KU LEUVEN

Applications of CPoU and uniform property Gamma

Banff International Research Station

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KU Leuven

September 12, 2019

1 – Outline

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$$T_\omega(A) = \{\text{limit traces}\}$$

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Definition. A has complemented partitions of unity (CPoU) if for any family of positive contractions $a_1, \ldots, a_k \in A$ and $\delta > 0$ such that

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$\Rightarrow \phi(e_{11}), \ldots, \phi(e_{nn})$ witness uniform property Γ.

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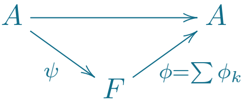

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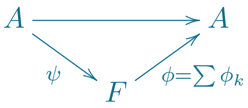

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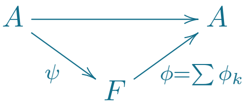

we produce elements $\hat{p}_1, \ldots, \hat{p}_k \in A^\omega \cap A'$ that almost witness CPoU but are not orthogonal. With uniform property Γ, we can replace them with orthogonal elements $p_1, \ldots, p_k$ that witness CPoU.

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The Toms–Winter Conjecture.

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Corollary. Separable simple unital nuclear $\mathcal{Z}$-stable C^*-algebras in the UCT class are classified by their Elliott invariant.

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Corollary. Separable simple unital nuclear $\mathcal{Z}$-stable C^*-algebras in the UCT class are classified by their Elliott invariant.

Corollary. Let A be a simple C^*-algebra. Then

$$\dim_{\mathrm{nuc}} A = \begin{cases} 0 & A \text{ is AF} \\ 1 & A \text{ is } \mathcal{Z}\text{-stable but not AF} \\ \infty & \text{otherwise} \end{cases}$$

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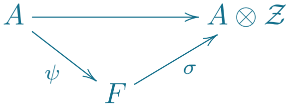

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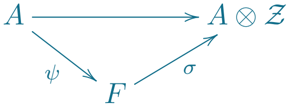

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$$a \otimes 1 = a \otimes h + a \otimes (1-h)$$

2 – Sketch of the proof - unique trace case

$$\begin{array}{ccc} A & \longrightarrow & A \otimes \mathcal{Z} \\ & {\scriptstyle\psi}\searrow \quad \nearrow{\scriptstyle\sigma} & \\ & F & \end{array}$$

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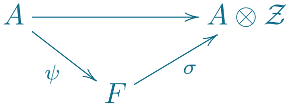

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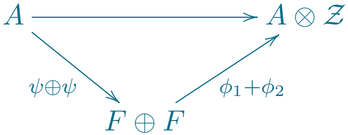

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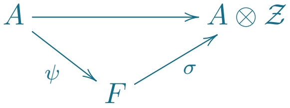

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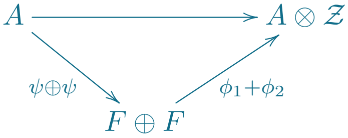

$$\phi_i(x) = u_i(\sigma(x) \otimes h_i)u_i^*$$

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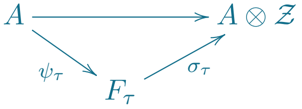

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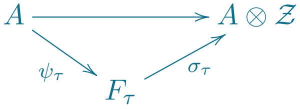

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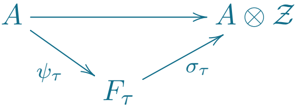

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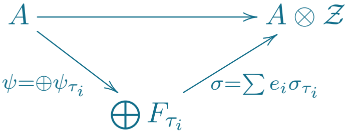

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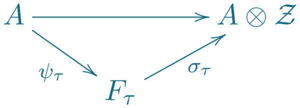

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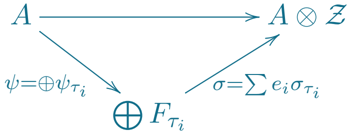

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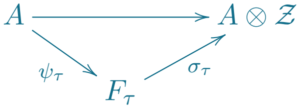

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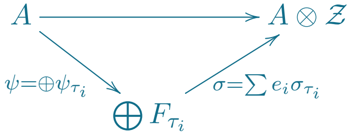

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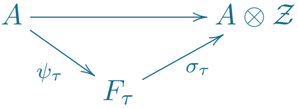

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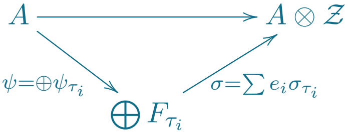

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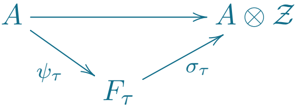

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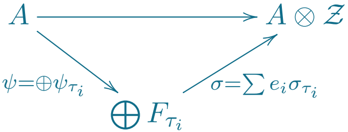

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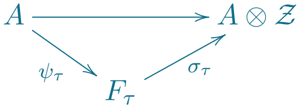

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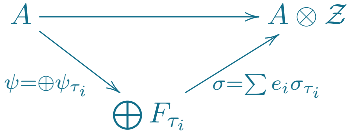

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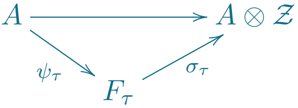

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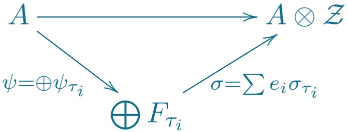

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As before, using the $h + (1-h)$ trick, we obtain $\dim_{\text{nuc}} A \leq 1$.

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By Matui-Sato, $A \otimes \mathcal{Z} \cong A$.

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3 – Outline

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By a Kaplansky density type argument

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Let A be a separable nuclear C^*-algebra and B a separable nuclear finite C*-algebra with uniform property Γ with $T(B)$ non-empty and compact. Let $\phi, \psi : A \to B^\omega$ be *-homomorphisms such that

$$\tau \circ \phi = \tau \circ \psi, \quad \tau \in T(B^\omega).$$

Then ϕ and ψ are unitarily equivalent in B^ω.

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Corollary
Let A be nuclear, separable with uniform property Γ and $T(A)$ non-empty and compact. Then $\overline{A}^{T(A)}$ is $(2, T(A))$-AFD, i.e. there is a simple unital AF-algebra B such that

$$\overline{A}^{T(A)} \cong \overline{B}^{T(B)}.$$

3 – Structure of tracial completions

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Remark

The inductive limit is induced by the decomposition of $T(A)$ as inverse limit decomposition of finite dimensional simplices.

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Morphisms between uniformly tracially complete C*-algebras $(\mathcal{M}, X)$ and $(\mathcal{N}, Y)$ are unital *-homomorphisms $\varphi : \mathcal{M} \to \mathcal{N}$ such that $\tau \circ \varphi \in X$ for all $\tau \in Y$.

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Thank you!