Convergence rates for quantum evolution & entropic continuity bounds in infinite dimensions

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Remark

Note: The title does not contain the phrase "resource theory"

But: The talk is pertinent to the topic of this workshop

Because: it concerns a norm which plays a key role in

- The resource theory of quantum channels

- It applies to any resource theory of channels which involves channels acting on states on an infinite-dimensional Hilbert space

- It provides the proper metric to measure the performance of the task of channel simulation for such channels

Questions

(1) How fast do infinite-dimensional quantum systems evolve?

(2) Do entropies in infinite-dimensions satisfy continuity bounds?

If so, what are the convergence rates?

(1) e.g.

Consider a closed system, governed by a time-independent Hamiltonian H; *Schrödinger's eqn.* $i\dot{\psi}(t) = H\psi(t)$;

$$\psi(t) = e^{-iHt}\psi(0) \qquad (\hbar = 1)$$

(Q) Is there a continuous function $c(t)$ satisfying

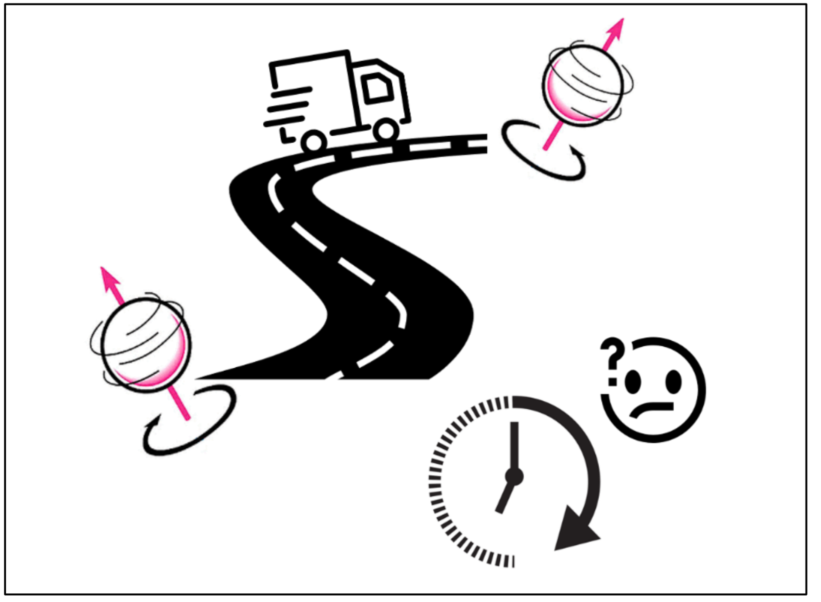
$||\psi(t) - \psi(0)|| \leq c(t)$ such that $c(t) \downarrow 0$ as $t \downarrow 0$

uniformly in the initial state $\psi(0)$?

Relevance of the Question

- Of fundamental interest
- Of importance in the study of Quantum Speed Limits

Quantum Speed Limits



- What is the minimum time t_{min} taken by a quantum system to evolve from a given initial state to a prescribed final state (or class of final states) ?

- Quantum Speed Limits provide bounds on t_{min}

- They have many applications: e.g. in quantum control, quantum communication, metrology,…..

Question (1)

(1) How fast do infinite-dimensional quantum systems evolve?

Which in the context of closed quantum systems can be phrased as follows:

(Q) Is there a continuous function $c(t)$ satisfying

$$||\psi(t) - \psi(0)|| \leq c(t) \text{ such that } c(t) \downarrow 0 \quad \text{as } t \downarrow 0$$

uniformly in the initial state $\psi(0)$?

- Let us first ask the above question for closed finite-dimensional quantum systems.

Evolution of finite-dimensional systems

The answer is simple for finite-dimensional systems:

$$\psi(t) = e^{-iHt}\psi(0)$$

$$||\psi(t) - \psi(0)|| = || \int_0^t \frac{d}{ds} e^{-isH}\psi(0) ds||$$

$$\leq ||H\psi(0)||t$$

$$\leq ||H||t$$

$$||H|| < \infty$$

(finite-dimensional systems)

$$||\psi(t) - \psi(0)|| \leq c(t) \equiv ||H||t$$

& $c(t) \downarrow 0$ as $t \downarrow 0$ uniformly in $\psi(0)$

(A) Closed finite-dimensional systems evolve linearly in time!

Infinite-dimensional Quantum System

e.g. Consider the quantum harmonic oscillator

$$H_{osc} = a^* a + 1/2$$

(scaled) Hamiltonian: $H = a^* a \equiv N,$

energy eigenvalues: $\lambda_n = n,$ energy eigenfunctions: φ_n

Choose $\psi(0) = (\varphi_n + \varphi_0)/\sqrt{2}$

$$\psi(t) = e^{-iHt}\psi(0) = (e^{-itn}\varphi_n + \varphi_0)/\sqrt{2}$$

$$||\psi(t) - \psi(0)|| = \frac{1}{\sqrt{2}}|e^{-itn}\varphi_n - \varphi_n|$$

$$= \frac{1}{\sqrt{2}}|e^{-i\pi} - 1| = \sqrt{2}$$

for time $t = \pi/n$

Note: for such a choice, $t \to 0$ as $n \to \infty$

- arbitrarily fast evolution!
- culprit : high energy states

Infinite-dimensional quantum systems

We consider the time evolution of both closed & open infinite-dimensional quantum systems

Examples:

Closed quantum systems

- Systems governed by Hamiltonians of the form

$$H = -\Delta + V$$

Open quantum systems

- Attenuator channels
- Amplifier channels
- Quantum Boltzmann eqn.
- Quantum Brownian motion
- Models from quantum optics
- etc.

