$$\textstyle\sum_{j=1}^N s_j p_j = 0, \quad \sum_{j=1}^N s_j\beta_j^i = 0 \quad (i = 1, \ldots, m), \tag{17}$$

$$\textstyle\sum_{j=1}^N p_j \otimes \alpha_j = 0, \quad \sum_{j=1}^N \beta_j^i \otimes \alpha_j = 0 \quad (i = 1, \ldots, m), \tag{18}$$

$$\beta_j^i \cdot \alpha_j = 0 \quad (j = 1, \ldots, N;\ i = 1, \ldots, m). \tag{19}$$

Define $\rho_1 = \rho,\ \rho_j = \rho + \gamma_1 + \cdots + \gamma_{j-1}$ for $j = 2, \ldots, N$, and

$$\tau(\xi_1, \ldots, \xi_N) = \cup_{j=1}^N (\xi_j, \rho_j]. \tag{20}$$

The main structural assumption

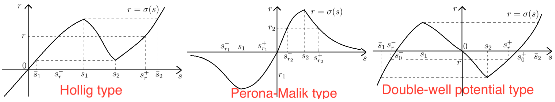
Definition: Let $\sigma\colon \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ and $\mathbb{K} = \{[A, (\sigma^i(A))] : A \in \mathbb{M}^{m\times n}\}$. We say that σ satisfies **Condition (OC)** if there exists a nonempty bounded open set Σ in $\mathbb{M}^{m\times n} \times (\mathbb{R}^n)^m$ such that

$$\begin{cases} \forall\ [A, (b^i)] \in \Sigma\ \ \exists\ N \geq 2 \text{ and } \tau_N\text{-configuration } (\xi_1, \ldots, \xi_N) \\ \text{such that } \xi_j \in \mathbb{K} \text{ for all } j \text{ and } [A, (b^i)] \in \tau(\xi_1, \ldots, \xi_N) \subseteq \Sigma. \end{cases} \tag{21}$$

Remarks: [*Comparison with Condition (C) in the previous works.*]

- Condition (OC) is substantially different from Condition (C) of [Müller & Šverák '03; Székelyhidi '04] because the τ_N-configurations required have *no matrix rank-1 structures*; moreover, it is defined for all dimensions m, n, while Condition (C) is only for $n = 2$.
- Even when $n = 2$, the τ_N-configurations are only equivalent to certain spatial T_N-configurations that are more restrictive than the usual T_N-configurations used for Condition (C); a general spatial T_N-configuration may not produce a τ_N-configuration at all.
- In addition, Condition (OC) is more analytic and suitable for the use of **Implicit Function Theorem**, which avoids the more geometrical transversality and stability analysis of Condition (C).

- For **scalar function cases** ($m = 1$), we allow $N = 2$ to include the following *forward-backward diffusion equations* (for $n = 1$):



- For **2-D cases** ($n = 2$), (19) becomes $(\beta^i_j)^\perp = q^i_j \alpha_j$ for $q^i_j \in \mathbb{R}$, where $\beta^\perp = \beta J$. Define $\mathcal{L} \colon \mathbb{M}^{m\times 2} \times (\mathbb{R}^2)^m \to \mathbb{M}^{2m\times 2}$ by

$$\mathcal{L}([A,(b^i)]) = \begin{bmatrix} A \\ BJ \end{bmatrix} \quad \forall\, B = (b^i_k) \in \mathbb{M}^{m\times 2}. \tag{22}$$

Then $(\xi_1, \dots, \xi_N)$ *is a* τ_N*-configuration in* $\mathbb{M}^{m\times 2} \times (\mathbb{R}^2)^m \iff (\mathcal{L}\xi_1, \dots, \mathcal{L}\xi_N)$ *is a* T_N*-configuration in* $\mathbb{M}^{2m\times 2}$ *with rank-1 matrices* $C_j = \begin{pmatrix} p_j \\ s_j q_j \end{pmatrix} \otimes \alpha_j$ *satisfying* **the more restrictive conditions:**

$$\begin{cases} \sum_{j=1}^N p_j \otimes \alpha_j = 0, \ \sum_{j=1}^N s_j q_j \otimes \alpha_j = 0, \\ \sum_{j=1}^N s_j p_j = 0, \ \sum_{j=1}^N q_j \otimes \alpha_j \otimes \alpha_j = 0. \end{cases} \tag{23}$$

- Thus a T_N-configuration in $\mathbb{M}^{2m\times 2}$ may not produce a τ_N-configuration at all; *this is the case for the T_5 example of [Székelyhidi '04] which does not produce a τ_5-configuration!*
- The set of T_N-configurations satisfying (23) may be degenerate and hard to study. We thus restrict ourselves to a set of even more special T_N-configurations, which turns out sufficient for our purpose.

Definition: Let $n=2$ and $N\geq 3$. Let M_N' be the set of T_N-configurations $(X_1,\dots,X_N)$ in $\mathbb{M}^{2m\times 2}$ whose determining rank-1 matrices are given by $C_j=\begin{pmatrix} p_j \\ (\alpha_j\cdot\delta)q_j\end{pmatrix}\otimes\alpha_j$, where $p_j,q_j\in\mathbb{R}^m$ and $\alpha_j,\delta\in\mathbb{R}^2$ satisfy that *at least three of α_j's are mutually noncollinear* and that

$$\sum_{j=1}^N p_j\otimes\alpha_j=0,\quad \sum_{j=1}^N q_j\otimes\alpha_j\otimes\alpha_j=0. \tag{24}$$

(Thus all conditions in (23) are automatically satisfied with $s_j=\alpha_j\cdot\delta$.) We define $\mathcal{M}_N'=\mathcal{L}^{-1}(M_N')$ to be the set of **special τ_N-configurations** in $\mathbb{M}^{m\times 2}\times(\mathbb{R}^2)^m$.

The general existence theorem under Condition (OC)

The main technical theorem to prove our main result is the following existence result under Condition (OC):

Theorem (B) (Y. '19)

Let $\sigma\colon \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ *be continuous and satisfy* Condition (OC), *with open set* $\Sigma \subset \mathbb{M}^{m\times n} \times (\mathbb{R}^n)^m$ *as given in the definition. Let* $\bar{\mathbf{u}} \in C^1(\bar{\Omega}_T; \mathbb{R}^m)$ *and* $\bar{\mathbf{v}}^i \in C^1(\bar{\Omega}_T; \mathbb{R}^n)$ *satisfy*

$$\bar{u}^i = div\,\bar{\mathbf{v}}^i, \quad [D\bar{\mathbf{u}}, (\bar{\mathbf{v}}^i_t)] \in \Sigma \quad on\ \bar{\Omega}_T \tag{25}$$

for $i = 1, \dots, m$*. Then there exists a sequence* $\{\mathbf{u}_\mu\}$ *of weak solutions of* (8) *in* $W^{1,\infty}(\Omega_T; \mathbb{R}^m)$ *satisfying* $\mathbf{u}_\mu|_{\partial\Omega_T} = \bar{\mathbf{u}}$ *that converges weakly* to* $\bar{\mathbf{u}}$ *in* $W^{1,\infty}(\Omega_T; \mathbb{R}^m)$.

