On the Sobolev quotient in CR geometry

Joint work with J.H.Cheng and P.Yang

Andrea Malchiodi (SNS)

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$$(Y) \qquad -c_n \Delta u + R_g = \overline{R}\, u^{\frac{n+2}{n-2}}; \qquad c_n = 4\frac{n-1}{n-2}, \quad \overline{R} \in \mathbb{R}.$$

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$$Q_{SY}(u) = \frac{\int_M \left(c_n |\nabla u|^2 + R_g u^2 \right) dV}{\left(\int_M |u|^{2^*} dV \right)^{\frac{2}{2^*}}}; \qquad 2^* = \frac{2n}{n-2}.$$

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The *Sobolev-Yamabe constant* is defined as

$$Y(M, [g]) = \inf_{u \not\equiv 0} Q_{SY}(u).$$

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- Since S^n is conformal to $\mathbb{R}^n$, one has that $Y(S^n,[g_{S^n}]) = S_n$.

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- In 1984 Schoen proved that $Y(M,[g]) < S_n$ in all other cases, i.e. $n \leq 5$ or (M,g) locally conformally flat, unless $(M,g) \simeq (S^n, g_{S^n})$.

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At large scales an approximate solution looks like the Green's function G_p of the operator L_g. If $G_p \simeq \frac{1}{|x|^{n-2}} + A$ at p, the correction is $-A/\lambda^{n-2}$.

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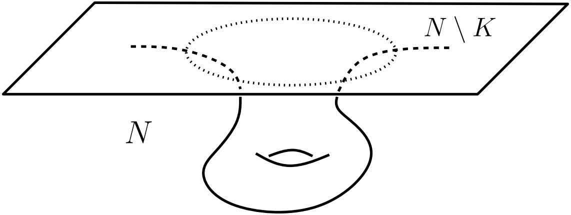

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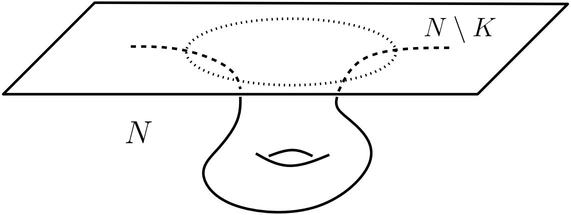

In general relativity these manifolds describe static gravitational systems.

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Then $(M \setminus \{p\}, \tilde{g})$ is asymptotically flat, and

$$m(\tilde{g}) = \lim_{x \to p} \left(G_p(x) - \frac{1}{d(x,p)} \right) = A.$$

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We also have a CR structure (complex rotation) $J : \xi \to \xi$ s.t. $J^2 = -1$.

Given J as above, we have locally a vector field Z_1 such that

$$JZ_1 = iZ_1; \qquad JZ_{\overline{1}} = -iZ_{\overline{1}} \qquad \text{where} \quad Z_{\overline{1}} = \overline{(Z_1)}.$$

Example 1: ***Heisenberg group*** $\mathbb{H}^1 = \{(z,t) \in \mathbb{C} \times \mathbb{R}\}$. Setting

$$\overset{\circ}{Z}_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z} + i\overline{z}\frac{\partial}{\partial t}\right); \qquad \overset{\circ}{Z}_{\overline{1}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \overline{z}} - iz\frac{\partial}{\partial t}\right).$$

Example 2: boundaries of complex domains. Consider $\Omega \subset \mathbb{C}^2$ and J_2 the standard complex rotation in $\mathbb{C}^2$. Given $p \in \partial\Omega$ one can consider the subset ξ_p of $T_p\partial\Omega$ which is invariant by J_2. We take ξ_p as contact distribution, and $J|_{\xi_p}$ as the CR structure J.

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Theorem ([Chanillo-Chiu-Yang, '12]) Let M^3 be a compact CR manifold. If $P \geq 0$ and $W > 0$, then M embeds into some $\mathbb{C}^N$.

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• Positivity of the mass implies that the Sobolev-Webster quotient of the manifold is lower than that of the sphere, and minimizers exist.

On the positivity condition for the Paneitz operator

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Consider S^3 in $\mathbb{C}^2$. Its *standard CR structure* $J_{(0)}$ is given by

$$J_{(0)} Z_1^{S^3} = i Z_1^{S^3}; \qquad\qquad Z_1^{S^3} = \bar{z}^2 \frac{\partial}{\partial z^1} - \bar{z}^1 \frac{\partial}{\partial z^2}.$$

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For small $s \neq 0$, the CR mass of S^3_s is negative ($m_s \simeq -18\pi s^2$).

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One then needs to verify that the two expansions match, obtaining then the asymptotic behaviour for $s \to 0$ of $A_{(s)}$, proportional to the mass. □

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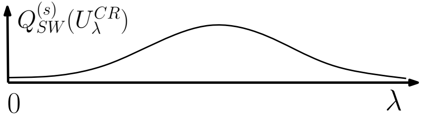
- For $|s| \neq 0$ small, the Webster quotient of the functions U_λ^{CR} has a profile of this kind (need to use Theorem 2 for λ large)



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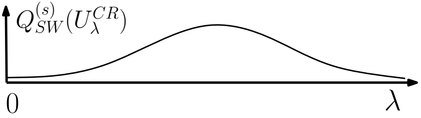

Remark. The CR Sobolev quotient of S^3_s, a closed manifold, behaves like that of a domain in $\mathbb{R}^n$!

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A related problem concerns the classification of

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Assuming finite volume, it is done in [Jerison-Lee, '88]. However we may not have this assumption, and moving planes do not work.

A related problem concerns the classification of

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In $\mathbb{R}^n$ it was shown in [Gidas-Spruck, '81] that $u \equiv 0$.

Some open problems

Another problem recently settled is compactness of solutions to Yamabe's equation ([Druet, '04], [Li-Zhang, '05-'06], [Brendle-Marques, '08], [Khuri-Marques-Schoen, '09]). Compactness holds if and only if $n \leq 24$.

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In $\mathbb{R}^n$ it was shown in [Gidas-Spruck, '81] that $u \equiv 0$. In $\mathbb{H}^n$, there are partial results in [Birindelli-Capuzzo Dolcetta-Cutrì, 97], for $p < \frac{Q}{Q-2}$.

Thanks for your attention