Single-mode Bosonic quantum-limited attenuator channel

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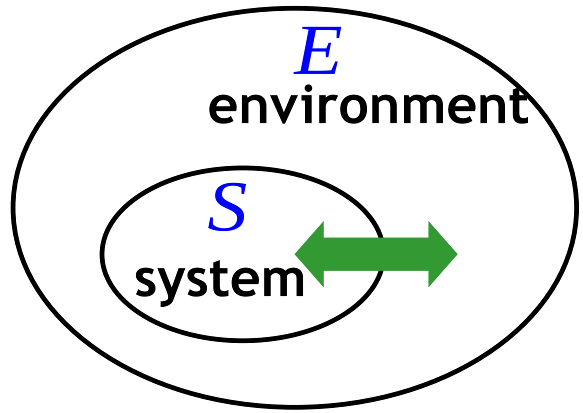
Mathematical Framework that we employ is that of

Quantum Dynamical Semigroups (QDS)

- Open quantum system:



A QDS describes its evolution under the so-called Markovian approximation, which is valid in the weak coupling limit.

- Closed quantum system : QDS ⟶ Unitary group

(since time evolution is unitary)

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Quantum Dynamical Semigroups (QDS)

In the Schrodinger picture: $(T_t)_{t\geq 0}$

one-parameter family of bounded linear, CPTP operators on the Banach space of trace class operators $\mathcal{T}_1(\mathcal{H})$

1. $T_0 = \mathrm{id}\,;$ (the identity operator)
2. $T_t \circ T_s = T_{t+s} \,\forall\, t, s \geq 0;$ (the semigroup property)

In the Heisenberg picture: $(T_{\star t})_{t\geq 0}$

one-parameter family of bounded linear, CP operators on $\mathcal{B}(\mathcal{H})$

- $T_{\star t}$: adjoint of T_t w.r.t. the Hilbert-Schmidt inner product.

$$\forall\, \rho \in \mathcal{T}_1(\mathcal{H}),\; A \in \mathcal{B}(\mathcal{H}) \quad \mathrm{Tr}(A T_t(\rho)) = \mathrm{Tr}(T_{\star t}(A)\rho)$$

- $T_{\star t}(I) = I$ (unital)

Dynamics of closed quantum systems

- QDSs reduces to one-parameter unitary groups $(T_t)_{t\in\mathbb{R}}$

(instead of $(T_t)_{t\geq 0}$)

e.g.: $(T_t^{vN})_{t\in\mathbb{R}}$ (vN for von Neumann)

$$\rho(t) = T_t^{vN}(\rho(0)) := e^{-itH}\rho(0)e^{itH},$$

(von Neumann eqn.) $\dot{\rho}(t) = -i[H, \rho(t)];$

Henceforth consider: $(T_t)_t$: QDS acting on a Banach space X;

- Generator of the QDS: $\mathcal{L}$

$$\mathcal{L}x := \frac{d}{dt}\Big|_{t=0} T_t x \; \forall\, x \in X; \qquad T_t = e^{t\mathcal{L}};$$

e.g.: $\mathcal{L}(\rho) = \frac{d}{dt}\Big|_{t=0} T_t^{vN}(\rho) = -i[H, \rho]$

Notions of continuity for semigroups

A QDS $(T_t)_t$ with generator $\mathcal{L}$ acting on a Banach space X;

- Uniformly continuous: if

$$\lim_{t\downarrow 0} \sup_{x\in X; ||x||=1} ||T_t x - x|| = 0$$

- if and only if the generator $\mathcal{L}$ is bounded
- the convergence is linear in t, for all $x \in X; ||x|| = 1.$

i.e. $||T_t x - x|| \leq \text{const.} t \quad \forall x \in X; ||x|| = 1.$

- Strongly continuous: if for all $x \in X$,

$$\lim_{t\downarrow 0} T_t x = x \quad \text{i.e.} \quad \lim_{t\downarrow 0} ||T_t x - x|| = 0$$

Notions of continuity for semigroups contd.

- Proof of the Claim: for a uniformly continuous QDS

$$||T_t x - x|| \leq \text{const}.t \qquad \forall\, x \in X; ||x|| = 1.$$

$$T_t x = e^{t\mathcal{L}} x; \quad \frac{d}{dt} T_t x = \mathcal{L} T_t x = T_t \mathcal{L} x;$$

$$||T_t x - x|| = ||T_t x - T_0 x|| \qquad (\because\ T_0 = \text{id})$$

$$= || \int_0^t \frac{d}{ds} T_s x ds|| = || \int_0^t T_s \mathcal{L} x ds||$$

$$\leq ||\mathcal{L} x|| \sup_{s\in[0,t]} ||T_s|| t$$

$$\leq \text{const.}\ t$$

(since $\mathcal{L}$ is bounded)

$$\forall\, x \in X, ||x|| = 1.$$

∎

$\Longrightarrow$

Finite-dimensional open quantum systems evolve linearly in time.

Strongly continuous QDSs

- Generator $\mathcal{L}$ is unbounded
- All we know is that

$$\lim_{t\downarrow 0} ||T_t x - x|| = 0 \quad \forall x \in X.$$

- No information about convergence rates
- There does not exist a uniform bound linear in t.