Remark: Condition (25) can be viewed as a **relaxation** for (11); any such $\bar{\mathbf{u}}$'s are called a **subsolution** of diffusion system (8). With an open set Σ as given in Condition (OC), we may construct many nontrivial functions $\bar{\mathbf{u}}$ and $\bar{\mathbf{v}}^i$ satisfying (25).

Existence/nonuniqueness/instability of the IBVP (6) is a simple consequence of Condition (OC). For example:

- Assume $[A, (b^i)] \in \Sigma$; define $\bar{\mathbf{u}} = (\bar{u}^1, \dots, \bar{u}^m)$, $\bar{\mathbf{v}}^i = (\bar{v}^i_1, \dots, \bar{v}^i_n)$ by

$$\bar{u}^i(x,t) = \sum_{k=1}^{n} a_{ik} x_k + \epsilon g(x) t, \quad \bar{v}^i_j(x,t) = \frac{1}{2} a_{ij} x_j^2 + b^i_j t + \epsilon h_j(x) t$$

for $i = 1, \dots, m$; $j = 1, \dots, n$, where

$$\mathbf{h}(x) = (h_1, \cdots, h_n) \in C_c^\infty(\Omega; \mathbb{R}^n), \quad g(x) = \operatorname{div} \mathbf{h}(x),$$

$$g(x) = \operatorname{div} \mathbf{h}(x) = 1 \quad \forall\, x \in \Omega' \subset\subset \Omega.$$

Then, for all sufficiently small $|\epsilon| > 0$, condition (25) holds.

- Each weak solution $\mathbf{u}_\mu$ in Theorem (B) solves the IBVP:

$$\begin{cases} \mathbf{u}_t = \operatorname{div} \sigma(D\mathbf{u}) & \text{in } \Omega_T, \\ \mathbf{u}(x,t) = Ax & (x \in \partial\Omega,\ 0 < t < T), \\ \mathbf{u}(x,0) = Ax & (x \in \Omega). \end{cases} \tag{26}$$

But the weak* limit $\bar{\mathbf{u}}$ is not a solution to (26) since $g(x) = 1$ on Ω'.

Proof of Theorem (B):

The proof is based on the following general existence theorem under a density assumption:

Theorem (C)

Let $\sigma\colon \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ be continuous, $\bar{\mathbf{u}} \in W^{1,\infty}(\Omega_T;\mathbb{R}^m)$, and let $\mathcal{U}$ be a nonempty bounded subset of $W^{1,\infty}_{\bar{\mathbf{u}}}(\Omega_T;\mathbb{R}^m)$. Assume, for each $\epsilon > 0$, there exists a set

$$\mathcal{U}_\epsilon \subset \{\mathbf{u} \in \mathcal{U} \,|\, \|\mathbf{u}_t - \mathit{div}\,\sigma(D\mathbf{u})\|_{H^{-1}(\Omega_T)} < \epsilon\}$$

that is dense in $\mathcal{U}$ in the $L^\infty(\Omega_T;\mathbb{R}^m)$-norm. Then the set

$$\mathcal{S} = \{\mathbf{u} \in W^{1,\infty}_{\bar{\mathbf{u}}}(\Omega_T;\mathbb{R}^m) \,|\, \mathbf{u} \text{ is Lipschitz solution of (8)}\}$$

is dense (thus nonempty) in $\mathcal{U}$ in the $L^\infty(\Omega_T;\mathbb{R}^m)$-norm.

This result is proved by the **Baire category method** similarly as in [Kim & Y. '15, '17, '18]. Note that, *if $u^i = \mathit{div}\,\mathbf{v}^i$, the H^{-1}-norm above can be bounded by $\|\mathbf{v}_t - \sigma(D\mathbf{u})\|_{L^2}$.*

The subsolution sets $\mathcal{U}$ and $\mathcal{U}_\epsilon$

Theorem (B) follows from Theorem (C) if we prove the following:

Theorem (Density Theorem)

Let $\Sigma, \bar{\mathbf{u}}, \bar{\mathbf{v}}^i$ be as given in Theorem (B); fix $m > \|\bar{\mathbf{u}}_t\|_{L^\infty(\Omega_T)}$. Define $\mathcal{U}$ to be the set of $\mathbf{u} \in C^1_{\bar{\mathbf{u}}}(\bar{\Omega}_T; \mathbb{R}^m)$ such that $\|\mathbf{u}_t\|_{L^\infty(\Omega_T)} < m$ and

$$\begin{cases} \exists\, \mathbf{v}^i \in C^1_{\bar{\mathbf{v}}^i, pc}(\Omega_T; \mathbb{R}^n) \text{ with pieces } \{E_j\}_{j=1}^{\mu} \text{ satisfying} \\ u^i = \operatorname{div} \mathbf{v}^i,\ [D\mathbf{u}, (\mathbf{v}^i_t)] \in \Sigma \text{ on } \bar{E}_j \quad \forall\, i = 1, \dots, m;\ j = 1, \dots, \mu, \end{cases}$$

and, for $\epsilon > 0$, define $\mathcal{U}_\epsilon$ to be the set of $\mathbf{u} \in \mathcal{U}$ such that

$$\begin{cases} \exists\, \mathbf{v}^i \in C^1_{\bar{\mathbf{v}}^i, pc}(\Omega_T; \mathbb{R}^n) \text{ with pieces } \{E_j\}_{j=1}^{\mu} \text{ satisfying} \\ u^i = \operatorname{div} \mathbf{v}^i,\ [D\mathbf{u}, (\mathbf{v}^i_t)] \in \Sigma \text{ on } \bar{E}_j;\ \|\mathbf{v}^i_t - \sigma^i(D\mathbf{u})\|_{L^2(\Omega_T)} < \epsilon. \end{cases}$$

Then, for each $\epsilon > 0$, $\mathcal{U}_\epsilon$ is dense in $\mathcal{U}$ in the L^∞-norm.

The proof relies on the **convex integration building block theorem**; property (21) of the open set Σ is critical.

Proof of Density Theorem:

Let $\epsilon > 0$, $\mathbf{u} \in \mathcal{U}$ and $\rho > 0$ be fixed. Then $\|\mathbf{u}_t\|_{L^\infty(\Omega_T)} < m$ and there exist $\mathbf{v}^i \in C^1_{\bar{\mathbf{v}}^i,pc}(\Omega_T;\mathbb{R}^n)$ with piecees $\{E_j\}_{j=1}^{\mu}$ such that

$$u^i = \operatorname{div} \mathbf{v}^i, \quad [D\mathbf{u}, (\mathbf{v}^i_t)] \in \Sigma \quad \text{on } \bar{E}_j$$

for $i = 1, \ldots, m$; $j = 1, \ldots, \mu$.

