How Zeno found the bomb

M. Wolf

Quantum Zeno effect

Repeated projective measurements freeze Hamiltonian evolution.

[Misra, Sudarshan JMP '77]

What about:

- open systems?
- time-dependent evolutions?
- non-projective measurements?

[Möbus, Wolf JMP '19]

see also:

[Burgarth et al. 1807.02036, 1809.09570]

[Barankai, Zimborás 1811.02509]

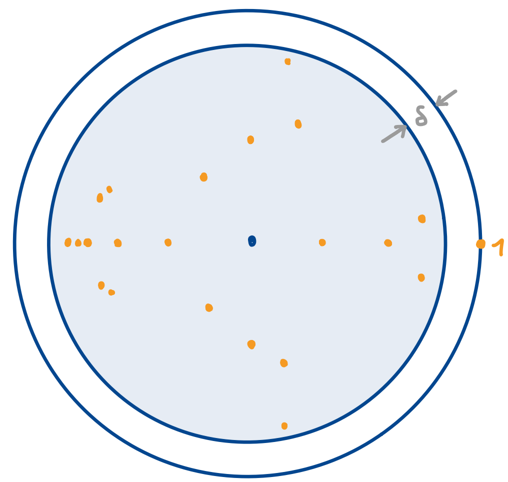
The framework (time-independent case)

Projective measurement

$\longrightarrow$ Quantum operation $T: \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$ c.p., trace non-increasing

Assumption: "spectral gap"

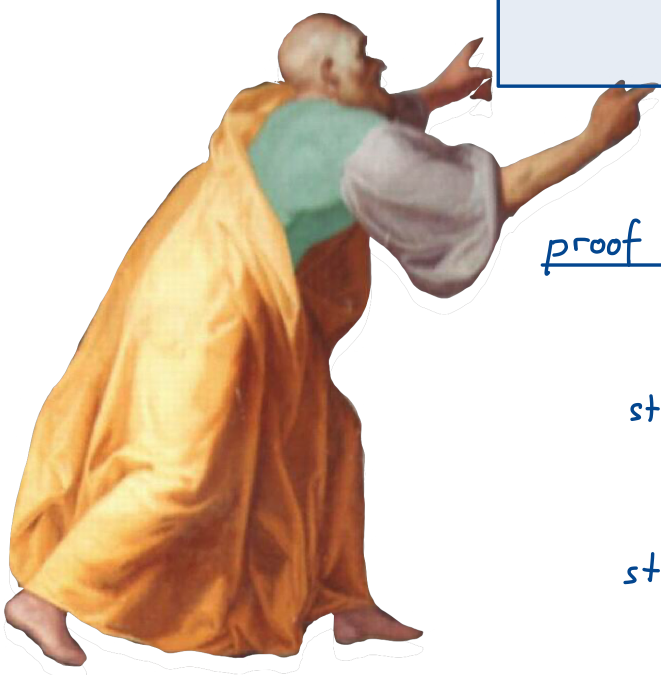
$$\boxed{1 \in \mathrm{spec}(T) \subseteq \{1\} \cup \mathbb{D}_{1-\delta}}$$



Hamiltonian evolution

$\longrightarrow$ Quantum dynamical semigroup $e^{t\mathcal{L}} : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$

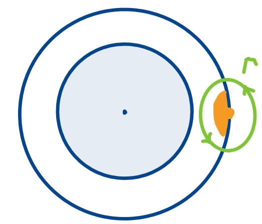
norm continuous



Thm: $\lim_{n\to\infty} \left(T \circ \exp\left[\frac{t}{n}\mathcal{L}\right]\right)^n = \exp\left[t\, P\circ\mathcal{L}\circ P\right] P$

in uniform operator topology. $P := \lim_{n\to\infty} T^n$.

