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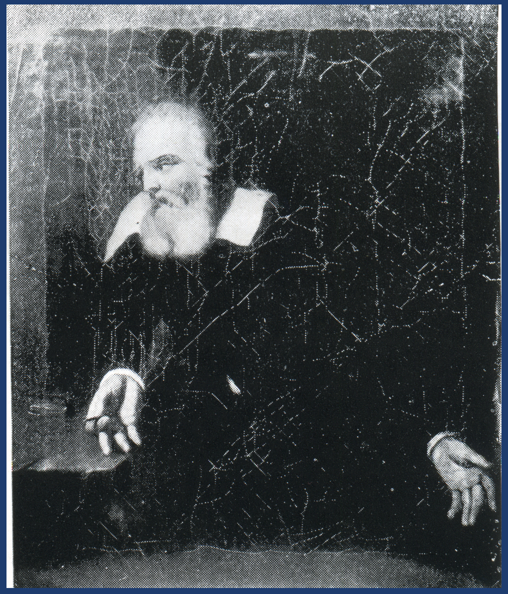
EPPUR CONVERGE

(AND YET IT CONVERGES)



BANFF, 27.11.2019

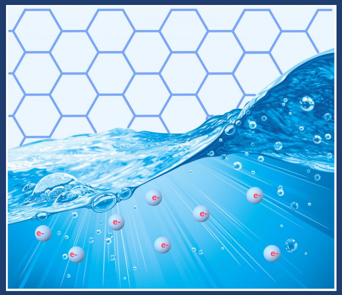
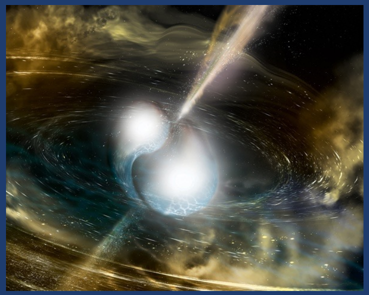
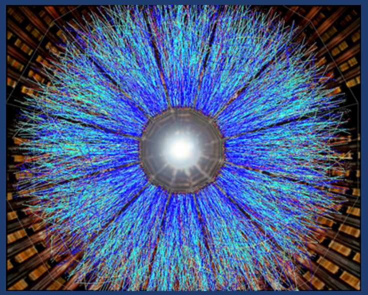
OUTLINE

- hydrodynamics
- complex spectral curves and convergence of hydrodynamics
- all-order hydrodynamics and microscopic quantum chaos: *pole-skipping*

HYDRODYNAMICS

HYDRODYNAMICS

- collective dynamics: liquids, graphene, neutron stars, quark-gluon plasma

- low-energy limit of QFTs – a Schwinger-Keldysh effective field theory [Grozdanov, Polonyi (2013); Crossley, Glorioso, Liu (2015); Haehl, Loganayagam, Rangamani (2015); Jensen, Pinzani-Fokeeva, Yarom (2017); …]
- expressed through conservation laws (equations of motion) of globally conserved operators

$$\nabla_\mu T^{\mu\nu} = 0 \qquad \nabla_\mu J^\mu = 0 \qquad \dots \nabla_\mu J^{\mu\nu_1\dots\nu_n} = 0$$

- tensor structures (symmetries and phenomenological gradient expansions) with transport coefficients (microscopic)

$$\partial u^\mu \sim \partial T \sim \sim \partial \mu \sim \epsilon$$

$$T^{\mu\nu}\left(u^\lambda, T, \mu\right) = (\varepsilon + P)\, u^\mu u^\nu + P g^{\mu\nu} - \eta\sigma^{\mu\nu} - \zeta \nabla\cdot u \Delta^{\mu\nu} + \dots$$

$$J^\mu(u^\lambda, T, \mu) = n u^\mu - \sigma T \Delta^{\mu\nu}\nabla_\nu(\mu/T) + \dots$$

HYDRODYNAMICS

- infinite, all-order hydrodynamic expansion

$$T^{\mu\nu} = \sum_{n=0}^{\infty} \left[\sum_{i}^{N} \lambda_i^{(n)} \mathcal{T}_{(n)}^{\mu\nu} \right] \xrightarrow[u^\mu \sim T \sim e^{-i\omega t + iqz}]{\nabla_\mu T^{\mu\nu} = 0}$$

$$\omega(q) = \sum_{n=0}^{\infty} \alpha_{n+1} q^{n+1}$$

- the series receives non-analytic corrections away form the large-N_c limit; long-time tails

- conformal symmetry constrains the series

CFT:
Weyl covariance
$T^\mu_{\ \mu} = 0$

- state of the art for relativistic neutral hydrodynamics

	max *N*	max *N* in CFT	
first order	*2*	*1*	Navier-Stokes (1821)
second order	*15*	*5*	BRSSS (2007)
third order	*68*	*20*	Grozdanov, Kaplis, PRD 93 (2016) 6, 066012

- **how does the all-order, infinite series behave?**

ANOTHER MOTIVATION

- expansion of the Bjorken flow solution (of energy density) in proper time is asymptotic [Heller, Janik, Witaszczyk (2013); Heller, Spalinski (2015)]

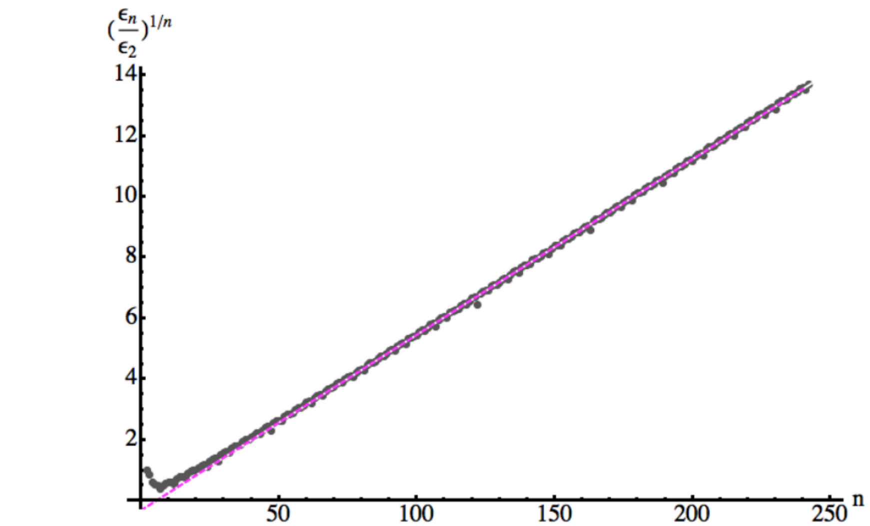


FIG. 1. Behavior of the coefficients of hydrodynamic series for the energy density as a function of the order. At large enough order the coefficients start exhibiting factorial growth. The radius of convergence of the Borel transformed series is estimated to be 6.37, in rough agreement with (9): $2/3 \times 6.37 = 4.25$.