The goal is to construct $\tilde{\mathbf{u}} \in \mathcal{U}_\epsilon$ with $\|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^\infty(\Omega_T)} < \rho$; that is,

(i) $\tilde{\mathbf{u}} \in C^1_{\bar{\mathbf{u}}}(\bar{\Omega}_T;\mathbb{R}^m)$, $\|\tilde{\mathbf{u}}_t\|_{L^\infty(\Omega_T)} < m$, $\|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^\infty(\Omega_T)} < \rho$, and

(ii) $\exists\, \tilde{\mathbf{v}}^i \in C^1_{\bar{\mathbf{v}}^i,pc}(\Omega_T;\mathbb{R}^n)$ with some pieces $\{P_j\}_{j=1}^{\kappa}$ such that

$$\begin{cases} \tilde{u}^i = \operatorname{div} \tilde{\mathbf{v}}^i \quad \text{on each } \bar{P}_j, \\ [D\tilde{\mathbf{u}}, (\tilde{\mathbf{v}}^i_t)] \in \Sigma \quad \text{on each } \bar{P}_j, \\ \|\tilde{\mathbf{v}}^i_t - \sigma^i(D\tilde{\mathbf{u}})\|_{L^2(\Omega_T)} < \epsilon. \end{cases} \tag{27}$$

Step 1: Fix $\nu \in \{1, \ldots, \mu\}$ and $\bar{y} \in E_\nu$. Let $A = D\mathbf{u}(\bar{y})$ and $b^i = \mathbf{v}^i_t(\bar{y})$; then $[A, (b^i)] \in \Sigma$. By (OC), $\exists\, \tau_N$-configuration $(\xi_1, \xi_2, \ldots, \xi_N)$ in $\mathbb{K}$ given by $\rho = [\tilde{A}, (\tilde{b}^i)], \gamma_j = [p_j \otimes \alpha_j, (s_j \beta^i_j)]$ and $\kappa_j > 1$ such that

$$[A, (b^i)] \in \tau(\xi_1, \ldots, \xi_N) \subset \Sigma.$$

Let $(\tilde{X}_1^s, \dots, \tilde{X}_N^s)$ be the T_N-configuration in $\mathcal{K}(\mathbf{0})$. Let $0 < \tau << 1$ be such that, for

$$X_j^{s,\tau} = (1-\tau)\tilde{X}_j^s + \tau \tilde{P}_j^s \quad (j = 1, 2, \dots, N), \tag{28}$$

the N-tuple $(X_1^{s,\tau}, \dots, X_N^{s,\tau})$ is an admissible T_N-configuration and that $[A, (b^i)] \in \mathbb{P}(T(X_1^{s,\tau}, \dots, X_N^{s,\tau}))$. Since $\mathbb{P}(X_j^{s,\tau}) = \mathbb{P}(\tilde{X}_j^{1,\tau})$ for $s \neq 0$ and

$$\lim_{\tau \to 0^+} \mathrm{dist}(\mathbb{P}(X_j^{1,\tau}); \mathbb{K}) = \mathrm{dist}(\mathbb{P}(X_j); \mathbb{K}) = 0,$$

there exists a further smaller $\tau > 0$ such that

$$\mathrm{dist}(\mathbb{P}(\bar{X}_j^{1,\tau}); \mathbb{K}) < \frac{\epsilon}{8(|\Omega_T|)^{1/2}} \quad (j = 1, 2, \dots, N). \tag{29}$$

Fix such a $\tau > 0$. Then

$$\mathbb{P}(\bar{T}(X_1^{1,\tau}, \dots, X_N^{1,\tau})) \subset \mathbb{P}(T(X_1, \dots, X_N)) \subset \Sigma.$$

Since Σ is open and $\mathbb{P}(\bar{T}(X_1^{1,\tau}, \dots, X_N^{1,\tau}))$ is compact, there exists a number $\delta_\tau > 0$ such that

$$[\mathbb{P}(\bar{T}(X_1^{1,\tau}, \dots, X_N^{1,\tau}))]_{\delta_\tau} \subset \Sigma.$$

Hence, for all $s \neq 0$,

$$\mathbb{P}([\bar{T}(X_1^{s,\tau}, \dots, X_N^{s,\tau})]_{\delta_\tau}) \subset [\mathbb{P}(\bar{T}(X_1^{s,\tau}, \dots, X_N^{s,\tau}))]_{\delta_\tau} \subset \Sigma. \tag{30}$$

Step 2: Apply the **Building Block Theorem** to unit cube $G = Q_0 \subset \mathbb{R}^{n+1}$ with $X^s \in T(\bar{X}_1^{s,\tau}, \ldots, \bar{X}_N^{s,\tau})$ to obtain a function $\omega = [\varphi, (\psi^i)] \in C_c^\infty(Q_0; \mathbb{R}^m \times (\mathbb{R}^n)^m)$ such that

$$\begin{cases} (a)\ \operatorname{div} \psi^i = 0,\ \|\varphi_t\|_{L^\infty(Q_0)} < \epsilon' + M'|s|,\ \|\varphi\|_{L^\infty(Q_0)} < \epsilon',\ \int_{\tilde{Q}_0} \varphi(x,t)\,dx = 0, \\ (b)\ |\{y \in Q_0 : [A + D\varphi(y), (b^i + \psi_t^i(y))] \notin \cup_{j=1}^N \{\mathbb{P}(X_j)\}| < \epsilon', \\ (c)\ [A + D\varphi(y), (b^i + \psi_t^i(y))] \in P([\bar{T}(\tilde{X}_1^{s,\tau}, \ldots, \tilde{X}_N^{s,\tau})]_{\epsilon'})\ \text{ for all } y \in Q_0. \end{cases}$$

Let $0 < l < 1$. Consider functions $[\tilde{\varphi}, (\tilde{\psi}^i)] = \mathcal{L}_{\bar{y},l}[\varphi, (\psi^i))]$ and $\tilde{g}^i = \mathcal{R}_{\bar{y},l}\varphi^i$ defined on $Q_{\bar{y},l}$, where $\mathcal{L}_{\bar{y},l}$ and $\mathcal{R}_{\bar{y},l}$ are defined in the **Building Block Theorem** above. Let

$$\tilde{\mathbf{u}} = \mathbf{u}_{\bar{y},l} = \mathbf{u} + \tilde{\varphi},\ \ \tilde{\mathbf{v}}^i = \mathbf{v}^i_{\bar{y},l} = \mathbf{v}^i + \tilde{\psi}^i + \tilde{g}^i \quad \text{on } Q_{\bar{y},l}. \tag{31}$$

Then $\tilde{\mathbf{u}} \in \mathbf{u} + C_c^\infty(Q_{\bar{y},l})$, $\tilde{\mathbf{v}}^i \in W^{1,\infty}_{\mathbf{v}^i}(Q_{\bar{y},l}) \cap C^1(Q_{\bar{y},l})$, $\operatorname{div} \tilde{\mathbf{v}}^i = \tilde{u}^i$; so

$$\begin{cases} \|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^\infty(Q_{\bar{y},l})} = \|\tilde{\varphi}\|_{L^\infty(Q_{\bar{y},l})} < l\epsilon' < \epsilon', \\ \|\tilde{\mathbf{u}}_t\|_{L^\infty(Q_{\bar{y},l})} < \|\mathbf{u}_t\|_{L^\infty(\Omega_T)} + \epsilon' + M'|s|, \\ \|\tilde{g}^i_t\|_{L^\infty(Q_{\bar{y},l})} \le C_n l(\epsilon' + M'|s|), \\ \|D\tilde{\varphi}\|_{L^\infty(Q_{\bar{y},l})} \le \epsilon' + M, \\ \|\tilde{\psi}^i_t\|_{L^\infty(Q_{\bar{y},l})} \le \epsilon' + M. \end{cases} \tag{32}$$