proof idea: $P_t := \frac{1}{2\pi i} \oint_\Gamma \left(z\,\mathrm{id} - T e^{t\mathcal{L}} \right)^{-1} dz$



step 1: $\left\| \left(P_{\frac{1}{n}} T e^{\frac{1}{n}\mathcal{L}} P_{\frac{1}{n}}\right)^n - \left(T e^{\frac{1}{n}\mathcal{L}}\right)^n \right\| \to 0$

step 2: $\left\| \left(P_{\frac{1}{n}} T e^{\frac{1}{n}\mathcal{L}} P_{\frac{1}{n}}\right)^n - e^{P\mathcal{L}P} P \right\| \to 0$

Lemma: [Chernoff J. Func. Ana. '68] X Banach space, $C \in \mathcal{B}(X)$ contraction, $n \in \mathbb{N}$. Then

$$\left\| C^n - e^{n(C-\mathbb{1})} \right\| \le \sqrt{n}\, \left\| C - \mathbb{1} \right\|$$

applied to $X := P_{\frac{1}{n}} \mathcal{B}_1(\mathcal{H})$, $C := P_{\frac{1}{n}} T e^{\frac{1}{n}\mathcal{L}} P_{\frac{1}{n}}$.

Time-dependent case

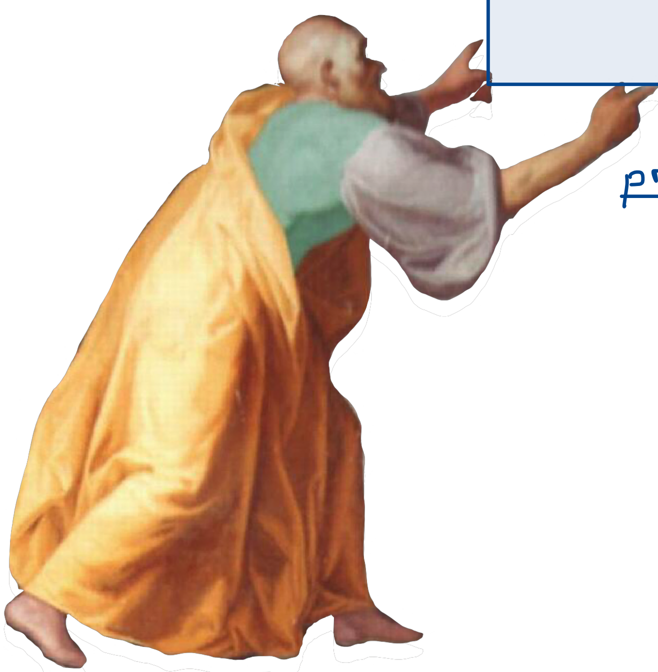
Time-dependent master equation: $\partial_t \rho(t) = \mathcal{L}_t(\rho(t))$, $\rho(0) = \rho_0$, $t \in [0,1]$

Lipschitz & bounded

$W_{[t,s]} : \rho(s) \mapsto \rho(t)$ propagator

$E_n := \prod_{i=1}^{n} T \circ W_{[\frac{i}{n}, \frac{i-1}{n}]}$ time-ordered

$\boxed{\text{Thm.: } \lim_{n\to\infty} E_n(\rho_0) \text{ coincides with solution of } \partial_t \tilde{\rho}(t) = P\mathcal{L}_t(\tilde{\rho}(t)),\ \tilde{\rho}(0) = P(\rho_0).}$



proof idea:
- two time-scales
- approximation with piecewise constant generators
- apply time-independent result

