Multiple single peak solutions to the Gelfand problem

joint paper with L. Battaglia and A. Pistoia

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Blow up for the Gelfand problem

We will study blowing up solutions for the **Gelfand problem**:

$$\begin{cases} -\Delta u = \varepsilon V(x) e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad \text{(G)}$$

- $\Omega \subset \mathbb{R}^2$ is a smooth bounded domain;
- $\varepsilon > 0$ is *small*;
- $0 < V_1 \leq V(x) \leq V_2$, V smooth.

Known results: the case of bounded solutions

Let us consider the "easy case"

Bounded solutions

For $0 < \varepsilon \leq \varepsilon_1$ there is a unique bounded solution u_ε. Moreover $u_\varepsilon \to 0$ uniformly in Ω as $\varepsilon \to 0$. These results are consequence of the implicit function theorem.

Then let us consider the richer and more interesting case of blow-up solutions.

Known results: the blow-up case

Blow-up phenomena for problem (G) are very well-known:

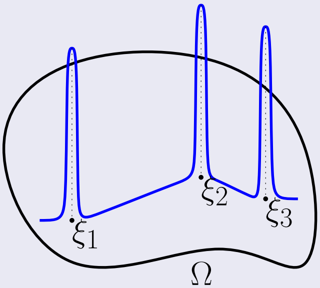
Brezis-Merle ('91); Li-Shafrir ('94); Ma-Wei ('01); Chen-Lin ('02)

Let $\{u_\varepsilon\}_{\varepsilon\to 0}$ be a non-compact family of solutions to (G) with $\varepsilon \int_\Omega V(x)e^{u_\varepsilon} dx \le C$. Then,

- **(Concentration at N points)**

$$\varepsilon V(x)e^{u_\varepsilon} \underset{\varepsilon\to 0}{\rightharpoonup} 8\pi \sum_{i=1}^{N} \delta_{\xi_i}$$

for some $\xi_1, \ldots, \xi_N \in \Omega$ with $\xi_i \neq \xi_j$ for $i \neq j$;



Single-peak solutions

Next let us focus on the case $N = 1$, the so-called single-peak solutions. Basically many of the results of this talk extend to the case $N > 1$ but for sake of simplicity we prefer avoid hard notations. Let us write explicitly the definition of single peak solution,

Definition of single-peak solution

We say that u_ε is a single-peak solution if it solves

$$\begin{cases} -\Delta u = \varepsilon V(x) e^u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

and

- $||u_\varepsilon||_\infty \leq C$ in $\Omega \setminus \{\xi\}$
- there exists $\xi_\varepsilon \to \xi$ such that $u_\varepsilon(\xi_\varepsilon) \to +\infty$ as $\varepsilon \to 0$

Known results: single-peak solutions

- **(Limiting profile)**

  $\left\| u_\varepsilon(x) - \mathrm{P}U_{\delta(\varepsilon),\xi(\varepsilon)}(x) \right\|_{H_0^1(\Omega)} \to 0$ as $\delta(\varepsilon) \to 0$, where
  $\mathrm{P} : H^1(\Omega) \to H_0^1(\Omega)$ is the standard projection i.e.

$$\begin{cases} -\Delta PU = -\Delta U & \text{in } \Omega \\ PU = 0 & \text{on } \partial\Omega \end{cases};$$

  and

$$U_{\delta,\xi}(x) = U\left(\frac{x-\xi}{\delta}\right) \qquad \text{with} \qquad U(x) = \log \frac{8}{(1+|x|^2)^2}$$

  which satisfies

$$\begin{cases} -\Delta U = e^U & \text{in } \mathbb{R}^2 \\ \displaystyle\int_{\mathbb{R}^2} e^U < +\infty & \end{cases};$$

Blow up for single-peak solutions

- **(Location of the peak)** $\xi(\varepsilon) \to \xi$ where ξ is a critical point of

$$\mathcal{F}(\xi) = R(\xi) + \frac{1}{4\pi} \log V(\xi) \quad (R \text{ is the Robin function})$$