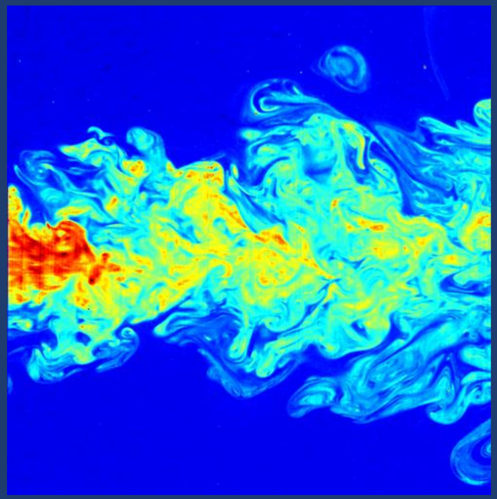
[from Heller, Janik, Witaszczyk (2013)]

- What does this actually mean? Is this a “good feature” of hydrodynamics or does this imply that hydrodynamics is inconsistent as an EFT? Is the Bjorken flow pathological (expansion around vacuum, boost invariance, …)? Can we conclude that hydro is “an asymptotic series” in something (proper time, momentum?), like perturbation theory?
- we need to understand this more systematically and precisely

A TOOL: HOLOGRAPHIC DUALITY

- duality: theory *A* = theory *B*
- holographic or gauge/gravity duality is a result of string theory, which is a quantum theory of gravity [Maldacena (1997)]

strongly coupled quantum theory	=	*weakly coupled gravity*
(extremely hard)	=	(much easier)



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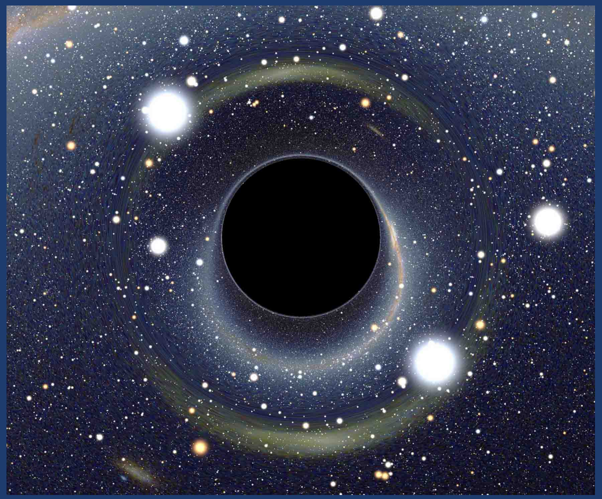
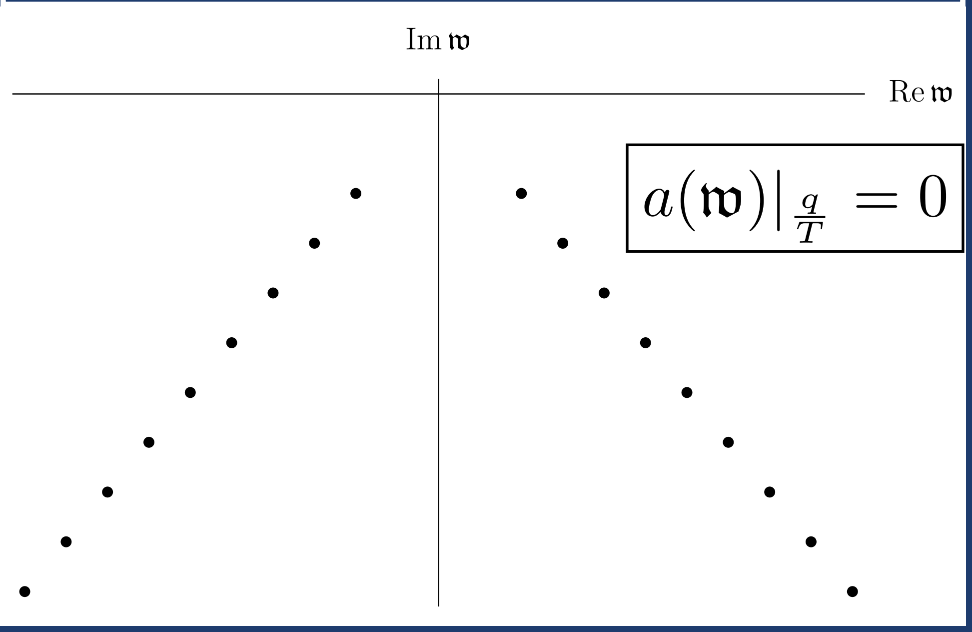


- weakly interacting gravity allows to analyse certain strongly coupled microscopic QFTs (e.g., *N=4* supersymmetric Yang-Mills theory, ABJM, free bosons and fermions, …?)

HYDRODYNAMICS

- holography is an extremely useful tool for studying the structure of thermal spectra
- the spectrum of field theory correlators equals the quasinormal spectrum of frequencies of dual black branes, plotted for $\mathfrak{w} \equiv \dfrac{\omega}{2\pi T} \in \mathbb{C}$:
- in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at $N_c \to \infty$:

$$\langle T_{\mu\nu}(-\omega, -q), T_{\rho\sigma}(\omega, q)\rangle_R \sim \frac{b(\omega, q)}{a(\omega, q)}$$



first order (1/1):	$\eta = \lambda_1^{(1)} = \# + \#/\lambda^{3/2} + \cdots$	Buchel, Liu, Starinets (2004)
second order (5/5):	$\lambda_i^{(2)} = \#_i + \#_i/\lambda^{3/2} + \cdots, \quad i = \{1, \ldots, 5\}$	Grozdanov, Starinets (2014)
third order (5/20):	$\lambda_i^{(3)} = \#_i + \cdots, \quad i = \{1, \ldots, 5\}$	Grozdanov, Kaplis (2016)

- sound:

$$\omega = \pm \frac{1}{\sqrt{3}} q - \frac{i}{6\pi T} q^2 \pm \frac{3 - 2\ln 2}{24\sqrt{3}\pi^2 T^2} q^3 - \frac{i\left(\pi^2 - 24 + 24\ln 2 - 12\ln^2 2\right)}{864\pi^3 T^3} q^4 \pm \cdots$$

- shear:

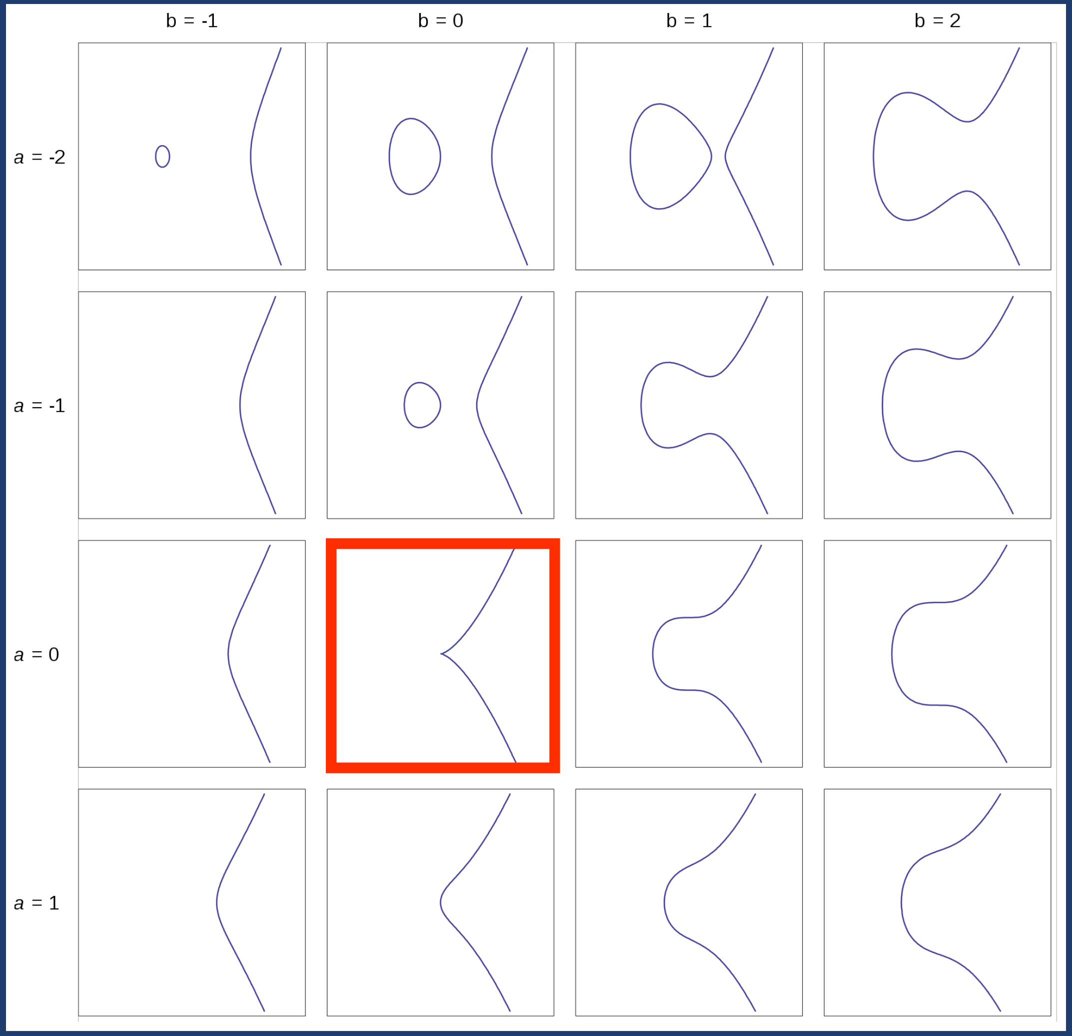
$$\omega = -\frac{i}{4\pi T} q^2 - \frac{i(1 - \ln 2)}{32\pi^3 T^3} q^4 - \frac{i(24\ln^2 2 - \pi^2)}{96\,(2\pi T)^5} q^6$$
$$- \frac{i\left[2\pi^2(\ln 32 - 1) - 21\zeta(3) - 24\ln 2(1 + \ln 2(\ln 32 - 3))\right]}{384\,(2\pi T)^7} q^8 + \cdots$$

COMPLEX SPECTRAL CURVES AND CONVERGENCE OF HYDRODYNAMICS

COMPLEX SPECTRAL CURVES

- algebraic curves are solutions to polynomial equations $P(x,y) = 0 \;\Rightarrow\; y(x)$

- e.g.: elliptic curves are non-singular solutions of $y^2 = x^3 + ax + b,\; x, y \in \mathbb{R}$



- we will be interested in critical points, such as cusps, self-intersections, …, of complex spectral curves (with *P(x,y)* not necessarily a polynomial)

$$P(x,y) = 0 \;\Rightarrow\; y(x),\;\; x, y \in \mathbb{C}$$

LOCAL ANALYSIS: PUISEUX SERIES

- **Taylor series** is a series in integer powers of the expansion parameter
- **Puiseux series** is a series in fractional powers of the expansion parameter
- consider a simple example of an algebraic curve for $x, y \in \mathbb{C}$

$$P(x,y) = x^2 + y^2 - 1 = 0$$

- we want to find series solutions for $y(x)$
- a regular point is defined by $P(x_r, y_r) = 0,\ \partial_y P(x_r, y_r) \neq 0$ at the regular point $(x_r, y_r) = (0,1)$, the solution gives a Taylor series

$$y = y^{(T)}(x) = 1 - \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

- a critical point (of order 2) is defined by $P(x_*, y_*) = 0,\ \partial_y P(x_*, y_*) = 0,\ \partial_y^2 P(x_*, y_*) \neq 0$ here, two such points, $(x_*, y_*) = (\pm 1, 0)$, each with two branches of Puiseux series, e.g.

$$\text{at } (x_*, y_*) = (1,0): \quad \begin{aligned} y &= y_1^{(P)}(x) = i\sqrt{2}(x-1)^{\frac{1}{2}} + i2^{-\frac{3}{2}}(x-1)^{\frac{3}{2}} + \cdots \\ y &= y_2^{(P)}(x) = -i\sqrt{2}(x-1)^{\frac{1}{2}} - i2^{-\frac{3}{2}}(x-1)^{\frac{3}{2}} + \cdots \end{aligned}$$

- **radius of convergence** is distance to the nearest critical point: $R_x^{(T)} = 1,\ R_x^{(P)} = 2$

CONVERGENCE OF HYDRODYNAMICS

- hydrodynamic modes as complex spectral (or infinite-order algebraic) curves
  [Grozdanov, Kovtun, Starinets, Tadić, PRL (2019) and JHEP (2019)]

hydro: $\det \mathcal{L}(\mathbf{q}^2, \omega) = 0$
QNM: $a(\mathbf{q}^2, \omega) = 0$ $\longrightarrow$ $\boxed{P(\mathbf{q}^2, \omega) = 0} \Longrightarrow \boxed{\omega_i(\mathbf{q}^2)}$ $\qquad \mathfrak{w} = \frac{\omega}{2\pi T}, \ \mathfrak{q} = \frac{|\mathbf{q}|}{2\pi T} \in \mathbb{C}$

- e.g., first-order hydrodynamics: $P_1(\mathbf{q}^2, \omega) = \left(\omega + iD\mathbf{q}^2\right)^2 \left(\omega^2 + i\Gamma\omega\mathbf{q}^2 - v_s^2\mathbf{q}^2\right) = 0$
- analytic implicit function theorem (a regular point)

$$\boxed{P(\mathbf{q}_*^2, \omega_*) = 0, \ \partial_\omega P(\mathbf{q}_*^2, \omega_*) \neq 0}$$

- Puiseux theorem: there exists a convergent series around a critical point $(\mathbf{q}_*^2, \omega_*)$

$$\boxed{P(\mathbf{q}_*^2, \omega_*) = 0, \ \partial_\omega P(\mathbf{q}_*^2, \omega_*) = 0, \ \ldots, \ \partial_\omega^p P(\mathbf{q}_*^2, \omega_*) \neq 0}$$

$$\boxed{\begin{aligned} p_{\text{shear}} &= 1 \\ p_{\text{sound}} &= 2 \end{aligned}}$$