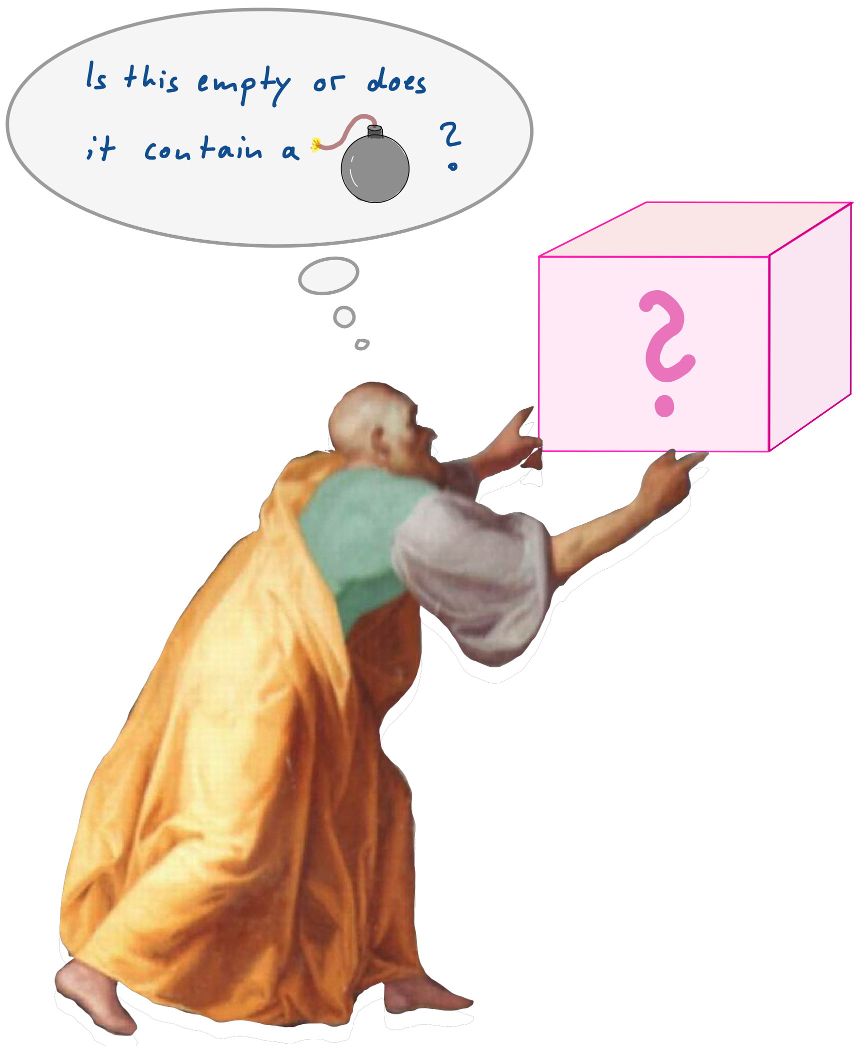
Semi summary

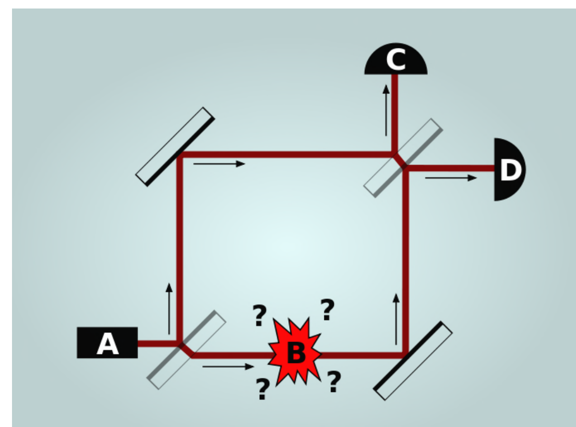
Generalization of the quantum Zeno effect
based on the assumptions:

- Banach space
- Interception by contractions with spectral gap
- Evolution with bounded generators

'Quantum structure' not really required.

'Interaction-free' channel discrimination



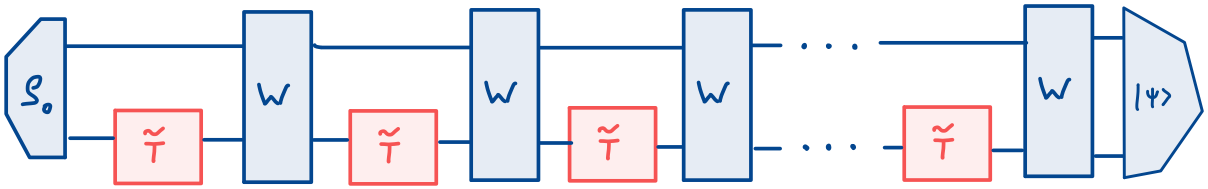


[Elitzur, Vaidman, Found. Phys. '93]
[Kwiat et al., Phys. Rev. Lett. '95]