Let us recall the definition of the Robin function. Denoting by $G(x, y)$ the Green function of $-\Delta$ with zero Dirichlet boundary conditions, we have the decomposition

$$G(x, y) = \frac{1}{2\pi} \log |x - y| + H(x, y).$$

The Robin function is defined as

$$R(x) = H(x, x).$$

Existence of single-peak solutions

As a counterpart, blowing-up solutions have been constructed:

Esposito-Grossi-Pistoia ('04); Del Pino-Kowalczyk-Musso ('04)

Let ξ_0 be a stable critical point of the function
$\mathcal{F}(\xi) = R(\xi) + \dfrac{1}{4\pi} \log V(\xi)$.
Then, there exists a family of solutions u_ε to (G) which blows up at ξ_0 so that

$$u_\varepsilon(x) = \mathrm{P}U\left(\frac{x - \xi(\varepsilon)}{\delta(\varepsilon)}\right) + \phi_\varepsilon(x), \qquad \phi_\varepsilon \underset{\varepsilon \to 0}{\to} 0 \text{ in } H_0^1(\Omega).$$

with $\delta(\varepsilon) \to 0$ and $\xi(\varepsilon) \to \xi_0$.
It can be proved that single peak solutions always exist

Examples and remark

Give some examples where the previous theorem applies,

The case $V \equiv 1$

In this case $\mathcal{F}(\xi) = R(\xi) + C$ and we construct solutions which concentrate at *stable* critical points of the Robin function. Of course the geometry of the domain plays a crucial role.

Examples and remark

Give some examples where the previous theorem applies,

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The case $V \not\equiv 1$

This case is more flexible and allows to choose V such that the stability condition is verified. For example we can choose V such that, in $B(0,\varepsilon)$

$$\mathcal{F}(\xi) = R(\xi) + \frac{1}{4\pi} \log V(\xi) \equiv \xi_1^2 + \xi_2^2 \Leftrightarrow V(\xi) = e^{4\pi\left(\xi_1^2+\xi_2^2-R(\xi)\right)}$$

$\Rightarrow$ 0 is a nondegenerate critical point of $\mathcal{F}$

Constructing the solutions

Recall the construction of the solution in Esposito-Grossi-Pistoia.

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Constructing the solutions

Recall the construction of the solution in Esposito-Grossi-Pistoia. We write $u(x) = PU_{\delta,\xi}(x) + \phi_{\delta,\xi}(x)$, with $\phi \to 0$ found via a **Lyapunov-Schmidt construction**.

General facts

GENERAL IDEA ABOUT THE LJAPOUNOV-SCHMIDT REDUCTION: Basically we have to find 3 objects:

- The function $\phi \in H_0^1(\Omega)$ such that $\phi \to 0$ in $H_0^1(\Omega)$
- The point $\xi \in \Omega$
- The positive number δ

UNFORTUNATELY (very) hard computations are involved!

Uniqueness of the solutions

The previous method allow us to find a solution for any critical point of the function

$$\mathcal{F}(\xi) = R(\xi) + \frac{1}{4\pi} \log V(\xi)$$

with $u_\varepsilon(x) = PU_{\xi(\varepsilon)}(x) + \phi_{\xi(\varepsilon)}(x)$, $\phi_{\xi(\varepsilon)}$ small in $H_0^1(\Omega)$ and $\xi(\varepsilon) \to \xi$ where ξ is a critical point of $\mathcal{F}$.
What about the multiplicity?
How many solutions blow up at a **given** critical point $\xi_0 \in \Omega$ of $\mathcal{F}$?