Our Aim

To find rates of convergence for strongly continuous QDSs (unbounded generators)

- Analytically richer case
- Includes all the examples mentioned previously

To study convergence rates for strongly continuous QDSs $(T_t)_t$:

(Schrödinger picture)

- We need a suitable norm on the space of quantum channels

$\because\ \forall\, t, T_t$: a linear CPTP map, i.e., a quantum channel

Or more generally,
on the space of real linear combinations of quantum channels

$$\left\{ \text{e.g.} \quad (T_t - T_s),\ t, s > 0, t \neq s; \right\}$$

i.e. on the space of Hermiticity-preserving maps

- Commonly used norm: Diamond norm

For a Hermiticity-preserving map T on $\mathcal{D}(\mathcal{H})$,

$$||T||_\diamond := \sup_{\rho \in \mathcal{D}(\mathcal{H} \otimes \mathcal{H})} ||(T \otimes \mathrm{id})\rho||_1$$

To study convergence rates for strongly continuous QDSs $(T_t)_t$:

Diamond norm:

$$||T||_\diamond := \sup_{\rho \in \mathcal{D}(\mathcal{H}\otimes\mathcal{H})} ||(T \otimes \mathrm{id})\rho||_1$$

- Note: for 2 quantum channels $T, T', \; ||T - T'||_\diamond \leq 2$

- $|| \bullet ||_\diamond$ is useful for the analysis of the continuity of channel capacities in finite dimensions *[Leung & Smith]*

$$||T - T'||_\diamond \approx 0 \quad \Longrightarrow \text{ e.g. } \; C(T) \approx C(T')$$

(classical capacities)

Unsuitability of the diamond norm when the underlying Hilbert space $\mathcal{H}$ is infinite-dimensional

e.g. Attenuator channel T_η η : *attenuation parameter*

- defined uniquely through its action on a coherent state

$$T_\eta(|\alpha\rangle\langle\alpha|) = |\sqrt{\eta}\alpha\rangle\langle\sqrt{\eta}\alpha|$$

- For $\eta \equiv \eta(t) := e^{-t},$ *(time-dependent attenuation parameter)*
- Let $T_t^{att} := T_{\eta(t)};$ $(T_t^{att})_t$: *strongly continuous QDS*
- But $||T_t^{att} - T_s^{att}||_\diamond = 2$ for any $t \neq s, t, s \geq 0$

- All attenuators are a maximum distance (=2) from each other w.r.t. $|| \bullet ||_\diamond$ no matter how close their attenuation parameters are !

$$||T_t^{att} - T_s^{att}||_\diamond = 2 \text{ for any } t \neq s,\, t, s \in \mathbb{R}$$

- What does this imply?

- It implies that the diamond norm $||\bullet||_\diamond$ is too strong a distance measure to capture the dynamics of the QDS $(T_t^{att})_t$

- To capture its dynamics & that of general infinite-dimensional systems a weaker distance measure is needed

Remedy: Consider instead Energy-Constrained Diamond norms

"ECD-norms" *[Winter], [Shirokov], [Pirandola]*

For a Hermiticity-preserving map T acting on $\mathcal{T}_1(\mathcal{H})$:

$$||T||_\diamond^{H,E} := \sup_{\substack{\rho \in \mathcal{D}(\mathcal{H}\otimes\mathcal{H}) \\ \mathrm{Tr}(\rho_1 H) \leq E}} ||(T \otimes \mathrm{id})\rho||_1$$

$\rho_1 = \mathrm{Tr}_2\, \rho;$ $H \geq 0$ (typically the Hamiltonian) $E > \inf \sigma(H)$ (spectrum)

- In the limit $E \to \infty$ one gets the usual $|| \bullet ||_\diamond$

Rationale: *[Winter '17]*

- To realize the maximal distance $||T_t^{att} - T_s^{att}||_\diamond = 2$ one needs to probe them with highly energetic states.
- But in most communications settings with such channels, there is an energy constraint on the input states.
- Hence, it is natural to put an energy constraint!

Remedy: Consider instead Energy-Constrained Diamond norms

"ECD-norms" *[Winter], [Shirikov], [Pirandola]*

For a Hermiticity-preserving map T acting on $\mathcal{T}_1(\mathcal{H})$:

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$\rho_1 = \mathrm{Tr}_2\,\rho;$ $H \geq 0$ (typically the Hamiltonian) $E > \inf\sigma(H)$ (spectrum)

In terms of ECD-norms, for attenuator channels: *[Winter]*

$||T_t^{att} - T_s^{att}||_\diamond^{H,E} \longrightarrow 0$ as $t \longrightarrow s$ $(H \equiv N := a^* a)$

Compare with: $||T_t^{att} - T_s^{att}||_\diamond = 2\ \forall\, t, s > 0\ t \neq s$

$$||T_t^{att} - T_s^{att}||_\diamond^{H,E} = ||T_{t-s}^{att} - T_0^{att}||_\diamond^{H,E}$$ (semigroup property)

Equivalently, $||T_t^{att} - \mathrm{id}||_\diamond^{H,E} \longrightarrow 0$ as $t \longrightarrow 0$

$$||T_t^{att} - \mathrm{id}||_\diamond^{H,E} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow 0$$

Note however: no information about rate of convergence

Our Aim: to make a refined analysis of convergence rates

Outline of the rest

Aim (1): To make a refined analysis of convergence rates of strongly continuous QDSs

- To do this: introduce a generalized family of ECD norms
- State our Main Results concerning convergence rates for
  - (I) Closed quantum systems
  - (II) Open quantum systems
  - (III) Quantum Speed limits
- Key mathematical ingredient of the proofs

- Address Question (2) : continuity bounds of entropies

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To find rates of convergence for such strongly continuous QDSs