Step 3: We estimate $\|\tilde{\mathbf{v}}_t^i - \sigma^i(D\tilde{\mathbf{u}})\|_{L^2(Q_{\bar{y},l})}$. Note that

$$\|\tilde{\mathbf{v}}_t^i - \sigma^i(D\tilde{\mathbf{u}})\|_{L^2(Q_{\bar{y},l})} = \|\mathbf{v}_t^i + \tilde{\psi}_t^i + \tilde{g}_t^i - \sigma^i(D\mathbf{u} + D\tilde{\varphi})\|_{L^2(Q_{\bar{y},l})}$$
$$\leq \|\mathbf{v}_t^i - b^i\|_{L^2(Q_{\bar{y},l})} + \|b^i + \tilde{\psi}_t^i - \sigma^i(A + D\tilde{\varphi})\|_{L^2(Q_{\bar{y},l})}$$
$$+\|\tilde{g}_t^i\|_{L^2(Q_{\bar{y},l})} + \|\sigma^i(A + D\tilde{\varphi}) - \sigma^i(D\mathbf{u} + D\tilde{\varphi})\|_{L^2(Q_{\bar{y},l})}.$$

By (32), $\|\tilde{g}_t^i\|_{L^2(Q_{\bar{y},l})} \leq C_n l(\epsilon' + M'|s|)|Q_{\bar{y},l}|^{1/2}$. Note that

$$\|b^i + \tilde{\psi}_t^i - \sigma^i(A + D\tilde{\varphi})\|^2_{L^2(Q_{\bar{y},l})} = \int_{F\cup F^c} |b^i + \tilde{\psi}_t^i - \sigma^i(A + D\tilde{\varphi})|^2 \, dy,$$

where $F = \{y \in Q_{\bar{y},l} \,|\, [A + D\tilde{\varphi}(y), (b^i + \tilde{\psi}_t^i(y))] \notin \{\cup_{j=1}^N \mathbb{P}(X_j)\}\}$.
By Step 2, $|F| < \epsilon'|Q_{\bar{y},l}|$ and, by (32), $|A + D\tilde{\varphi}| \leq 1 + 3M$ and $|Du + D\tilde{\varphi}| \leq 1 + 3M$ on $Q_{\bar{y},l}$. Hence

$$\int_F |b^i + \psi_t^i - \sigma^i(A + D\varphi)|^2 \, dy < \epsilon'(1 + 3M + \tilde{M})^2|Q_{\bar{y},l}|,$$

$$\int_G |b^i + \psi_t^i - \sigma^i(A + D\varphi)|^2 \, dy \leq \frac{\epsilon^2}{32|\Omega_T|}|Q_{\bar{y},l}|,$$

$$\|b^i+\tilde{\psi}_t^i-\sigma^i(A+D\tilde{\varphi})\|^2_{L^2(Q_{\bar{y},l})} \leq \left[(1 + 3M + \tilde{M})\sqrt{\epsilon'} + \frac{\epsilon}{4(|\Omega_T|)^{1/2}}\right] |Q_{\bar{y},l}|^{1/2}.$$

Let

$$m(l) = \max_{1\le j\le N;\ y\in Q_{\bar y,l}} \left(|v_t^i(y) - b^i| + |Du(y) - A|\right).$$

Then $m(l) \to 0$ as $l \to 0^+$. We have the following estimates:

$$\|v_t^i - b^i\|_{L^2(Q_{\bar y,l})} \le m(l)|Q_{\bar y,l}|^{1/2};$$

$$\|\sigma^i(A + D\tilde\varphi) - \sigma^i(Du + D\tilde\varphi)\|_{L^2(Q_{\bar y,l})} \le \alpha(m(l))|Q_{\bar y,l}|^{1/2},$$

where $\alpha(s)$ is the module of continuity of σ. Hence, we obtain

$$\|\tilde{\mathbf{v}}_t^i - \sigma^i(D\tilde{\mathbf{u}})\|_{L^2(Q_{\bar y,l})} \le \Big[(1 + 3M + \tilde M)\sqrt{\epsilon'} + C_n l\epsilon'$$

$$+m(l) + \alpha(m(l)) + 2MC_n l|s| + \frac{\epsilon}{4(|\Omega_T|)^{1/2}}\Big]|Q_{\bar y,l}|^{1/2}.$$

Step 4: We estimate $\mathrm{dist}([D\tilde{\mathbf{u}}, (\tilde{\mathbf{v}}_t^i)];\ \mathbb{P}(\bar T(X_1^{1,\tau}, \dots, X_N^{1,\tau})))$ on $Q_{\bar y,l}$. Since $D\tilde{\mathbf{u}} = D\mathbf{u} + D\varphi$ and $\tilde{\mathbf{v}}_t^i = \mathbf{v}_t^i + \tilde\psi_t^i + \tilde g_t^i$, we have on $Q_{\bar y,l}$,

$$\mathrm{dist}([D\tilde{\mathbf{u}}, (\tilde{\mathbf{v}}_t^i)];\ \mathbb{P}(\bar T(X_1^{1,\tau}, \dots, X_N^{1,\tau})))$$

$$\le \mathrm{dist}([A+D\tilde\varphi, (b^i+\tilde\psi_t^i)];\ \mathbb{P}(\bar T(X_1^{1,\tau}, \dots, X_N^{1,\tau}))) + |[D\mathbf{u}-A, (\mathbf{v}_t^i - b^i + \tilde g_t^i)]|$$

$$\le \mathrm{dist}([A+D\tilde\varphi, (b^i+\tilde\psi_t^i)];\ \mathbb{P}(\bar T(X_1^{1,\tau}, \dots, X_N^{1,\tau}))) + |D\mathbf{u}-A| + |(\mathbf{v}_t^i - b^i)| + |\tilde g_t^i|,$$

$$< (1 + C_n l)\epsilon' + 2m(l) + 2MC_n l|s|.$$

Step 5: In this step, we select the small numbers $\epsilon' \in (0,1)$ and $s \neq 0$ in the previous estimates to ensure that, for all sufficiently small $l \in (0,1)$, it holds that

$$\begin{cases} \|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^\infty(Q_{\bar{y},l})} < \rho, \\ \|\tilde{\mathbf{u}}_t\|_{L^\infty(Q_{\bar{y},l})} < m, \\ [D\tilde{\mathbf{u}}, (\tilde{\mathbf{v}}_t^i)] \in \Sigma \ \text{ on } Q_{\bar{y},l}, \\ \|\tilde{\mathbf{v}}_t^i - \sigma^i(D\tilde{\mathbf{u}})\|_{L^2(Q_{\bar{y},l})} < \dfrac{\epsilon}{2(|\Omega_T|)^{1/2}} |Q_{\bar{y},l}|^{1/2}. \end{cases} \tag{33}$$