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What about the multiplicity?
How many solutions blow up at a **given** critical point $\xi_0 \in \Omega$ of $\mathcal{F}$?
We have uniqueness, if ξ_0 is **non-degenerate**:

Gladiali-Grossi,'04 (case $V \equiv 1$ and Ω symmetric),
Bartolucci-Jevnikar-Lee-Yang '18 (the general case)

Assume ξ_0 is a nondegenerate critical point of the function $\mathcal{F}(\xi)$.
Then, there is a **unique** family of solutions to (G) blowing up at ξ_0

Remarks on the uniqueness

In the previous example we chose

$$\mathcal{F}(\xi) = R(\xi) + \frac{1}{4\pi} \log V(\xi) \equiv \xi_1^2 + \xi_2^2$$

By the Bartolucci-Jevnikar-Lee-Yang's result this solution is *locally* unique around 0.

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What does it happen if we choose V such that

$$\mathcal{F}(\xi) = R(\xi) + \frac{1}{4\pi} \log V(\xi) \equiv \xi_1^3 - \xi_1\xi_2^2$$

(note that here 0 is a stable saddle point for $\mathcal{F}$).
We will show that in this case uniqueness fails! This means that the nondegeneracy assumption of Bartolucci-Jevnikar-Lee-Yang is *sharp*.

We will consider the case when ξ_0 is a **degenerate** critical point of $\mathcal{F}$ and we look for **multiple blowing-up solutions** to (G).
We are inspired by Grossi ('02) and Grossi-Neves ('13) who find local multiplicity respectively for

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u = u^p & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \end{cases} ; \quad \begin{cases} -\varepsilon^2\Delta u + u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega \end{cases} .$$

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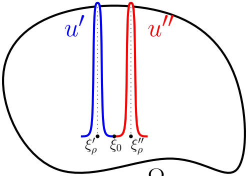
Here the concentration occurs at some critical point ξ_0 of $V(x)$ (or the mean curvature in the case of the Neumann problem).
The idea to get more solutions concentrating at the same point is to make expansions at the *second order*. In this setting a crucial one is

$$\xi_\varepsilon = \xi_0 + h(\varepsilon)\tau$$



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In which way could we find *more* solutions concentrating at a critical point of the function $\mathcal{F}(\xi) = R(\xi) + \frac{1}{4\pi} \log V(\xi)$?

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In which way could we find *more* solutions concentrating at a critical point of the function $\mathcal{F}(\xi) = R(\xi) + \frac{1}{4\pi} \log V(\xi)$?

Again we look for solutions as

$$u_\varepsilon(x) = PU_{\xi(\varepsilon)}(x) + \phi_{\xi(\varepsilon)}(x)$$

but in this case we set

$$\boxed{\xi = \xi_0 + h(\varepsilon)\tau}$$

where ξ_0 is a critical point of $R(\xi) + \frac{1}{4\pi} \log V(\xi)$ and h is a suitable function (to find!) such that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$.

So if we find more values of τ we get more solutions concentrating at ξ_0.

More precisely our result is..

Theorem (Battaglia, Grossi, Pistoia, Arxiv)

Recall that $\mathcal{F}(\xi) = R(\xi) + \frac{1}{4\pi} \log V(\xi)$, ξ_0 verifies $\nabla \mathcal{F}(\xi_0) = 0$

Theorem (Battaglia, Grossi, Pistoia, Arxiv)

Recall that $\mathcal{F}(\xi) = R(\xi) + \frac{1}{4\pi}\log V(\xi)$, ξ_0 verifies $\nabla\mathcal{F}(\xi_0) = 0$ and assume V is such that:

- $\nabla\mathcal{F}(\xi_0) = (0,0)$, $D^2\mathcal{F}(\xi_0) \equiv 0$ (fully degeneracy)
- The polynomial $\mathcal{L}(\tau) = \frac{4\pi^2}{3}\sum_{i,j,k=1}^{2}\left(\partial^3_{\tau_i\tau_j\tau_k}\mathcal{F}(0)\right)\tau_i\tau_j\tau_k$ has $\xi = 0$ as its only critical point.