- hydrodynamic series are Puiseux series around $(\mathbf{q}^2, \omega)^{(\text{regular})}_{\text{shear}} = (\mathbf{q}_*^2, \omega_*)^{(\text{critical})}_{\text{sound}} = (0, 0)$

$$\boxed{\mathfrak{w}_{\text{shear}} = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \ldots}$$

$$\boxed{\mathfrak{w}_{\text{sound}} = -i \sum_{n=1}^{\infty} a_n e^{\pm \frac{i\pi n}{2}} \left(\mathfrak{q}^2\right)^{n/2} = \pm v_s \mathfrak{q} - \frac{i}{2}\mathfrak{G}\mathfrak{q}^2 + \ldots}$$

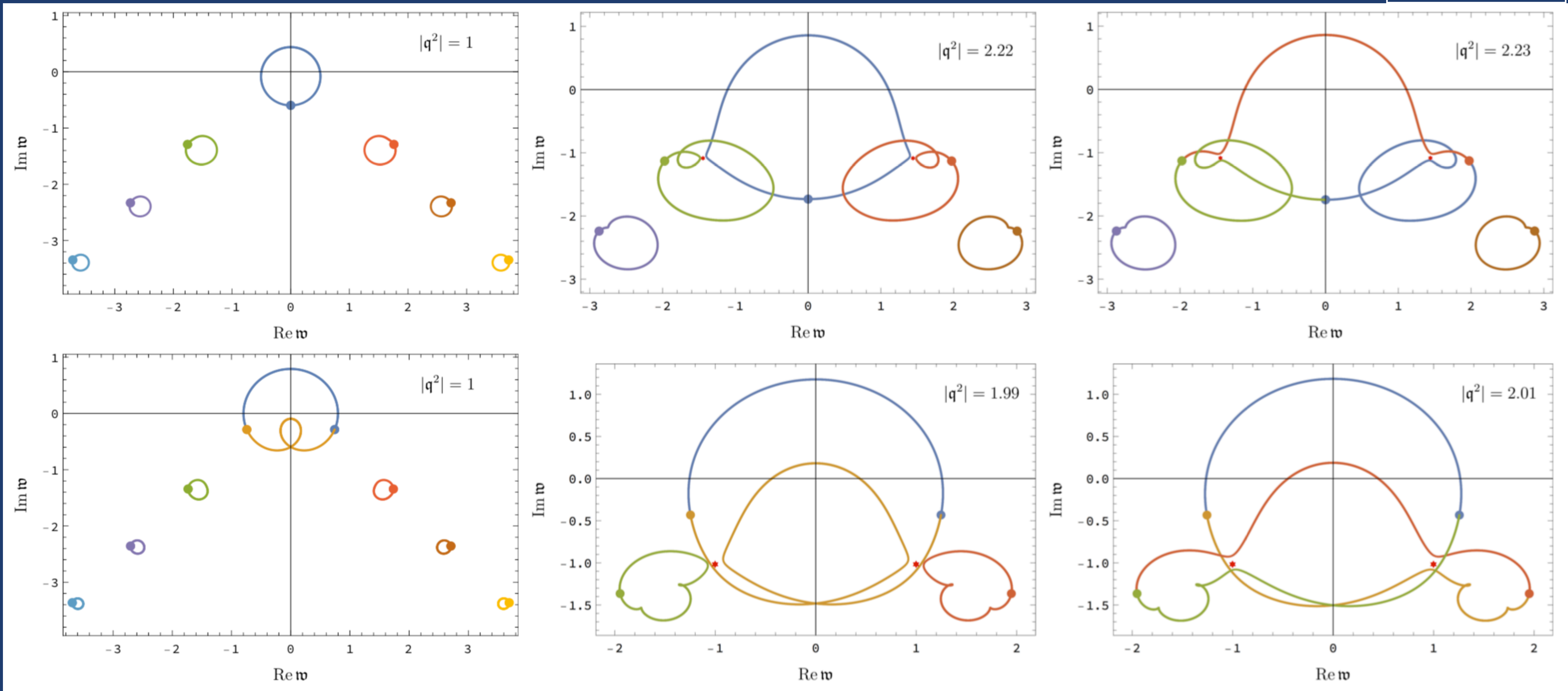
CONVERGENCE OF HYDRODYNAMICS

- radius of convergence of $\mathfrak{w}(\mathfrak{q}) = \sum_{i=1}^{\infty} c_n \mathfrak{q}^n$, i.e. $|\mathfrak{q}| < \mathfrak{q}_*$, is set by the lowest momentum at which the hydro pole collides (level-crossing): $\boxed{\mathfrak{q}_* = \min\left[|\mathfrak{q}_{\text{collision}}|\right]}$