- We introduce a generalized family of ECD norms labelled by a parameter $\alpha \in (0,1]$; α-ECD norms

$$||T||_{\diamond^{2\alpha}}^{H,E} := \sup_{\substack{\rho \in \mathcal{D}(\mathcal{H}\otimes\mathcal{H}) \\ \mathrm{Tr}(\rho_1 H^{2\alpha}) \le E^{2\alpha}}} ||(T \otimes \mathrm{id})\rho||_1$$

T : a Hermiticity-preserving map acting on $\mathcal{T}_1(\mathcal{H})$

$\rho_1 = \mathrm{Tr}_2\,\rho;$ $H \geq 0,$ $E > \inf \sigma(H)$

- For $\alpha = 1/2,\ (2\alpha = 1)$ it reduces to the usual ECD norm

$$||T||_{\diamond^1}^{H,E} \equiv ||T||_{\diamond}^{H,E} := \sup_{\substack{\rho \in \mathcal{D}(\mathcal{H}\otimes\mathcal{H}) \\ \mathrm{Tr}(\rho_1 H) \le E}} ||(T \otimes \mathrm{id})\rho||_1$$

Studying the entire family of α-ECD norms leads to a more refined analysis of convergence rates of QDSs

Properties of α-ECD norms

(1) $||\bullet||_{\diamond^{2\alpha}}^{H,E}$ is a norm for Hermiticity-preserving maps

$$||T||_{\diamond^{2\alpha}}^{H,E} = 0 \Leftrightarrow T = 0 \qquad \forall \alpha \in (0,1];$$

(2) $E \mapsto |\bullet||_{\diamond^{2\alpha}}^{H,E}$ is non-decreasing and concave:

for $E' \geq E > \inf(\sigma(H))$,

$$||\bullet||_{\diamond^{2\alpha}}^{H,E} \leq ||\bullet||_{\diamond^{2\alpha}}^{H,E'} \leq \left(\frac{E'}{E}\right)^{2\alpha} ||\bullet||_{\diamond^{2\alpha}}^{H,E}$$

& in the limit $E \to \infty$, one recovers the usual diamond norm:

$$\sup_{E > \inf \sigma(H)} ||\bullet||_{\diamond^{2\alpha}}^{H,E} = ||\bullet||_{\diamond}$$

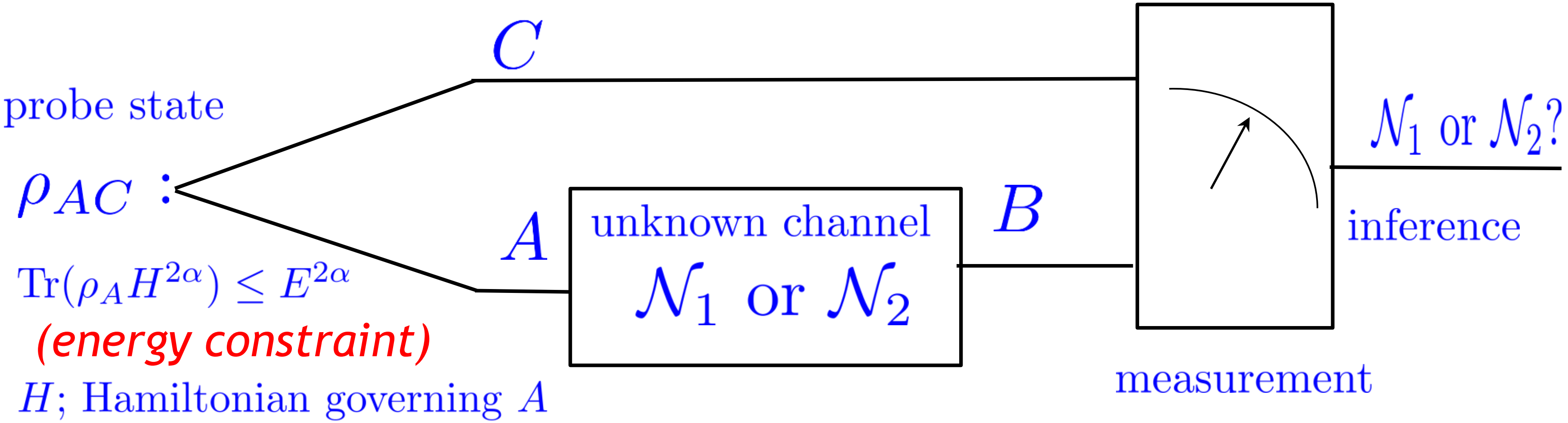
For $\alpha = 1/2$, (1), (2), & a host of other properties were found by Shirokov & Winter

(3) For $\alpha \leq \beta$, $||T||_{\diamond^{2\beta}}^{H,E} \leq ||T||_{\diamond^{2\alpha}}^{H,E}$

Etc. ....

Operational interpretation of α-ECD norms

- In binary hypothesis testing between quantum channels
- You are given a quantum channel & told it is either $\mathcal{N}_1$ or $\mathcal{N}_2$
- You need to determine which one it is!



Minimum probability of error in inferring whether the channel is $\mathcal{N}_1$ or $\mathcal{N}_2$ $= \frac{1}{2} - \frac{1}{2}||\mathcal{N}_1 - \mathcal{N}_2||_{\diamond^{2\alpha}}^{H,E}$

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Main Results (I): Dynamics of Closed Quantum Systems contd.