Then if the equation

$$\nabla\mathcal{L} = \frac{V(\xi_0)}{8}e^{8\pi(\xi_0,\xi_0)}\Big[64\pi^2\Big(H_{x_1y_1}(\xi_0,\xi_0) + H_{x_2y_2}(\xi_0,\xi_0)\Big)\nabla_x H(\xi_0,\xi_0) - \pi\nabla(\Delta\log V)(\xi_0)\Big] = \eta$$

has *two* solutions $\tau_1 \neq \tau_2$ then, there exist two distinct solutions $u_{1,\varepsilon}$ and $u_{2,\varepsilon}$ such that

$$u_{1,\varepsilon} = PU_{\xi_0+h(\varepsilon)\tau_{1,\varepsilon}}(x) + \phi_{1,\varepsilon}(x), \quad u_{2,\varepsilon} = PU_{\xi_0+h(\varepsilon)\tau_{2,\varepsilon}}(x) + \phi_{2,\varepsilon}(x),$$

where $h(\varepsilon) \sim \varepsilon\sqrt{\log\frac{1}{\varepsilon}}$, $\boxed{\tau_{1,\varepsilon} \to \tau_1}$ $\boxed{\tau_{2,\varepsilon} \to \tau_2}$

Idea of the proof

We will look for solutions in the form

$$\boxed{u_\varepsilon = PU_{\xi_0+h(\varepsilon)\tau_\varepsilon} + \varepsilon^2\left(\widehat{W} + \widetilde{W}\right) + \phi}$$

Note that, if we do not consider the functions $\widehat{W}$ and $\widetilde{W}$ then the remainder term ϕ interacts with the leading term of the expansion (Esposito-Grossi-Pistoia case).

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Note that, if we do not consider the functions $\widehat{W}$ and $\widetilde{W}$ then the remainder term ϕ interacts with the leading term of the expansion (Esposito-Grossi-Pistoia case).
The other terms are needed to *improve* our expansion:

$$-\Delta \widehat{W} - \frac{8\delta^2}{(\delta^2 + |x - \xi_0|^2)^2} \widehat{W} = \frac{\left(8\pi D^2 H(\xi_0, \xi_0) + D^2 \log V(\xi_0), x - \xi_0, x - \xi_0\right)}{(\delta^2 + |x - \xi_0|^2)^2}$$

$$-\Delta \widetilde{W} = 8 \frac{\mathcal{M}(x) - 1 - (\nabla \mathcal{M}(x), x - \xi_0) - \frac{1}{2}\left(D^2 \mathcal{M}(x), x - \xi_0, x - \xi_0\right)}{|x - \xi_0|^4}$$

$\widehat{W}$ is *singular* at 0 while $\widetilde{W}$ is regular, as $-\Delta \widetilde{W} \in L^p(\Omega)$ for $p > 1$. with

$$\mathcal{M}(x) = e^{8\pi(H(x,\xi_0) - H(\xi_0,\xi_0)) + \log \frac{V(x)}{V(\xi_0)}}.$$

Comments on the proof

What about the proof?

- There are not relevant differences to find the function ϕ. The linearization technique and the fixed point argument are very similar.
- The equation which allow to find τ (instead of ξ) is not so simple like in the previous case. It worth to note that for some suitable $V(x)$ we have different solutions τ. In this way we get the multiplicity results.

An example in the ball

Take $\Omega = B_1$, $\xi_0 = (0, 0)$ and V such that $\mathcal{F}(\xi) = \xi_1^3 - \xi_1 \xi_2^2$.

