Log-determinants in conformal geometry

(joint work with A. Malchiodi)

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Application: compactness of isospectral domains/surfaces:

- B. Osgood, R. Phillips, P. Sarnak, JFA 80 ('88)
- B. Osgood, R. Phillips, P. Sarnak, Ann. Math. 129 ('89)

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Examples:

- conformal Laplacian $L = -\Delta + \frac{(n-2)}{4(n-1)}R$
- Paneitz operator $P = \Delta^2 - \mathrm{div}(\frac{2}{3}Rg - 2Ric) \circ \nabla$
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$$\Rightarrow \gamma_2(L) = 6\gamma_3(L), \quad \gamma_2(\not{D}_g^2) = \frac{132}{7} \gamma_3(\not{D}_g^2)$$

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Gauss-Bonnet formula: $4\pi^2 \chi(M) = \int \left(\frac{|W|^2}{8} + Q\right) dv$, $\chi(M)$ Euler characteristic of M

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- $\gamma_2, \gamma_3 < 0$ and $k_A < 8\pi^2(-\gamma_2)$
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[M. Gursky, CMP 207 ('99)] $k_L < 32\pi^2$ if $R \geq 0$ except on $(\mathbb{S}^4, g_0)$

Main results

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Theorem 1

$\frac{\gamma_2}{\gamma_3} \geq 6$, w_n blow-up sequence of $\mathcal{N}(w_n) + U = \mu_n e^{4w_n}$ in M
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We find conformal metrics with $U = const.$ as saddle points of F_A:

Theorem 2

$\frac{\gamma_2}{\gamma_3} \geq 6$, M compact manifold s.t. $k_A = -\int U \notin 8\pi^2(-\gamma_2)\mathbb{N}$
Then $\exists\ \tilde{g} \in [g]$ with $U_{\tilde{g}} = const.$

On the standard sphere

- T. Branson, A. Chang, P. Yang, CMP 149 ('92) ["unique" maximizer for F_L and $F_{\not{D}_g^2}$]
- M. Gursky, CMP 189 ('97) ["unique" c.p. for F_L and $F_{\not{D}_g^2}$]
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Via stereographic projection $F_L'(w) = 0$ on $(\mathbb{S}^4, g_0)$ is equivalent to

$$3\Delta^2 W + 2\Delta(|\nabla W|^2) - 4\text{div}\,[(\Delta W + |\nabla W|^2)\nabla W] = 16\pi^2 e^{4W}$$

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Question: $\int_{\mathbb{R}^4} e^{4W} < +\infty$ does it implies $W \sim -2\log|x|$ at ∞?

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Any solution W of (P) has the form $W_{\lambda,p}$. In particular $\int_{\mathbb{R}^n} e^{nW} = n(\frac{n}{n-1})^{n-1}\omega_n$

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In a radial setting proved, among other things, in

- B. Kawohl, M. Lucia, Indiana Univ. Math. J. 57 ('08)

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The classification of positive $\mathcal{D}^{1,n}(\mathbb{R}^N)-$solutions to $-\Delta_n W = W^{\frac{nN}{N-n}-1}$, $n < N$, is very recent:

- L. Damascelli, S. Merchán, L. Montoro, B. Sciunzi, Adv. Math. 265 ('14)
- J. Vétois, J. Differential Equations 260 ('16)
- B. Sciunzi, Adv. Math. 291 ('16)

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- P.-L. Lions, Appl. Anal. 12 ('81) [on the ball, $n = 2$]
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see J. Prajapat, G. Tarantello, Proc. Edinburgh 131 ('01) when $n = 2$

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$$\delta \int_{\{|w|<k\}\cap B_\rho} [(\Delta w)^2 + |\nabla w|^4] \leq \int_{B_r \setminus B_\rho} \frac{1+|w|^4}{(r-\rho)^4} + k\int_{B_r} |f|$$

BMO and blow-up estimates

Use such Caccioppoli-type estimates as in

- G. Dolzmann, N. Hungerbühler, S. Müller, J. Reine Angew. Math. 520 ('00)

to show BMO-estimates

$$[w]_{BMO} = \Big(\sup_{0<r<r_0} \fint_{B_r} (w - \overline{w}_r)^4 \Big)^{\frac{1}{4}} \leq C, \quad r_0 = \text{injectivity radius}$$

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Epsilon regularity: Adams ineq. $\Rightarrow \exists\ \epsilon, C_0 > 0$ s.t.