$$\mathfrak{q}^2 = \left|\mathfrak{q}^2\right| e^{i\theta}$$

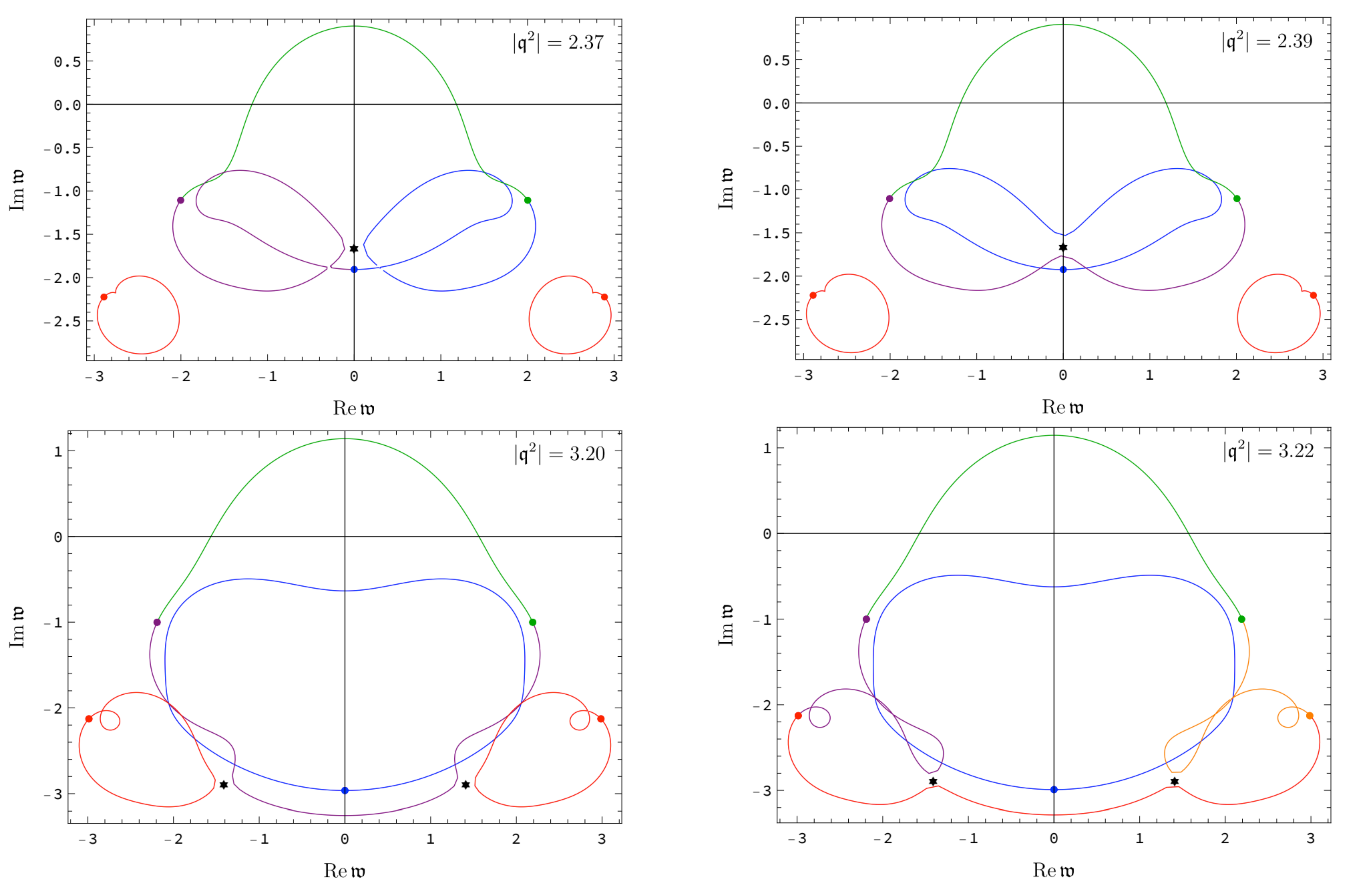
shear

sound



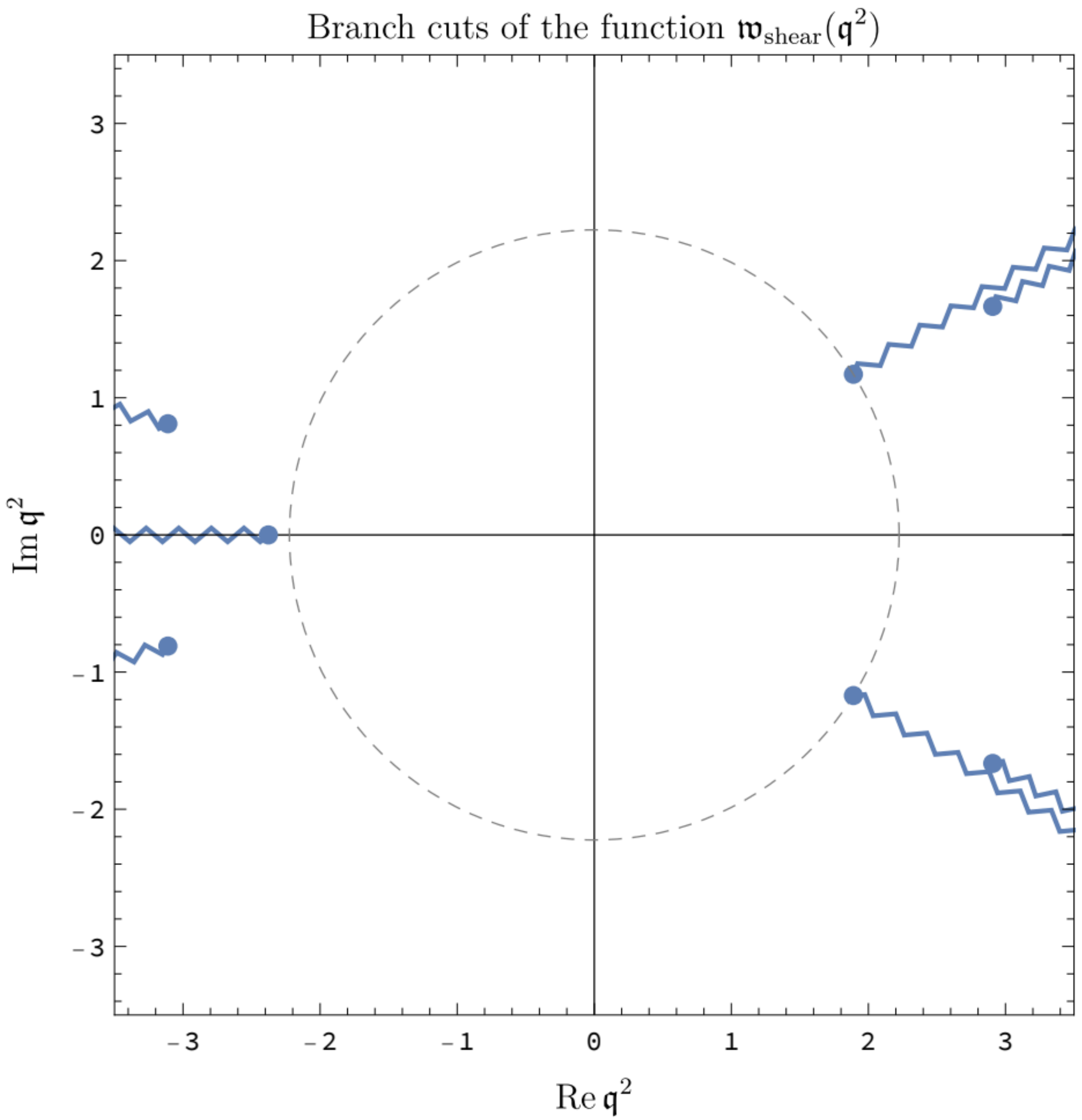
shear:	$\mathfrak{q}_* \approx 1.49131$ $\mathfrak{w}(\mathfrak{q}_*) \approx \pm 1.4436414 - 1.0692250i$	$\mathcal{N} = 4$ SYM	sound:	$\mathfrak{q}_* = \sqrt{2} \approx 1.41421$ $\mathfrak{w}(\mathfrak{q}_*) = \pm 1 - i$