Step 6: Fixed ν, the family $\{Q_{\bar{y},l} \mid \bar{y} \in E_\nu,\ 0 < l < l_{\bar{y}}\}$ forms a **Vitali covering** of the set E_ν by closed cubes. There exists a countable subfamily of disjoint closed cubes $\{P_{\nu,k} = Q_{\bar{y}_k,l_k} \mid k = 1, 2, \ldots\}$ such that

$$E_\nu = (\cup_{k=1}^\infty P_{\nu,k}) \cup R_\nu, \quad |R_\nu| = 0.$$

Let $\tilde{\mathbf{u}}_{\nu,k} = \mathbf{u}_{\bar{y}_k,l_k}$ and $\tilde{\mathbf{v}}^i_{\nu,k} = \mathbf{v}^i_{\bar{y}_k,l_k}$ be defined by (31) on $P_{\nu,k} = Q_{\bar{y}_k,l_k}$. For each $\nu = 1, 2, \ldots, \mu$, let N_ν be such that

$$\left|\cup_{k=N_\nu+1}^\infty P_{\nu,k}\right| = \sum_{k=N_\nu+1}^\infty |P_{\nu,k}| < \frac{\epsilon^2}{2\mu M^2}. \tag{34}$$

Consider the partition

$$\Omega_T = \left(\cup_{\nu=1}^{\mu} \cup_{k=1}^{N_\nu} P_{\nu,k}\right) \cup P, \tag{35}$$

where $P = \Omega_T \setminus \left(\cup_{\nu=1}^{\mu} \cup_{k=1}^{N_\nu} P_{\nu,k}\right) = \left(\cup_{\nu=1}^{\mu} \cup_{k=N_\nu+1}^{\infty} P_{\nu,k}\right) \cup R$ with $|R| = 0$. Using partition (35), define

$$\tilde{\mathbf{u}} = \mathbf{u}\chi_P + \sum_{\nu=1}^{\mu}\sum_{k=1}^{N_\nu} \tilde{\mathbf{u}}_{\nu,k}\chi_{P_{\nu,k}}, \quad \tilde{\mathbf{v}}^i = \mathbf{v}\chi_P + \sum_{\nu=1}^{\mu}\sum_{k=1}^{N_\nu} \tilde{\mathbf{v}}^i_{\nu,k}\chi_{P_{\nu,k}}.$$

Then $\tilde{\mathbf{u}} - \mathbf{u} \in C_c^\infty(P_{\nu,k})$, $\tilde{\mathbf{v}}^i - \mathbf{v}^i \in C^1(P_{\nu,k})$, $\tilde{\mathbf{u}} \in W^{1,\infty}_{\tilde{\mathbf{u}}}(\Omega_T) \cap C^1(\bar{\Omega}_T; \mathbb{R}^m)$ and $\tilde{\mathbf{v}}^i \in C^1_{\tilde{\mathbf{v}}^i,pc}(\Omega_T; (\mathbb{R}^n)^m)$ with pieces $\{P,\ P_{\nu,k} \mid \nu = 1, \ldots, \mu,\ k = 1, \ldots, N_\nu\}$. Then, all requirements in (i) and (ii) at the start of the proof are satisfied because

$$\|\tilde{\mathbf{v}}^i_t - \sigma^i(D\tilde{\mathbf{u}})\|^2_{L^2(\Omega_T)}$$

$$= \sum_{\nu=1}^{\mu}\sum_{k=1}^{N_\nu} \|\tilde{\mathbf{v}}^i_t - \sigma^i(D\tilde{\mathbf{u}})\|^2_{L^2(P_{\nu,k})} + \sum_{\nu=1}^{\mu}\sum_{k=N_\nu+1}^{\infty} \|\mathbf{v}^i_t - \sigma^i(D\mathbf{u})\|^2_{L^2(P_{\nu,k})}$$

$$\le \sum_{\nu=1}^{\mu}\sum_{k=1}^{N_\nu} \frac{\epsilon^2}{4|\Omega_T|}|P_{\nu,k}| + \sum_{\nu=1}^{\mu}\sum_{k=N_\nu+1}^{\infty} M^2|P_{\nu,k}| \le \frac{\epsilon^2}{4|\Omega_T|}|\Omega_T| + \frac{\mu M^2\epsilon^2}{2\mu M^2} < \epsilon^2.$$

III. Compatibility of Condition (OC) with Polyconvexity

In this final part we discuss the following compatibility result on $\mathbb{M}^{2\times 2}$.

Theorem (D) (Y. '18)

There exist strongly polyconvex functions F on $\mathbb{M}^{2\times 2}$ such that $\sigma = DF$ satisfies Condition (OC) with $N = 5$.

Remark:

- The search for a $\mathcal{T}_5$-configuration supported by a strongly polyconvex function is greatly aided by the `linear programming` and `jacobian` computations *using MATLAB*, but our computations are **more restrictive** than those in [Székelyhidi '04].
- Also, for the special $\mathcal{T}_5$-configuration constructed, the required polyconvex functions F can be constructed for "**generic values**" of $\{D^2F(A_i^0)\}$; we derive such a result directly from the construction of F as the result of [Sz '04] on stably embedded T_N-configurations may not be available for the special T_N-configurations due to dimension deficiency.

A τ_5-configuration in M_5' supported by a polyconvex F_0

Let $F(A) = \frac{\epsilon}{2}|A|^2 + G(A, \det A)$ on $\mathbb{M}^{2\times 2}$ with a smooth G. Then

$$\sigma = DF(A) = \epsilon A + G_A(\tilde{A}) + G_\delta(\tilde{A}) \operatorname{cof} A; \quad \tilde{A} = (A, \det A). \tag{36}$$

Suppose $(X_1, \ldots, X_5) \in M_N'$ with $X_j = \begin{bmatrix} A_j \\ B_j \end{bmatrix}$. Then $X_j \in K_F \iff$

$$\epsilon A_j + G_A(\tilde{A}_j) + G_\delta(\tilde{A}_j) \operatorname{cof} A_j = -B_j J. \tag{37}$$

It is well known that $\exists$ smooth convex $G\colon \mathbb{M}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ with

$$G(\tilde{A}_j) = c_j, \;\; G_A(\tilde{A}_j) = Q_j, \;\; G_\delta(\tilde{A}_j) = d_j$$

provided $c_j - c_i > \langle Q_i, A_j - A_i\rangle + d_i(\det A_j - \det A_i)$ for $i \neq j$.
Under (37), this condition holds for sufficiently small $\epsilon > 0$ provided

$$c_i - c_j + d_i \det(A_i - A_j) + \langle A_i - A_j, B_i J\rangle < 0 \quad (i \neq j). \tag{38}$$

Lemma (MATLAB Lemma 1)

There exists $(X_1^0, \ldots, X_5^0) \in M_5'$ *such that* (38) *holds for some* $c_1, \ldots, c_5$; $d_1, \ldots, d_5$. *Also,* $\forall\, 0 < \epsilon << 1$, $\exists$ *smooth convex* $G\colon \mathbb{M}^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ *such that* $F_0(A) = \frac{\epsilon}{2}|A|^2 + G(A, \det A)$ *satisfies that* $X_j^0 \in K_{F_0}$ *for all* j.