Warmup: the Kwiat et al. protocol

[Kwiat et al., Phys. Rev. Lett. '95]

$T : \mathbb{C}^{2\times 2} \to \mathbb{C}^{2\times 2}$, $T(\rho) := \mathrm{tr}[\rho] \underbrace{|0\rangle\langle 0|}_{\text{"vacuum"}}$



$$= \langle\psi| \left[(\mathrm{id} \otimes \tilde{T}) \circ W \right]^n (\rho_0) |\psi\rangle$$

Input state $\rho_0 := |1\rangle\langle 1| \otimes |0\rangle\langle 0|$

W : rotation by angle $\frac{\pi}{2n}$ in span$\{ |01\rangle, |10\rangle \}$

$\tilde{T} = \mathrm{id} \Rightarrow |\psi\rangle := |01\rangle$ is measured

$\tilde{T} = T \Rightarrow$ For $n \to \infty$ Zeno freezes evolution so that $|10\rangle$ is measured

$$\rho_k := \left[(\mathrm{id} \otimes T) \circ W \right]^k (\rho_0) \in \mathrm{conv}\{ |10\rangle\langle 10| , |00\rangle\langle 00| \}$$

Evolution governed by stochastic matrix $\begin{pmatrix} c^2 & 0 \\ 1-c^2 & 1 \end{pmatrix}^k = \begin{pmatrix} c^{2k} & 0 \\ * & 1 \end{pmatrix}$, $c := \cos\left(\frac{\pi}{2n}\right)$

Absorption with prob. $1 - c^{2n} \simeq \boxed{\frac{\pi^2}{4n}}$.

Generalization I

$T : \mathcal{B}_1(\mathcal{H}) \to \mathcal{B}_1(\mathcal{H})$
- cptp
- spectral gap
- unique fixed point $|0\rangle\langle 0| \overset{!}{=}$ vacuum , $P := \mathbb{1} - |0\rangle\langle 0|$

$\boxed{\text{Can } T \text{ be discriminated 'interaction-free' from id ?}}$

Possible meanings of 'interaction-free':

1) Demon in the box detects non-vacuum input with negligible total probability.

2) Total absorbed energy / # photons is negligible.

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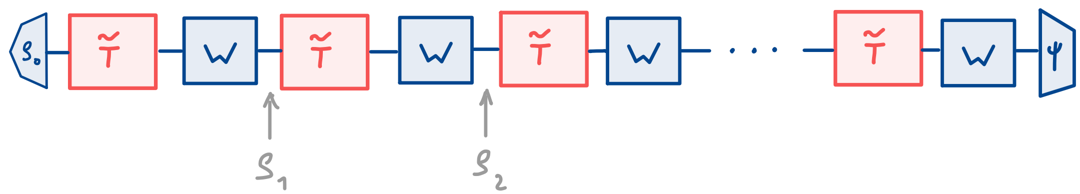
$H \in \mathcal{B}(\mathcal{H})$ Hamiltonian not commuting with $|0\rangle\langle 0|$, $W(\rho) := e^{-iH/n}\rho\, e^{iH/n}$