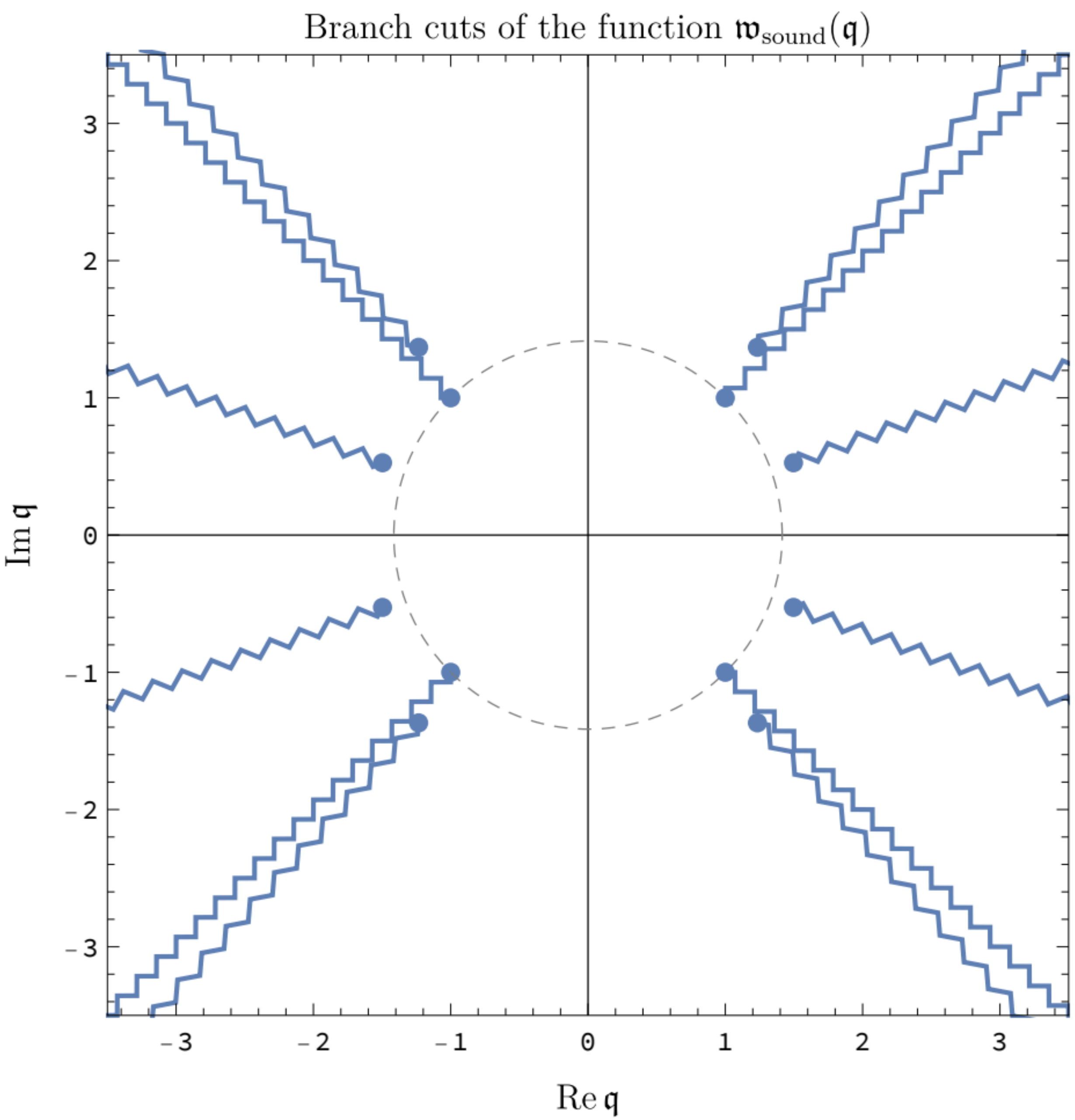
HIGHER CRITICAL POINTS (E.G. SHEAR CHANNEL)



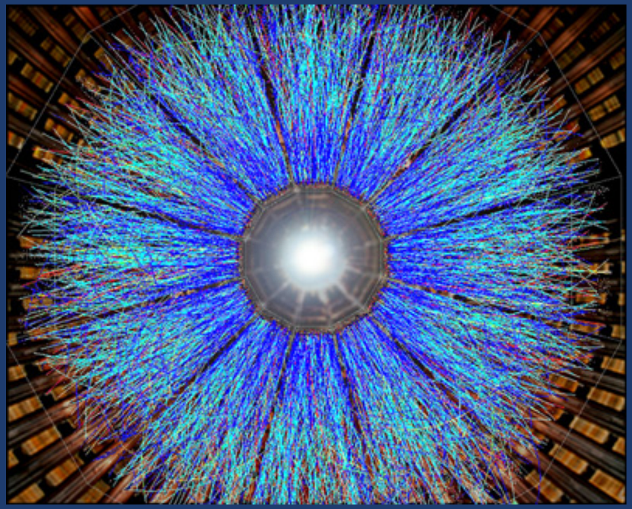
ANALYTIC STURCTURE

- analytic structure of dispersion relations in momentum space





UNREASONABLE EFFECTIVENESS



"**unreasonable**": hydro works for large ∂

dispersion relations are infinite series $\omega(q) = \sum_{n=1}^{\infty} \alpha_n q^n$

radius of convergence → $q/T \sim O(10)$

microscopic input from holography

- orders of magnitude larger than naive $q/T \ll 1$ – if this fact is generically true in hydrodynamic theories, this may help explain the **"unreasonable effectiveness of hydrodynamics"** which is the question of why hydrodynamics works in quark-gluon plasma and other systems where derivatives are large

SO… WHAT NOW

some of what we know

Bjorken flow expansion in proper time diverges

dispersion relations converge in momentum space

why? options:

Fourier transform "generically" converts convergent series to divergent series

essential singularity at $t \to \infty$ implies an asymptotic series

$$G(\omega, q) \sim \frac{1}{\omega^2 - q^2 - m^2}$$

the Bjorken flow is sick, dispersion relations are well defined

Bjorken flow is non-linear whereas dispersion relations are only linear

coefficients in the series may grow faster but may also not; who knows…

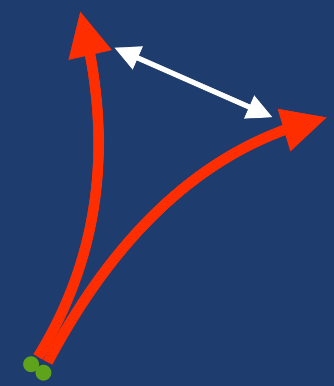
ALL-ORDER HYDRODYNAMICS AND MICROSCOPIC QUANTUM CHAOS

CHAOS

Lyapunov exponent   butterfly velocity

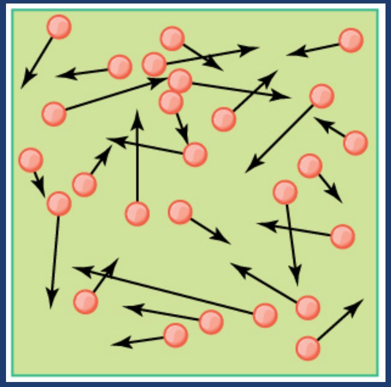
$$|\Delta Z(t,\mathbf{x})| \approx |\Delta Z(t_i,\mathbf{x}_i)|\, e^{\lambda_L (t-|\mathbf{x}|/v_B)}$$

- classical chaos means extreme sensitivity to initial conditions
- exponential **Lyapunov** divergence of trajectories and **the butterfly effect**



$t = t_i$

- in quantum systems, molecules collide chaotically



- the effect can be diagnosed with special "out-of-time-ordered" correlation functions [Larkin, Ovchinnikov; Kitaev]