Perturbations of $(X_1^0, \ldots, X_5^0)$ and F_0

To embed more T_5-configurations on $(K_F)_5$, we perturb $(X_1^0, \ldots, X_5^0)$ and F_0.

Perturbation of F_0: Let $B_1(0) \subset \mathbb{M}^{2\times 2}$, $\zeta \in C_c^\infty(B_1(0))$ with $0 \le \zeta(A) \le 1$, $\zeta(0) = 1$. Given $r > 0$ and tensor $H = (H^{pqij})$ with $H^{pqij} = H^{ijpq} \in \mathbb{R}$, define

$$V_{H,r}(A) = \frac{1}{2}\zeta(A/r) \sum_{i,j,p,q \in \{1,2\}} H^{ijpq} a_{ij} a_{pq} \quad (A = (a_{ij}) \in \mathbb{M}^{2\times 2}).$$

Let $r_0 = \min_{i \neq j} |A_i^0 - A_j^0| > 0$. Let F be a perturbation of F_0 of the form:

$$F(A) = F_0(A) + \sum_{j=1}^{5} V_{\tilde{H}_j, r_0}(A - A_j^0) \quad \text{(with } \tilde{H}_j \text{ to be chosen).} \tag{39}$$

Then

$$DF(A_j^0) = DF_0(A_j^0), \quad D^2F(A_j^0) = D^2F_0(A_j^0) + \tilde{H}_j; \tag{40}$$

thus, $X_j^0 \in K_F$, and F will be strongly polyconvex if

$$\sum_{j=1}^{5} |\tilde{H}_j| < \frac{\epsilon}{C} \quad \text{(with a } C \text{ independent of } r_0 \text{ and } \{\tilde{H}_j\}). \tag{41}$$

Perturbations of $(X_1^0, \dots, X_5^0)$**:** Perturb $(X_1^0, \dots, X_5^0)$ around each vertex of the "pentagon" $[P_1^0 \cdots P_5^0]$ by the **parameters:**

$$\begin{cases} Q \in \mathbb{M}^{4\times 2} \cong \mathbb{R}^8, \ \delta = \delta^0 = (1,1), \\ \alpha_1 = (-1, z_1), \alpha_2 = (y_2, -1), \alpha_3 = (1, z_3), \alpha_4 = (1, z_4), \alpha_5 = (y_5, 1), \\ p_3 = (p_{31}, p_{32}), \ p_4 = (p_{41}, p_{42}), \ p_5 = (p_{51}, p_{52}), \\ q_4 = (q_{41}, q_{42}), \ q_5 = (q_{51}, q_{52}), \ \ \kappa_1, \ \kappa_2, \ \kappa_3, \ \kappa_4, \ \kappa_5. \end{cases}$$

The resulting p_1, p_2, q_1, q_2 and q_3 from (24) are thus given by:

$$\begin{cases} p_1 = \frac{y_2 z_3+1}{1-y_2 z_1} p_3 + \frac{y_2 z_4+1}{1-y_2 z_1} p_4 + \frac{y_2+y_5}{1-y_2 z_1} p_5, \\ p_2 = \frac{z_1+z_3}{1-y_2 z_1} p_3 + \frac{z_1+z_4}{1-y_2 z_1} p_4 + \frac{y_5 z_1+1}{1-y_2 z_1} p_5, \\ q_1 = \frac{(y_2 z_4+1)(z_3-z_4)}{(z_1+z_3)(y_2 z_1-1)} q_4 + \frac{(y_2+y_5)(y_5 z_3-1)}{(z_1+z_3)(y_2 z_1-1)} q_5, \\ q_2 = -\frac{(z_1+z_4)(z_3-z_4)}{(y_2 z_1-1)(y_2 z_3+1)} q_4 - \frac{(y_5 z_1+1)(y_5 z_3-1)}{(y_2 z_1-1)(y_2 z_3+1)} q_5, \\ q_3 = -\frac{(z_1+z_4)(y_2 z_4+1)}{(z_1+z_3)(y_2 z_3+1)} q_4 - \frac{(y_2+y_5)(y_5 z_1+1)}{(z_1+z_3)(y_2 z_3+1)} q_5. \end{cases} \tag{42}$$

Let $Y = (z_1, y_2, z_3, z_4, y_5, p_3, p_4, p_5, q_4, q_5, \kappa_1, \dots, \kappa_5) \in \mathbb{R}^{20}$ and

$$C_j = C_j(Y) = \begin{pmatrix} p_j \\ (\alpha_j \cdot \delta^0) q_j \end{pmatrix} \otimes \alpha_j \quad (j = 1, \dots, 5).$$

For each $\nu = 1, \ldots, 5$, define

$$\begin{cases} Z_1^\nu(Y) = \kappa_\nu C_\nu, \\ Z_2^\nu(Y) = C_\nu + \kappa_{\nu+1} C_{\nu+1}, \\ Z_3^\nu(Y) = C_\nu + C_{\nu+1} + \kappa_{\nu+2} C_{\nu+2}, \\ Z_4^\nu(Y) = C_\nu + C_{\nu+1} + C_{\nu+2} + \kappa_{\nu+3} C_{\nu+3}, \\ Z_5^\nu(Y) = C_\nu + C_{\nu+1} + C_{\nu+2} + C_{\nu+3} + \kappa_{\nu+4} C_{\nu+4}. \end{cases} \tag{43}$$

Define $X_j^\nu(Y,Q) = Q + Z_j^\nu(Y)$ for all ν and j. Let

$$P_1^\nu(Y,Q) = Q, \ \ P_2^\nu(Y,Q) = Q + C_\nu, \ P_3^\nu(Y,Q) = Q + C_\nu + C_{\nu+1},$$
$$P_4^\nu(Y,Q) = Q + C_\nu + C_{\nu+1} + C_{\nu+2},$$
$$P_5^\nu(Y,Q) = Q + C_\nu + C_{\nu+1} + C_{\nu+2} + C_{\nu+3}.$$

Then, $(X_1^\nu, \cdots, X_5^\nu) \in M_5'$ with pentagon $[P_1^\nu P_2^\nu \cdots P_5^\nu]$ for all (Y,Q). *For all* $\nu, j, i \mod 5$, *with* $j \geq i$, the invariance property holds:

$$X_j^\nu(Y,Q) = X_{j-i+1}^{\nu+i-1}(Y, P_i^\nu(Y,Q)). \tag{44}$$

To embed $X_j^\nu(Y,Q)$ on K_F, define $\Phi\colon \mathbb{M}^{4\times 2} \cong \mathbb{R}^8 \to \mathbb{M}^{2\times 2} \cong \mathbb{R}^4$ by

$$\Phi(X) = DF(A) + BJ, \tag{45}$$

where $X = \begin{pmatrix} A \\ B \end{pmatrix} \in \mathbb{M}^{4\times 2}$. Then $X \in K_F \iff \Phi(X) = 0$. We have $A = PX$ and $BJ = EX$, where

$$P = \begin{pmatrix} I & O & O & O \\ O & O & I & O \end{pmatrix}, \quad E = \begin{pmatrix} O & O & O & I \\ O & -I & O & O \end{pmatrix}.$$