Consider the evolution of density operators: $(T_t^{vN})_t$

$$\rho(t) = T_t^{vN}(\rho_0) := e^{-itH}\rho_0 e^{itH}, \text{ with } \dot{\rho}(t) = -i[H, \rho(t)];$$

- *Winter proved [2017]:* For $E > \inf(\sigma(H)), \forall, t, s \geq 0.$

$$||T_t^{vN} - T_s^{vN}||_\diamond^{H,E} \leq (4E)^{\frac{1}{3}}(|t-s|)^{\frac{1}{3}} \quad \text{....(a)}$$

Theorem 2: Let $\alpha \in (0, 1]$; then for $E > \inf(\sigma(H))$,

$$||T_t^{vN} - T_s^{vN}||_{\diamond^{2\alpha}}^{H,E} \leq 2g_\alpha E^\alpha |t-s|^\alpha \quad \forall, t, s \geq 0.$$

In particular, for $\alpha = 1/2, g_\alpha = 2$,

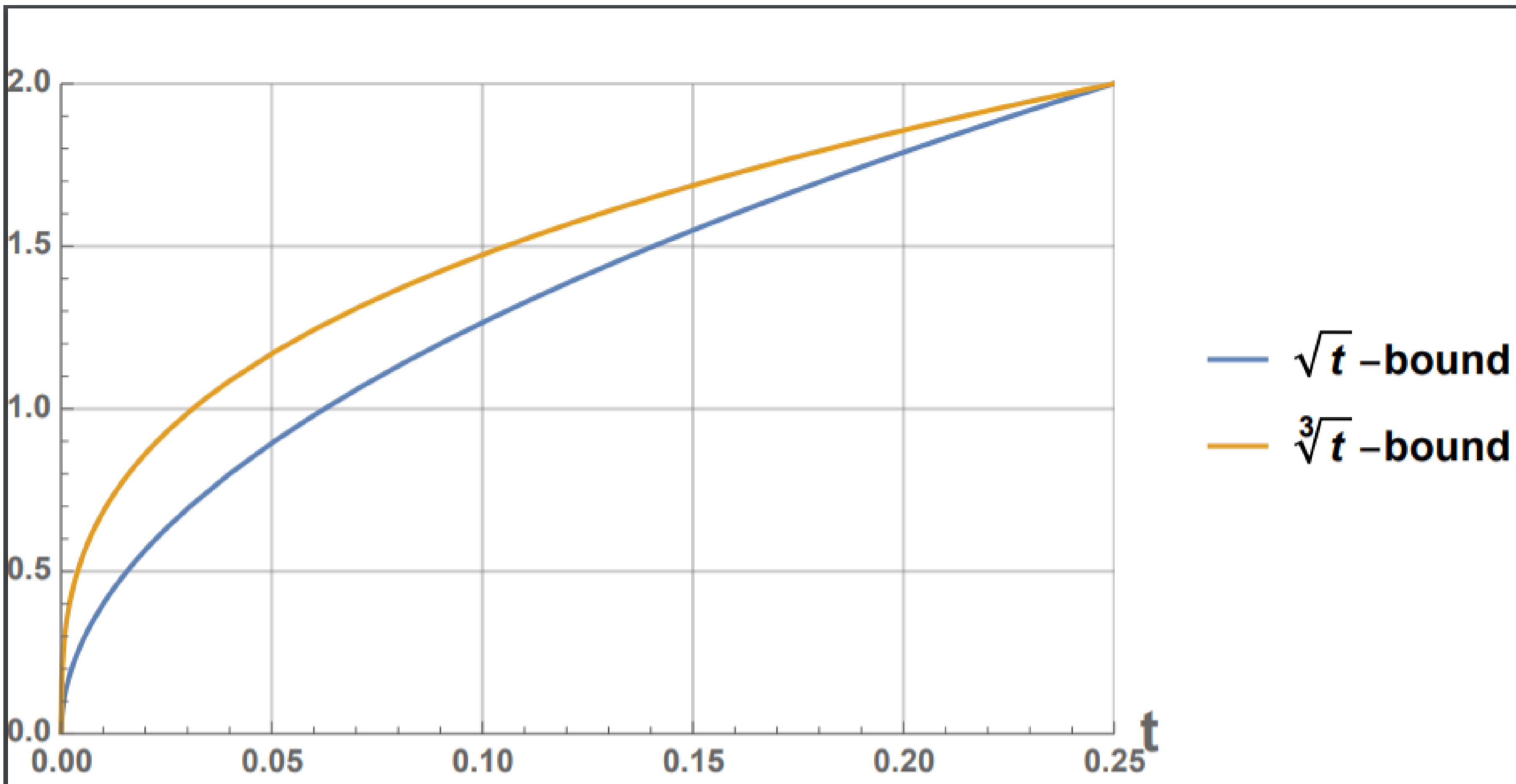
$$||T_t^{vN} - T_s^{vN}||_\diamond^{H,E} \leq 4\sqrt{E}|t-s|^{\frac{1}{2}} \quad \text{....(b)}$$

*(compare with **(a)**)*

Comparison of results

[Winter '17] $$||T_t^{vN} - \mathrm{id}||_\diamond^{H,E} \leq (4E)^{\frac{1}{3}} \sqrt[3]{t}$$

[Becker, D '19] $$||T_t^{vN} - \mathrm{id}||_\diamond^{H,E} \leq 4\sqrt{E}\sqrt{t}$$



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Main Results (II): Dynamics of Open Quantum Systems

- infinite-dimensional open quantum systems
- governed by strongly continuous QDS $(T_t)_t$ *(Schrodinger pic.)*
- s.t. adjoint semigroup $(T_{*t})_t$ *(Heisenberg pic.)* has a generator