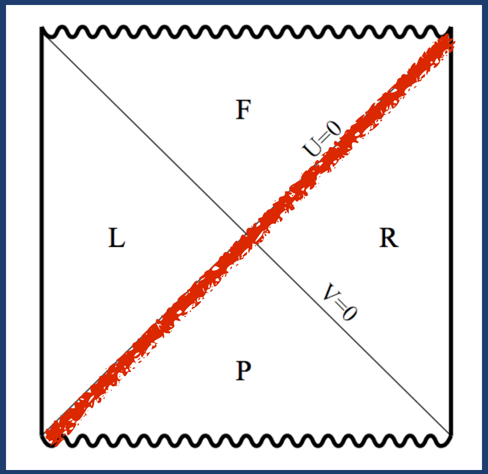
$$C(t,\mathbf{x}) = \left\langle [W(t,\mathbf{x}),V(0,\mathbf{0})]^\dagger [W(t,\mathbf{x}),V(0,\mathbf{0})] \right\rangle_T \sim \epsilon\, e^{\lambda_L (t-|\mathbf{x}|/v_B)}$$

scrambling time $\boxed{t_* = \frac{1}{\lambda_L} \ln N}$ ⟵ typically, $\epsilon = 1/N_c^2 \ll 1$ Lyapunov exponent   butterfly velocity

- its "build-up" describes the **quantum butterfly effect**
- standard lore: *"microscopic quantum information is smeared out at large distances"*

CHAOS IN HOLOGRAPHY

- Lyapunov exponent and butterfly velocity follow from the holographic shock wave on the horizon of a two-sided black hole



$$ds^2 = A(UV)dUdV + B(UV)d\vec{x}^2 - A(UV)\delta(U)h(x)dU^2$$

- Lyapunov exponent saturates the Maldacena-Shenker-Stanford bound

$$\lambda_L \leq 2\pi T$$

- what is the precise connection between hydrodynamics and chaos?

POLE-SKIPPING

- the phenomenon of pole-skipping makes precise the analytic connection between hydrodynamics and chaos; **true in "all" classical holographic theories** [Grozdanov, Schalm, Scopelliti, PRL 120 (2018) 23, 231601 arXiv:1710.00921; Blake, Lee, Liu, JHEP 10, 127 (2018), arXiv:1801.00010; Blake, Davison, Grozdanov, and Liu, JHEP 10, 035 (2018), arXiv:1809.01169]
- resumed all-order hydrodynamic series (e.g. the sound channel)

$$\omega_\pm(k) = \pm \sum_{n=0}^{\infty} \mathcal{V}_{2n+1} k^{2n+1} - i \sum_{n=0}^{\infty} \Gamma_{2n+2} k^{2n+2}$$

passes through the "point of chaos"

$$\mathcal{P}_c : \qquad \omega(k = ik_0) = i\lambda_L, \qquad \lambda_L = 2\pi T, \qquad k_0 = \lambda_L / v_B$$

which is defined through the fact that the (longitudinal) retarded energy density two point function has both a pole and a zero at this point

$$G^R_{T^{00}T^{00}}(\omega, k) = \frac{b(\omega, k)}{a(\omega, k)}, \qquad \lim_{(\omega,\, k) \to \mathcal{P}_c} a(\omega, k) = \lim_{(\omega,\, k) \to \mathcal{P}_c} b(\omega, k) = 0$$

POLE-SKIPPING

- simple example: the SYK chain [Gu, Qi, Stanford (2017)]

$$G^R_{T^{00}T^{00}}(\omega, k) = C\frac{i\omega\left(\frac{\omega^2}{\lambda_L^2}+1\right)}{-i\omega + D_E k^2}$$

pole (diffusion): $\omega = -iD_E k^2$

zero : $\omega = \pm i\lambda_L$

$$|k_0| = \frac{\lambda_L}{v_B} = \frac{\lambda_L}{\sqrt{\lambda_L D_E}}$$

- in $\mathcal{N}=4$ SYM theory at infinite N_c:

$$k_0 = \sqrt{6}\pi T$$
$$v_B = \lambda_L/k_0 = \sqrt{2/3}$$

point of chaos is inside the radius of convergence

- the reason for pole-skipping in holography is a special, new property of Einstein's equations at the horizon, which shall remain unexplained for today [Blake, Davison, Grozdanov, and Liu, JHEP 10, 035 (2018), arXiv:1809.01169]

POLE-SKIPPING

- pole-skipping occurs in all channels for complex values of frequencies and momenta on the circles with fixed absolute values $|k_c|, |\omega(k = k_c)|$

  [Grozdanov, Kovtun, Starinets, Tadić (2019); Blake, Davison, Vegh (2019)]

$\mathcal{P}_c^{sound}$: $\langle T_{tt}(-\omega,-k_z), T_{tt}(\omega,k_z)\rangle$	$\omega(k=k_c) = i\lambda_L,$	$k_c = ik_0$
$\mathcal{P}_c^{shear}$: $\langle T_{xz}(-\omega,-k_z), T_{xz}(\omega,k_z)\rangle$	$\omega(k=k_c) = -i\lambda_L$	$k_c = k_0$
$\mathcal{P}_c^{scalar}$: $\langle T_{xy}(-\omega,-k_z), T_{xy}(\omega,k_z)\rangle$	$\omega(k=k_c) = -i\lambda_L$	$k_c = ik_0$

- in the scalar channel, there are no hydrodynamic modes
- a gapped mode experiences pole-skipping

POLE-SKIPPING

- in $\mathcal{N}=4$ SYM theory at infinite N_c and infinite coupling

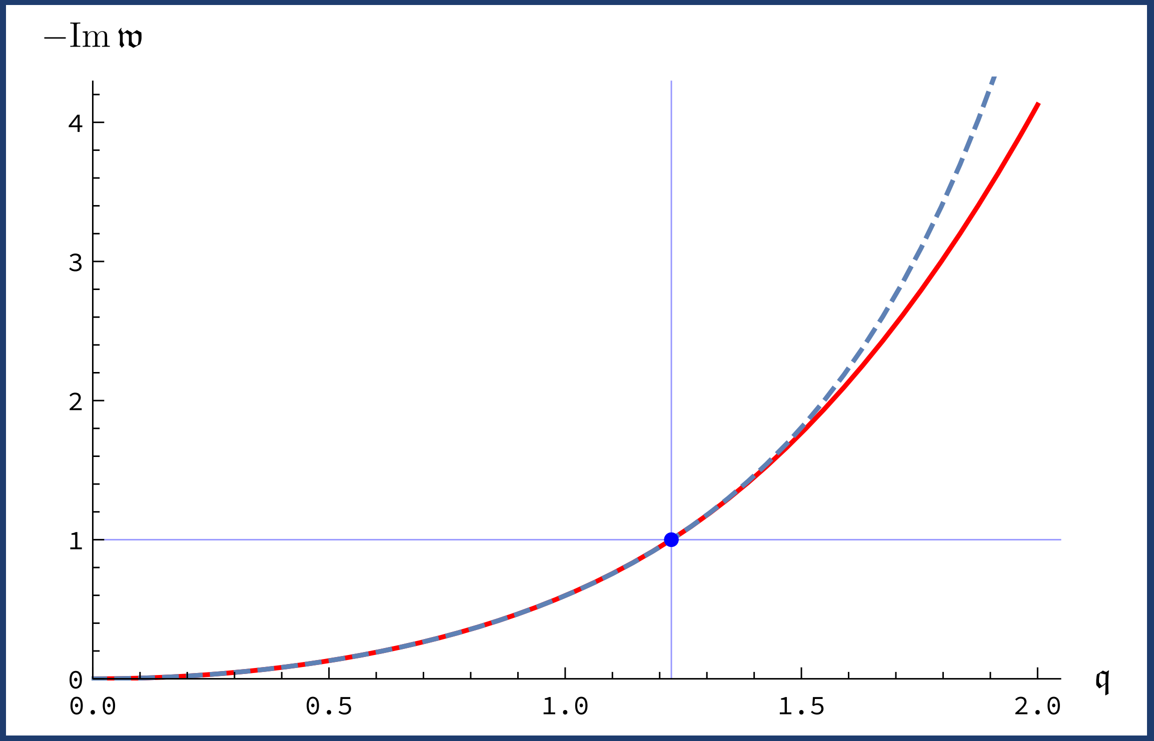
$$\text{diffusion}: \ \omega_c = \omega(q_c = q_0) = -i\lambda_L$$
$$\text{sound}: \ \omega_c = \omega(q_c = iq_0) = i\lambda_L$$