Thus, $D\Phi(X) = D^2F(A)P + E$; so $\operatorname{rank}(D\Phi(X)) = 4 \quad \forall\, X \in \mathbb{M}^{4\times 2}$.
Define the functions:

$$\Psi^\nu(Y,Q) = (\Phi(X_1^\nu(Y,Q)), \ldots, \Phi(X_5^\nu(Y,Q))). \tag{46}$$

To study $\Psi^\nu(Y,Q) = 0$ near (Y^0, P_ν^0), compute partial Jacobian matrix

$$\frac{\partial \Psi^\nu}{\partial Y}(Y,Q) = \begin{bmatrix} D\Phi(X_1^\nu)\dfrac{\partial Z_1^\nu}{\partial Y} \\ \vdots \\ D\Phi(X_5^\nu)\dfrac{\partial Z_5^\nu}{\partial Y} \end{bmatrix}. \tag{47}$$

Nondegeneracy of functions Ψ^ν

Note that $\frac{\partial \Psi^\nu}{\partial Y}(Y,Q)$ depends *affinely* on the Hessians $\{D^2F(PX_k^\nu)\}_k$ and is otherwise independent of F and Q. Let $J_\nu = \det \frac{\partial \Psi^\nu}{\partial Y}(Y^0, P_\nu^0)$. Since $X_j^\nu(Y^0, P_\nu^0) = X_{\nu+j-1}^0$ for all $\nu, j = 1, \dots, 5$, we have

$$D^2F(PX_j^\nu(Y^0, P_\nu^0)) \in \{D^2F(A_1^0), \dots, D^2F(A_5^0)\} \quad \forall \nu, j = 1, \dots, 5.$$

Thus J_ν is a polynomial of tensors $H_1 = D^2F(A_1^0), \dots, H_5 = D^2F(A_5^0)$ whose coefficients are independent of F. We write this polynomial as

$$J_\nu = j_\nu(H_1, H_2, H_3, H_4, H_5). \tag{48}$$

Lemma (MATLAB Lemma 2)

Given s, t, let $h_1(s) = \begin{pmatrix} sI & O \\ O & I \end{pmatrix}$ and $h_2(t) = \begin{pmatrix} I & O \\ O & tI \end{pmatrix}$, and $g_\nu(s,t) = j_\nu(h_1(s), h_2(t), h_1(s), h_1(s), h_2(t))$. Then

$$g_1(1,0) \neq 0, \; g_2(0,0) \neq 0, \; g_3(0,1) \neq 0, \; g_4(0,0) \neq 0, \; g_5(0,0) \neq 0.$$

Thus $j_\nu(H_1, \dots, H_5)$ is not identically zero for each $\nu = 1, \dots, 5$.

We first select $(H_1^0, \ldots, H_5^0)$ with the property:

$$\begin{cases} j_\nu(H_1^0, \ldots, H_5^0) \neq 0 \ \ \forall\, \nu = 1, 2, \ldots, 5; \\ \tilde{H}_j = H_j^0 - D^2 F_0(A_j^0) \text{ satisfy (41).} \end{cases} \tag{49}$$

Since $\Psi^\nu(Y^0, P_\nu^0) = 0$, $\det \frac{\partial \Psi^\nu}{\partial Y}(Y^0, P_\nu^0) = j_\nu(H_1^0, \ldots, H_5^0) \neq 0$, by the **Implicit Function Theorem**, $\exists\, \eta > 0$ and smooth functions

$$Y_\nu \colon B_\eta(P_\nu^0) \subset \mathbb{M}^{4\times 2} \cong \mathbb{R}^8 \to B_\eta(Y^0) \subset \mathbb{R}^{20}$$

for $\nu = 1, \cdots, 5$, such that for $Y \in B_\eta(Y^0)$ and $Q \in B_\eta(P_\nu^0)$,

$$\det \frac{\partial \Psi^\nu}{\partial Y}(Y, Q) \neq 0; \quad \Psi^\nu(Y, Q) = 0 \iff Y = Y_\nu(Q). \tag{50}$$

We may also select $\eta > 0$ sufficiently small so that, for all ν, i (modulo 5)

$$P_i^\nu(Y_\nu(Q), Q) \in B_\eta(P^0_{\nu+i-1}) \quad \forall\, Q \in B_\eta(P_\nu^0). \tag{51}$$

Lemma (Eigenvalue Lemma)

Let $z^\nu(Q) = Z_1^\nu(Y_\nu(Q))$ *for* $Q \in B_\eta(P_\nu^0) \subset \mathbb{R}^8$. *Then* $M = Dz^\nu(Q) \in \mathbb{M}^{8\times 8}$ *has* -1 *as eigenvalue of multiplicity at least 4 and 0 as eigenvalue of multiplicity at least 3, and all eigenvalues of* M *consist of* $\{-1, 0, \mu_M\}$, *where* $\mu_M = 4 + tr(M)$. *Furthermore, if* $\mu_M \notin \{0, -1\}$, *then* $\operatorname{rank}[adj(I - \mu_M^{-1} M)] = 1$ *and, for any* $b \in \mathbb{R}^8$,

$$\det(I - \mu_M^{-1} M + z^\nu \otimes b) = [adj(I - \mu_M^{-1} M) z^\nu] \cdot b. \tag{52}$$

Let $M^0 = Dz^\nu(P_\nu^0)$. Then $M^0 = \frac{W(H_1^0, \dots, H_5^0)}{j_\nu(H_1^0, \dots, H_5^0)}$, where $H_j^0 = D^2 F(A_j^0)$ ($j = 1, \dots, 5$), and $W(H_1, \dots, H_5)$ is a 8×8 matrix whose entries are polynomials of tensors $(H_1, \dots, H_5)$. Both W and j_ν are independent of F. Therefore, both $\mu_{M^0}(1 + \mu_{M^0})$ and $|\mathrm{adj}(I - \mu_{M^0}^{-1} M^0) z_0^\nu|^2$, where $z_0^\nu = \kappa_\nu^0 C_\nu^0 \in \mathbb{R}^8$, are rational functions of $(H_1^0, \dots, H_5^0)$ that are independent of the function F.