$\rho_k := (W \circ T)^k (\underbrace{\rho_0}_{= |0\rangle\langle 0|})$



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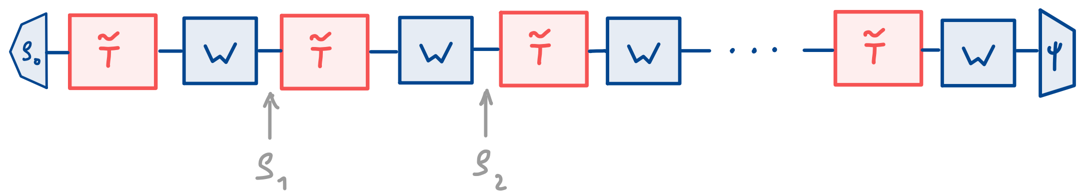
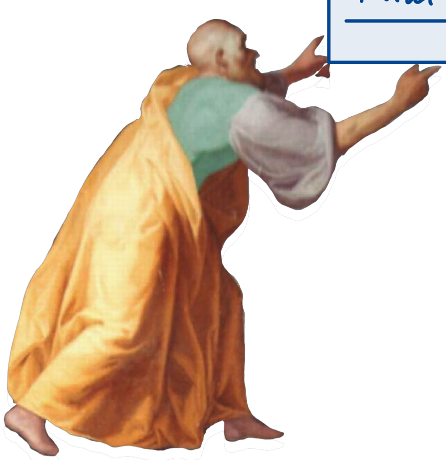




Thm.: $\displaystyle\sum_{k=1}^{n} tr[P \rho_k] = O(1/n)$

note : $\boxed{T}$ detects non-vacuum input ρ with prob.

$$tr[\,T^c(\rho)\, Q\,] = tr[\,V \rho V^* (\mathbb{1} \otimes Q)\,] = tr[\,\underbrace{V^*(\mathbb{1} \otimes Q) V}_{\leq P}\, \rho\,]$$

$0 \leq Q \leq \mathbb{1}$

conjugate channel Stinespring isometry

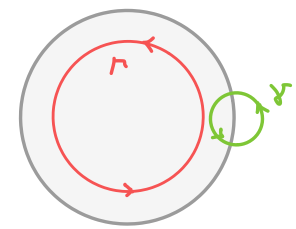
$$\rho_k := (W \circ T)^k(\rho_0)\,, \quad \rho_0 = |0\rangle\langle 0|$$

$$\boxed{\underline{\text{Thm.:}} \quad \sum_{k=1}^{n} \mathrm{tr}[P\rho_k] = \mathcal{O}(1/n)}$$

<u>proof idea:</u> 1) $(W \circ T)^k = \Phi + \Delta^k$

$\uparrow$ ergodic projection $\Phi := \frac{1}{2\pi i} \oint_\gamma R(z, W\circ T)\, dz$

$$\Delta^k = \frac{1}{2\pi i} \oint_\Gamma z^k R(z, W \circ T)\, dz$$



2) <u>Lemma:</u> $\mathrm{tr}[P\, R(z, W\circ T)(\rho_0)] = \mathcal{O}(1/n^2)$.

$\frac{1}{n}$-terms vanish due to properties implied by existence of pure fixed point.

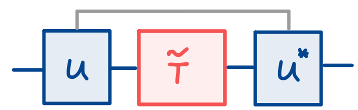
Generalization II

What if there are more fixed points?

Solution 1: find H s.t. degeneracy is not lifted

Solution 2: $(\dim(\mathcal{H}) < \infty)$ symmetrize $\tilde{T}$ using the symmetry group of the vacuum:

$$\tilde{T}(\varsigma) \mapsto \int_G u\, \tilde{T}(u^* \varsigma\, u)\, u^*\, du \;=$$

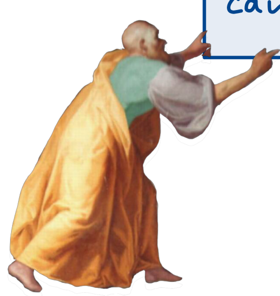


$\longrightarrow$ 7-dim. commutant with embedded relevant qubit

If fixed point still not unique add qubit ancilla and encode into qubit space $\mathrm{span}\{|01\rangle, |00\rangle + |11\rangle\}$

$\dim(\mathcal{H}) < \infty$:

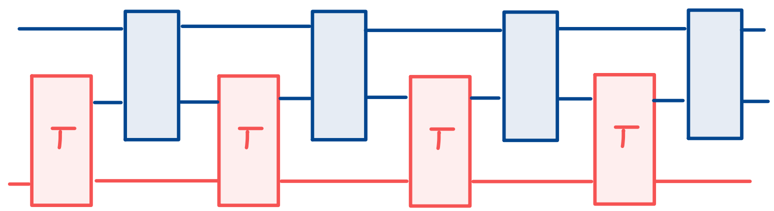
$\boxed{\text{Any pair of a unitary and a quantum channel can be discriminated interaction-free.}}$