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Since $\nabla H(0,0) = 0$, we get $\nabla(\Delta \log V)(0,0) = (4,0)$. Therefore the "terrible" equation $\nabla \mathcal{L} = \eta$, namely

$$\nabla\left(\frac{4\pi^2}{3}\sum_{i,j,k=1}^{2}\left(\partial^3_{\tau_i\tau_j\tau_k}\mathcal{F}(0)\right)\tau_i\tau_j\tau_k\right) =$$

$$\left(H_{x_1y_1}(\xi_0,\xi_0) + H_{x_2y_2}(\xi_0,\xi_0)\right)\nabla_x H(\xi_0,\xi_0) - \frac{1}{64\pi^2}\nabla(\Delta \log V)(\xi_0)$$

becomes

$$\begin{cases} 3\tau_1^2 - \tau_2^2 + 4 = 0 \\ \tau_1\tau_2 = 0 \end{cases} .$$

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We get $\tau_1 = (0,2)$ $\tau_2 = (0,-2)$, corresponding to two solutions of Gelfand problem. They write as, for $i = 1,2$,

$$u_\varepsilon \sim PU_{\xi_0 + \varepsilon\sqrt{\log \frac{1}{\varepsilon}}\tau_i} + \varepsilon^2 \left(\widehat{W} + \widetilde{W} \right)$$

Some extensions

We can generalize the argument to the case

$$\nabla \mathcal{F}(0) \equiv 0, \quad D^2 \mathcal{F}(0) \equiv 0, \quad D^3 \mathcal{F}(0) \equiv 0, \quad \dots \quad D^N \mathcal{F}(0) \equiv 0;$$

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$D^{N+1}\mathcal{F}(0)$ is non-degenerate.

In this case, we get $|\tau_\varepsilon| \sim \varepsilon^{\frac{2}{N}} \left(\log \frac{1}{\varepsilon}\right)^{\frac{1}{N}}$ and we have to solves the $N \times N$ system

$$\nabla \mathcal{P}(\tau) = \eta.$$

If $N \geq 3$ we may find up to N solutions, depending on $\deg(\nabla \mathcal{P})$.

An intereseting extension

To get actual multiplicity we need to solve the equation

$$\nabla \mathcal{P}(\tau) = \eta.$$

and so we need $\eta \neq 0$.

If $N \geq 3$, then $\mathcal{F}(\xi) = H(\xi, \xi) + \frac{1}{4\pi} \log V(\xi)$ vanishes up to the third order, therefore

$$\eta = -32\pi^2 (\Delta H(\xi_0, \xi_0)) \nabla H(\xi_0, \xi_0) - 4\pi \nabla(\Delta H(\xi_0, \xi_0))(\xi_0).$$

If Ω is **simply connected**, then $-\Delta H(\xi, \xi) = \frac{2}{\pi} e^{-4\pi H(\xi,\xi)}$ for any ξ, hence $\eta = 0$ and the argument fails.

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If Ω is **simply connected**, then $-\Delta H(\xi, \xi) = \frac{2}{\pi} e^{-4\pi H(\xi,\xi)}$ for any ξ, hence $\eta = 0$ and the argument fails.

On the other hand, if Ω is **not** simply connected, then H does not satisfy the above equation (w.l.o.g. in $\xi = 0$), hence $\eta \neq 0$.

We do not know if the obstruction is real or just technical.

Some extensions

We can extend the previous result:

Battaglia, Grossi, Pistoia (Arxiv)

Assume Ω is not simply connected and V is such that

- $D^i\mathcal{F}(\xi_0) \equiv 0$ for all $i = 1, \dots, N$;
- $D^{N+1}\mathcal{F}(\xi_0)$ is not degenerate;
- $\eta = \pi^3 \left(\partial^2_{x_1 y_1} H(\xi_0, \xi_0) + \partial^2_{x_2 y_2} H(\xi_0, \xi_0)\right) \nabla_x H(\xi_0, \xi_0) + 4\pi^2 \nabla_x \left(\partial^2_{x_1 y_1} H + \partial^2_{x_2 y_2} H\right)(\xi_0, \xi_0) \neq 0.$

Then, there exist at least $|\deg(\nabla\mathcal{P})|$ distinct solutions to (G) blowing up at 0.

THANK YOU FOR YOUR ATTENTION...

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AND THANK YOU AGAIN

Angela, Monica and Pierpaolo!!!!!!