Lemma (**MATLAB Lemma 3**)

Similar to the MATLAB computations in Lemma 2, one verifies that the rational functions of $(H_1, \dots, H_5)$ *representing* $\mu_{M^0}(1 + \mu_{M^0})$ *and* $|adj(I - \mu_{M^0}^{-1} M^0) z_0^\nu|^2$ *are not identically zero.*

We then select the values of $(H_1^0, \dots, H_5^0) = (D^2F(A_1^0), \dots, D^2F(A_5^0))$ to satisfy (49) and the property:

$$\begin{cases} \mu_{M^0} \notin \{-1, 0\}; \\ |\mathrm{adj}(I - \mu_{M^0}^{-1} M^0) z_0^\nu|^2 \neq 0. \end{cases} \tag{53}$$

Remark: *Such values of $(H_1^0, \dots, H_5^0)$ are* **generic** *near $(D^2F_0(A_1^0), \dots, D^2F_0(A_5^0))$.*

We finally define F by (39) with the chosen $(H_1^0, \dots, H_5^0)$.
Then select $\eta > 0$ further small so that, by continuity,

$$\mu_{M(Q)} \notin \{-1, 0\}, \ \ \mathrm{adj}\Big[I - \mu_{M(Q)}^{-1} M(Q)\Big] z^\nu(Q) \neq 0 \tag{54}$$

for all $Q \in B_\eta(P_\nu^0)$ and $\nu = 1, \dots, 5$, where $M(Q) = Dz^\nu(Q)$. Let

$$\hat{X}_j^\nu(Q) = Q + Z_j^\nu(Y_\nu(Q)), \quad \hat{P}_j^\nu(Q) = P_j^\nu(Y_\nu(Q), Q).$$

Then $(\hat{X}_1^\nu(Q), \dots, \hat{X}_5^\nu(Q)) \in M_5' \cap (K_F)_5$. Define

$$\tilde{\Sigma} = \bigcup_{\nu=1}^{5} \{T(\hat{X}_1^\nu(Q), \dots, \hat{X}_5^\nu(Q)) \colon Q \in B_\eta(P_\nu^0)\}, \quad \Sigma = \mathcal{L}^{-1}(\tilde{\Sigma}).$$

The openness of Σ and Proof of Theorem (D):

Clearly, $\tilde{\Sigma}$ and Σ are nonempty, bounded, and Σ satisfies (21). To finish the proof, we need to show Σ is open, which is equivalent to showing $\tilde{\Sigma}$ is open. Let $\bar{X} \in \tilde{\Sigma}$; then $\bar{X} \in T(\hat{X}_1^\nu(\bar{Q}), \dots, \hat{X}_5^\nu(\bar{Q}))$ for some $\nu \in \{1, \dots, 5\}$, $\bar{Q} \in B_\eta(P_\nu^0)$; thus for some $i \in \{1, \dots, 5\}$ and $0 < \bar{\lambda} < 1$,

$$\bar{X} = \bar{\lambda}\hat{X}_i^\nu(\bar{Q}) + (1 - \bar{\lambda})\hat{P}_i^\nu(\bar{Q})$$

(See Figure below.) By (51), $\hat{P}_i^\nu(\bar{Q}) \in B_\eta(P^0_{\nu+i-1})$. Let $z(U) = z^{\nu+i-1}(U) = Z_1^{\nu+i-1}(Y_{\nu+i-1}(U))$. Then

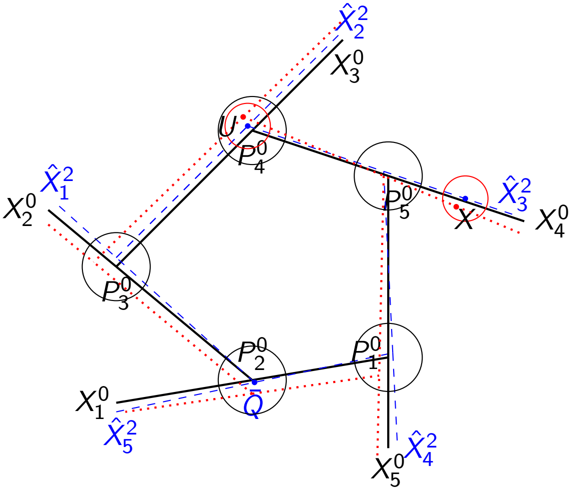
$$\bar{X} = \hat{P}_i^\nu(\bar{Q}) + \bar{\lambda}z(\hat{P}_i^\nu(\bar{Q})) = \bar{U} + \bar{\lambda}z(\bar{U}) \quad (\bar{U} \equiv \hat{P}_i^\nu(\bar{Q})). \tag{55}$$

Case 1: $\det(I + \bar{\lambda}Dz(\bar{U})) \neq 0.$

Let $F(U, X) = U + \bar{\lambda}z(U) - X$. Then, by (55), one has $F(\bar{U}, \bar{X}) = 0$, and $\det \frac{\partial F}{\partial U}(\bar{U}, \bar{X}) = \det(I + \bar{\lambda}Dz(\bar{U})) \neq 0$. Thus, by the **ImFT**, there are balls $B_{\eta'}(\bar{U}) \subset B_\eta(P^0_{\nu+i-1})$ and $B_\rho(\bar{X})$ such that, for each $X \in B_\rho(\bar{X})$, $\exists\, U \in B_{\eta'}(\bar{U}) \subset B_\eta(P^0_{\nu+i-1})$ such that $F(U, X) = 0$; that is,

$$X = U + \bar{\lambda}Z_1^{\nu+i-1}(Y_{\nu+i-1}(U)) \in T(\hat{X}_1^{\nu+i-1}(U), \dots, \hat{X}_5^{\nu+i-1}(U)) \in \tilde{\Sigma}.$$

This proves $B_\rho(\bar{X}) \subset \tilde{\Sigma}$.



Here $\nu = 2$, $i = 3$, $\bar{Q} = \hat{P}_1^2 = \hat{P}_1^2(\bar{Q})$, $\bar{U} = \hat{P}_3^2 = \hat{P}_3^2(\bar{Q})$. Blue dashed lines represent T_5-configuration $(\hat{X}_1^2, \ldots, \hat{X}_5^2)$ with $\bar{X} \in (\hat{X}_3^2, \hat{P}_3^2)$. Two smaller red circles represent $B_\rho(\bar{X})$, $B_{\eta'}(\bar{U})$. Red dotted lines represent a special T_5-configuration to be found determined by some $U \in B_{\eta'}(\bar{U})$.

Case 2: $\det(I + \bar{\lambda} Dz(\bar{U})) = 0$.

Let $\bar{M} = Dz(\bar{U})$. Since $0 < \bar{\lambda} < 1$, by the **Eigenvalue Lemma**, one has $\bar{\lambda} = -\mu_{\bar{M}}^{-1}$. Let

$$\bar{b} = \operatorname{adj}\big(I - \mu_{\bar{M}}^{-1} \bar{M}\big) z(\bar{U}).$$

By (54), $\bar{b} \neq 0$. Let

$$G(U, X) = U + \big(\bar{\lambda} + (U - \bar{U}) \cdot \bar{b}\big) z(U) - X.$$

Then $G(\bar{U}, \bar{X}) = 0$ and

$$\frac{\partial G}{\partial U}(\bar{U}, \bar{X}) = I + \bar{\lambda} \bar{M} + z(\bar{U}) \otimes \bar{b}.$$

Hence $\det \frac{\partial G}{\partial U}(\bar{U}, \bar{X}) = \big(\operatorname{adj}(I - \mu_{\bar{M}}^{-1} \bar{M}) z(\bar{U})\big) \cdot \bar{b} = |\bar{b}|^2 \neq 0$.
The rest of the proof of $B_\rho(\bar{X}) \subset \tilde{\Sigma}$ follows the same way as in Case 1.

Thank you very much for your attention!