$$|\mu| \int_{B_r} e^{4w} \leq \epsilon \ \Rightarrow \ \int_{B_{\frac{r}{2}}} [(\Delta w)^2 + |\nabla w|^4] \leq C_0$$

if $\mathcal{N}(w) + U = \mu e^{4w}$ in B_r and $\|w - \overline{w}_r\|_{L^4(B_r)} \leq C$

Monotone case: $\frac{\gamma_2}{\gamma_3} \geq 6$

Up to a factor, $\langle \mathcal{N}(w_1) - \mathcal{N}(w_2), \varphi \rangle$ writes as

$$\int \Delta_{\hat{g}} p \, \Delta_{\hat{g}} \varphi \, dv_{\hat{g}} + 2 \int \langle \nabla^2_{\hat{g}} p, \nabla^2_{\hat{g}} \varphi \rangle_{\hat{g}} dv_{\hat{g}} + \int |\nabla p|^2_{\hat{g}} \langle \nabla p, \nabla \varphi \rangle_{\hat{g}} dv_{\hat{g}}$$

$$+ (\frac{\gamma_2}{6\gamma_3} - 1) \left[\int \Delta p \Delta \varphi \, dv - \int (2\text{Ric}(\nabla p, \nabla \varphi) - \frac{2R}{3} \langle \nabla p, \nabla \varphi \rangle) dv \right]$$

where $p = w_1 - w_2$ and $\hat{g} = e^{w_1 + w_2} g$

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$\varphi = p$ not admissible $\Rightarrow$ admissible approximations of p following

- L. Greco, T. Iwaniec, C. Sbordone, Manuscripta Math. 92 ('97)
- T. Iwaniec, C. Sbordone, J. Reine Angew. Math. 454 ('94)
- T. Iwaniec, Ann. of Math. 136 ('92)

Hodge decomposition

$$\frac{\nabla p}{(\delta^2 + |\nabla w_1|^2 + |\nabla w_2|^2)^{2\epsilon}} = \nabla \varphi + h$$

with $\epsilon > 0$, $0 < \delta \leq 1$, $\overline{\varphi} = 0$ and $\Delta \text{div } h = 0$

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Use φ as test function thanks to

$$\Delta \varphi = \frac{\Delta p}{(\delta^2 + |\nabla w_1|^2 + |\nabla w_2|^2)^{2\epsilon}} - 4\epsilon \frac{\nabla^2 w_1(\nabla w_1, \nabla p) + \nabla^2 w_2(\nabla w_2, \nabla p)}{(\delta^2 + |\nabla w_1|^2 + |\nabla w_2|^2)^{2\epsilon+1}} - \text{div } h$$

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with $\epsilon > 0$, $0 < \delta \leq 1$, $\overline{\varphi} = 0$ and $\Delta \text{div } h = 0$

$$\Rightarrow \qquad \|h\|_{\frac{4(1-\epsilon)}{1-4\epsilon}} \leq K\epsilon \left(\delta^{1-4\epsilon} + \|\nabla w_1\|_{4(1-\epsilon)}^{1-4\epsilon} + \|\nabla w_2\|_{4(1-\epsilon)}^{1-4\epsilon} \right)$$

Use φ as test function thanks to

$$\Delta \varphi = \frac{\Delta p}{(\delta^2 + |\nabla w_1|^2 + |\nabla w_2|^2)^{2\epsilon}} - 4\epsilon \frac{\nabla^2 w_1(\nabla w_1, \nabla p) + \nabla^2 w_2(\nabla w_2, \nabla p)}{(\delta^2 + |\nabla w_1|^2 + |\nabla w_2|^2)^{2\epsilon+1}} - \text{div } h$$

Then $\mathcal{N}(w_1) = \text{div } F_1$ and $\mathcal{N}(w_2) = \text{div } F_2$ imply for $\epsilon > 0$ small

$$\int \frac{|\nabla_{\hat{g}}^2 p|^2 + |\nabla p|^4}{(|\nabla w_1|^2 + |\nabla w_2|^2)^{2\epsilon}} dv \leq C(\|F_1 - F_2\|_{\frac{4(1-\epsilon)}{3}}^{\frac{4(1-\epsilon)}{3}} + \eta \|\nabla p\|_{2-4\epsilon}^{2-4\epsilon} + 1)$$

where $\eta = |\gamma_2 - 6\gamma_3| \sup_M(|R| + \|\text{Ric}\|)$

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- K. Uhlenbeck, J. Viaclovsky, Math. Res. Lett. 7 ('00)

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Thanks for your attention