[Davies '77] $$\mathcal{L}_*(S) = \frac{d}{dt}\Big|_{t=0} T_{*t}(S), \; \forall\, S \in \mathcal{B}(\mathcal{H})$$

$$\mathcal{L}_*(S) = \sum_{j\in\mathbb{N}} L_j^* S L_j + G^* S + SG$$

GKLS-type form (Gorini, Kossakowski, Lindblad, Sudarshan)

with $\sum_{j\in\mathbb{N}} L_j^* L_j + G^* + G = 0$

In particular, $G = -\frac{1}{2}\sum_{j\in\mathbb{N}} L_j^* L_j - iH \equiv K - iH,$

H : self-adjoint but unbounded (results in unitary dynamics)

$\{L_j\}_{j\in\mathbb{N}}$: Lindblad-type operators (result in dissipative dynamics)

Main Results (II): Dynamics of Open Quantum Systems

- To state our results, we need to introduce:

A notion of smallness of one operator w.r.t another

Relative boundedness: for positive operators $A, B,$

B is relatively A-bounded if $D(A) \subset D(B)$ & $\forall \psi \in D(A)$
$\exists\, a > 0,\, b \geq 0$ s.t.

$$||B\psi|| \leq a||A\psi|| + b||\psi||$$

Main Results (II): Dynamics of Open Quantum Systems

Theorem 3 [Open systems]:

Assumptions: governed by a strongly continuous QDS of GKLS-type form

H : Self-adjoint operator (e.g. Hamiltonian)

$\{L_j\}_{j\in\mathbb{N}}$: Lindblad-type operators; $K := -\frac{1}{2}\sum_{j\in\mathbb{N}} L_j^* L_j$

1. If H is relatively K-bounded

$$||T_t - T_s||_{\diamond 2\alpha}^{|K|,E} \leq \omega_K(E)|t-s|^\alpha$$

e.g. for the QDS $(T_t^{att})_t$:

$$H = 0, \quad K = N = a^* a$$ *(number operator)*

Attenuator channel $(T_t^{att})_t$ with attenuation parameter $\eta(t) = e^{-t}$

- Action on coherent states: $T_t^{att}(|\alpha\rangle\langle\alpha|) = |e^{-t}\alpha\rangle\langle e^{-t}\alpha|$
- Its generator: $\mathcal{L}_t^{att}(\rho) = \frac{d}{dt}|_{t=0} T_t^{att}(\rho) \; = a\rho a^* - \frac{1}{2}(N\rho + \rho N)$
- The generator of the adjoint semigroup $(T_{\star t}^{att})_t$

$$\mathcal{L}_{\star t}^{att}(A) = a^* A a - \frac{1}{2}(NA + AN) \qquad \forall\, A \in \mathcal{B}(\mathcal{H})$$

$$\because \mathrm{Tr}(A\mathcal{L}_t^{att}(\rho)) = \mathrm{Tr}(\mathcal{L}_{\star t}^{att}(A)\rho)$$

- Comparing with the GKLS-type form

$$\sum_{j\in\mathbb{N}} L_j^* A L_j + G^* A + AG \quad G = -\frac{1}{2}\sum_{j\in\mathbb{N}} L_j^* L_j - iH \equiv K - iH,$$

we see that $H = 0, \quad K = N = a^* a$ *(number operator)*

- No unitary dynamics; evolution entirely dissipative

So H is relatively K-bounded & $K = N.$

Attenuator channel $(T_t^{att})_t$ with attenuation parameter e^{-t}

Theorem 3 2. If H is relatively K-bounded

$$\|T_t - T_s\|_{\diamond^{2\alpha}}^{|K|,E} \leq \omega_K(E)|t-s|^\alpha \qquad \forall, t, s \geq 0.$$

$\because K = N,$

$$\|T_t^{att} - T_s^{att}\|_{\diamond^{2\alpha}}^{N,E} \leq \omega_N(E)|t-s|^\alpha \quad \forall, t, s \geq 0.$$

In particular, for $\alpha = 1/2,\ s = 0,$

$$\|T_t^{att} - \mathrm{id}\|_{\diamond}^{N,E} \leq \omega_N(E)\sqrt{t} \qquad \forall t \geq 0.$$

It provides a refinement of the asymptotic result: *[Winter '17]*

$$\|T_t^{att} - \mathrm{id}\|_{\diamond}^{N,E} \longrightarrow 0 \quad \text{as} \quad t \longrightarrow 0$$

Main Results (II): Dynamics of Open Quantum Systems

Theorem 3 [Open systems] contd.:

Under the same assumptions as before:

2. If K is relatively H-bounded

$$\|T_t - T_s\|_{\diamond^{2\alpha}}^{|H|,E} \leq \omega_H(E)|t-s|^\alpha$$

K : *small dissipative perturbation of the Hamiltonian dynamics e.g. for quantum Brownian motion*

Theorem 3 applies to various examples

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Quantum Speed Limits

Previously known: *[Mandelstam & Tamm '91], [Levitin & Toffoli '09]*

For closed quantum systems there is a sharp bound on the minimum time $t = t_{\min}$

over which $|\psi(0)\rangle \rightarrow |\psi(t)\rangle$ i.e. $\langle\psi(t)|\psi(0)\rangle = 0$

orthogonal

$$t_{\min} \geq \max\left\{\frac{\pi}{2\Delta E}, \frac{\pi}{2E}\right\}$$

E : energy of the initial state

ΔE : energy variance of the initial state

Main Results (III) Quantum Speed Limits

Theorem 4 [Closed systems]