Generalization III

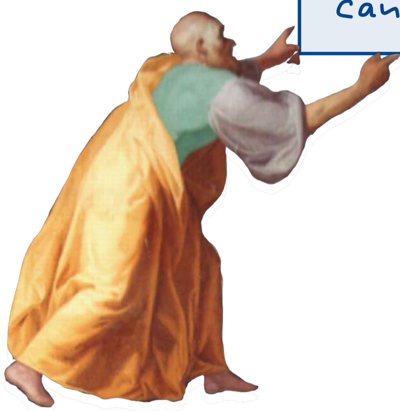
Interaction-free discrimination still works for a unitary and

- a compact set of channels
- a memory channel that is semicausal



proof ingredient: semicausal $\Rightarrow$ semilocalizable

Generalization IV



Interaction-free discrimination of two channels that have the vacuum as fixed point and are otherwise generic can not work.

problem: both channels 'freeze' the evolution.

Summary & open problems

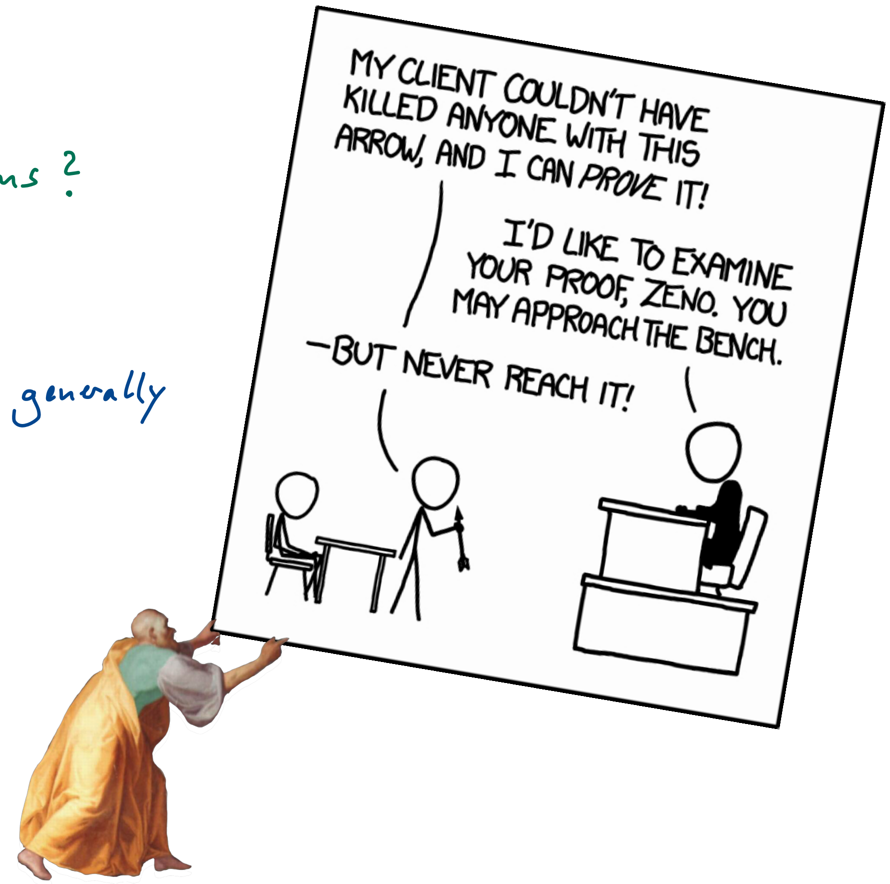
- Quantum Zeno effects generalizes to time-dependent, open dynamics and general quantum operations.

- Speed of convergence?
- Unbounded generators? α-EC norms?

- Interaction-free channel discrimination is generally possible if one hypothesis is a unitary.

- Quantitative bounds for imperfect cases.
- Asymmetric scenaria?

- GPT & RT?



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