(a) Let $\psi(0) = \psi_0$: initial state to the Schrodinger eqn., with

$$E \geq \mathrm{Tr}(|H|\psi_0)$$

The minimal time needed for it to evolve to a state $\psi(t)$ with angle $\theta = \cos^{-1}(\mathrm{Re}(\psi(t), \psi(0)) \in [0, \pi]$:

$$t_{\min} \geq (1 - \cos\theta)/2E$$

(b) Let $\rho(0) = \rho_0$: initial state with

$$E^{2\alpha} \geq \mathrm{Tr}(|H|^{2\alpha}\rho_0) \qquad \alpha \in (0, 1];$$

The minimal time needed for it to evolve to a state $\rho(t)$ with relative Bures angle $\theta = \cos^{-1} ||\sqrt{\rho(0)}\sqrt{\rho(t)}||_1 \in [0, \pi/2]$

$$t_{\min} \geq \left(\frac{2-2\cos\theta}{g_\alpha}\right)^{1/\alpha} \cdot \frac{1}{E}$$

Main Results (III) Quantum Speed Limits

Theorem 5 [Open systems]: (governed by a strongly continuous QDS of GKLS form)

H : Hamiltonian

$\{L_j\}_{j\in\mathbb{N}}$: Lindblad-type operators; $K := -\frac{1}{2}\sum_{j\in\mathbb{N}} L_j^* L_j$

s.t. K is relatively H-bounded

Let ρ_0 : initial state with purity $p_i = \mathrm{Tr}(\rho_0^2)$ for which

$$E^{2\alpha} \geq \mathrm{Tr}(|H|^{2\alpha}\rho_0) \qquad \alpha \in (0,1];$$

The minimal time needed for it to evolve to a state

(a) with relative Bures angle θ :

$$t_{\min} \geq \left(\frac{2-2\cos\theta}{\omega_H(E)}\right)^{1/\alpha}$$

(b) with purity p_f :

$$t_{\min} \geq \left(\frac{|p_f - p_i|}{2\omega_H(E)}\right)^{1/\alpha}$$

Key mathematical ingredient of the proofs: Favard spaces

For any QDS $(T_t)_t$ acting on a Banach space X, there exist Favard spaces, $F_\alpha, \alpha \in (0,1]$:

$$F_\alpha := \{x \in X \ : \ |x|_{F_\alpha} := \sup_{t>0} ||\frac{1}{t^\alpha}(T_t x - x)|| < \infty\}$$

i.e. for $x \in F_\alpha$, $||(T_t x - x)|| \leq |x|_{F_\alpha} t^\alpha$ α-Hölder continuity

How does the energy constraint arise?

- Favard spaces can be equivalently described in terms of the resolvent of the generator $\mathcal{L}$ of the QDS

Key Lemma: $x \in F_\alpha \Leftrightarrow \sup_{\lambda>0} ||\lambda^\alpha \mathcal{L}(\lambda I - \mathcal{L})^{-1} x|| < \infty$

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Interlude

Let us now move onto the second question:

(2) Continuity of entropies in infinite dimensions

- Do entropies in infinite-dimensions satisfy continuity bounds ?

- If so, what are the convergence rates?

Continuity of entropies

In finite dimensions, the entropies satisfy continuity bounds

e.g. Let $\rho, \sigma \in \mathcal{D}(\mathcal{H}), \quad \dim \mathcal{H} = d < \infty$

von Neumann entropy: $S(\rho) = -\operatorname{Tr}(\rho \log \rho)$

- Audenaert-Fannes inequality: If $\frac{1}{2}||\rho - \sigma||_1 \leq \varepsilon$

  then $|S(\rho) - S(\sigma)| \leq \varepsilon \log(d-1) + h(\varepsilon)$

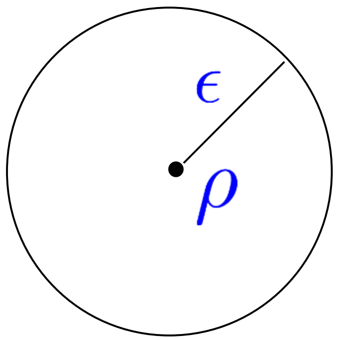
  $h(\varepsilon)$: binary entropy $\quad h(x) := -x \log x - (1-x)\log(1-x)$

- For infinite-dimensional spaces, continuity fails dramatically

Entropies of infinite-dimensional quantum systems

Let $\rho \in \mathcal{D}(\mathcal{H})$, $\mathcal{H}$: infinite-dimensional Hilbert space

$S(\rho)$ is not continuous & is unbounded in every neighbourhood!!

- 

  $B_\epsilon(\rho)$: ϵ-*ball in trace distance*
  $\forall \rho, \exists\, \rho' \in B_\epsilon(\rho)$, for which $S(\rho')$ is infinite
- $(\rho_n)_{n\in\mathbb{N}}\,;\, ||\rho_n - \rho||_1 \to 0$, then $S(\rho_n) \not\to S(\rho)$

How can one prove continuity bounds for the entropy if the entropy is discontinuous?

- Continuity bounds hold under additional assumptions!

Entropies of infinite-dimensional quantum systems

Let H : Hamiltonian, such that

- For any $\beta > 0, \quad e^{-\beta H} \in \mathcal{T}_1(\mathcal{H})$ & satisfies:

The Gibbs' Hypothesis :

$$\gamma(\beta) := \frac{e^{-\beta H}}{\mathrm{Tr}\, e^{-\beta H}}$$ exists

- It is well-known that the Gibbs state $\gamma(\beta)$ maximizes the von Neumann entropy among all states ρ s.t. $\mathrm{Tr}(\rho H) \leq E \quad (E > \inf(\sigma(H)))$ with $\beta \equiv \beta(E)$ s.t.

$$\mathrm{Tr}\left(e^{-\beta(E)H}(H - E)\right) = 0$$

Denote $\gamma(E) \equiv \gamma(\beta(E))$

Entropies of infinite-dimensional quantum systems

Theorem *[Winter '15] :* For $\rho, \sigma \in \mathcal{D}(\mathcal{H})$ s.t. $\frac{1}{2}||\rho - \sigma||_1 \leq \varepsilon$

& both $\mathrm{Tr}(\rho H), \mathrm{Tr}(\sigma H) \leq E$ energy constraint $(E > \inf(\sigma(H)))$

with H satisfying the Gibbs' hypothesis,

$$|S(\rho) - S(\sigma)| \leq 2\varepsilon S(\gamma(E/\varepsilon)) + h(\varepsilon)$$

?

- $\lim_{\varepsilon \downarrow 0} \varepsilon S(\gamma(E/\varepsilon)) = 0$ *[Shirokov]*

- To get a more refined/explicit bound, one needs to determine the high-energy asymptotics of Gibbs states

$$E > \inf(\sigma(H)), \; E/\varepsilon \to \infty \quad \text{as} \quad \varepsilon \to 0,$$

Main Results (IV): High energy asymptotics of entropy of Gibbs states

Theorem 6: under the assumptions of the previous theorem & $(\star)$

$$S(\gamma(E)) = \eta \log E(1 + o(1)) \text{ as } E \to \infty$$

?

?   *(logarithmic divergence)*

e.g. If $H = N = a^* a$, $(\star)$ holds & $\eta = 1.$

Corollary: *[Becker, D]*

$$|S(\rho) - S(\sigma)| \leq 2\varepsilon\eta \log(E/\varepsilon)(1 + o(1)) \quad \text{as } \varepsilon \to 0$$

(more quantitative/explicit bound)

Logarithmic divergence!

Compare with:

$$|S(\rho) - S(\sigma)| \leq 2\varepsilon S(\gamma(E/\varepsilon)) + h(\varepsilon)$$

[Winter '17]

- Fact: For the case of a quantum harmonic oscillator

$$H = H_{osc} = a^* a + 1/2,$$

we can explicitly evaluate the high energy asymptotics of the Gibbs state & this yields:

$$S(\gamma(E)) \sim \log E \text{ as } E \to \infty$$

- Hence, Theorem 6 shows that the logarithmic divergence of the entropy is not a special feature of H_{osc}, but is universal for many classes of Hamiltonians

- Technical tool: Weyl's law

Main Technical Ingredient: Weyl's Law

- It concerns the quantity:

$N_H(E)$: the number of eigenvalues of H that are at most of energy E (counted with multiplicities).

- gives an asymptotic description of $N_H(E)$ for certain classes of operators in the limit of high energies
- & shows that this distribution is universal

Main Results (IV): High energy asymptotics of entropy of Gibbs states

Theorem 6: under the assumptions of the previous theorem & $(\star)$

$$S(\gamma(E)) = \eta \log E(1 + o(1)) \text{ as } E \to \infty \quad ?$$

$?$

Consider 2 auxiliary functions:

$$N_H^{\uparrow}(E) := \sum_{\substack{\lambda, \lambda' \in \sigma(H) \\ \lambda + \lambda' \leq E}} \lambda^2 \quad \& \quad N_H^{\downarrow}(E) := \sum_{\substack{\lambda, \lambda' \in \sigma(H) \\ \lambda + \lambda' \leq E}} \lambda\lambda'$$

$(\star)$: is the assumption that $\xi := \lim_{E \to \infty} \frac{N_H^{\uparrow}(E)}{N_H^{\downarrow}(E)}$ exists

$\eta = (\xi - 1)^{-1}$

Weyl's Law ensures that these 2 functions have a universal asymptotic behavior for a large classes of operators as $E \to \infty$

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Corollary :

$$|S(\rho) - S(\sigma)| \leq 2\varepsilon\eta \log(E/\varepsilon)(1 + o(1)) \quad \text{as } \varepsilon \to 0$$

Similar bounds holds for other types of entropies & for capacities too *[Becker &D], [Winter], [Shirokov]*

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Summary

Two issues concerning infinite-dimensional quantum systems:

(1) Dynamics resulting from strongly continuous QDSs

- Since the generators of such QDSs are unbounded, bounds on the dynamics need to be considered in norms weaker than the commonly used diamond norm
- We introduced a family of α-ECD norms to obtain a more refined picture of quantum evolution.
- We improved previously known bounds on the dynamics of closed quantum systems & obtained, for the first time, bounds on the dynamics of open quantum systems.
- **Technical tool:** Favard spaces
- **Applications:** Quantum speed limits

Summary contd.

(2): • Studied the high energy asymptotics of the entropy of Gibbs states

- This allowed us to make previously known continuity bounds [Winter '17] on entropies (& on capacities [Shirokov '17]) more quantitative in the asymptotic regime of small distances between states.

- Technical tool: Weyl's Law

Thanks to Simon Becker

arXiv:1810.00863 (to appear in CMP)

Thanks to Andreas Winter

This work was entirely inspired by his paper *arXiv:1712.10267*

Energy-constrained diamond norm with applications to the uniform continuity of continuous variable channel capacities

Thank you for your attention!