$$q_0 \in \mathbb{R}$$

$$\text{Lyapunov exponent}: \ \lambda_L = |\omega_c| = 2\pi T$$
$$\text{butterfly velocity}: \ v_B = |\omega_c / q_c|$$

shear (diffusion):

$$\langle T_{xz}(-\omega, -q_z), T_{xz}(\omega, q_z)\rangle$$



$$\omega = -\frac{i}{4\pi T}q^2 - \frac{i(1-\ln 2)}{32\pi^3 T^3}q^4 - \frac{i(24\ln^2 2 - \pi^2)}{96\,(2\pi T)^5}q^6 - \frac{i\left[2\pi^2(\ln 32 - 1) - 21\zeta(3) - 24\ln 2(1+\ln 2(\ln 32 - 3))\right]}{384\,(2\pi T)^7}q^8 + \cdots$$

POLE-SKIPPING

- in $\mathcal{N} = 4$ SYM theory at infinite N_c and infinite coupling

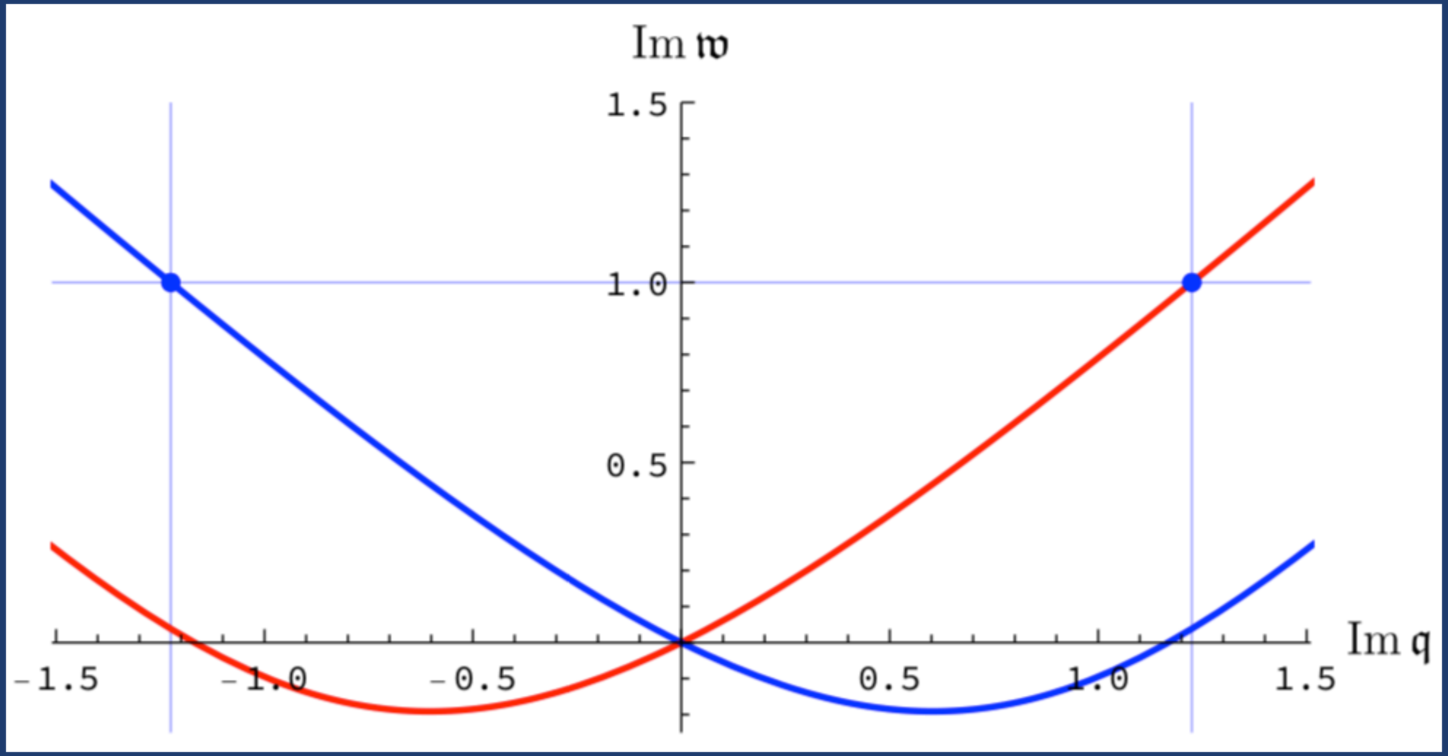
$$\text{diffusion}: \; \omega_c = \omega(q_c = q_0) = -i\lambda_L$$
$$\text{sound}: \; \omega_c = \omega(q_c = iq_0) = i\lambda_L$$

$$q_0 \in \mathbb{R}$$

$$\text{Lyapunov exponent}: \; \lambda_L = |\omega_c| = 2\pi T$$
$$\text{butterfly velocity}: \; v_B = |\omega_c / q_c|$$

sound:

$$\langle T_{tt}(-\omega, -q_z), T_{tt}(\omega, q_z)\rangle$$



$$\omega = \pm\frac{1}{\sqrt{3}}q - \frac{i}{6\pi T}q^2 \pm \frac{3 - 2\ln 2}{24\sqrt{3}\pi^2 T^2}q^3 - \frac{i\left(\pi^2 - 24 + 24\ln 2 - 12\ln^2 2\right)}{864\pi^3 T^3}q^4 \pm \cdots$$

POLE-SKIPPING

- in $\mathcal{N}=4$ SYM theory at infinite N_c and infinite coupling

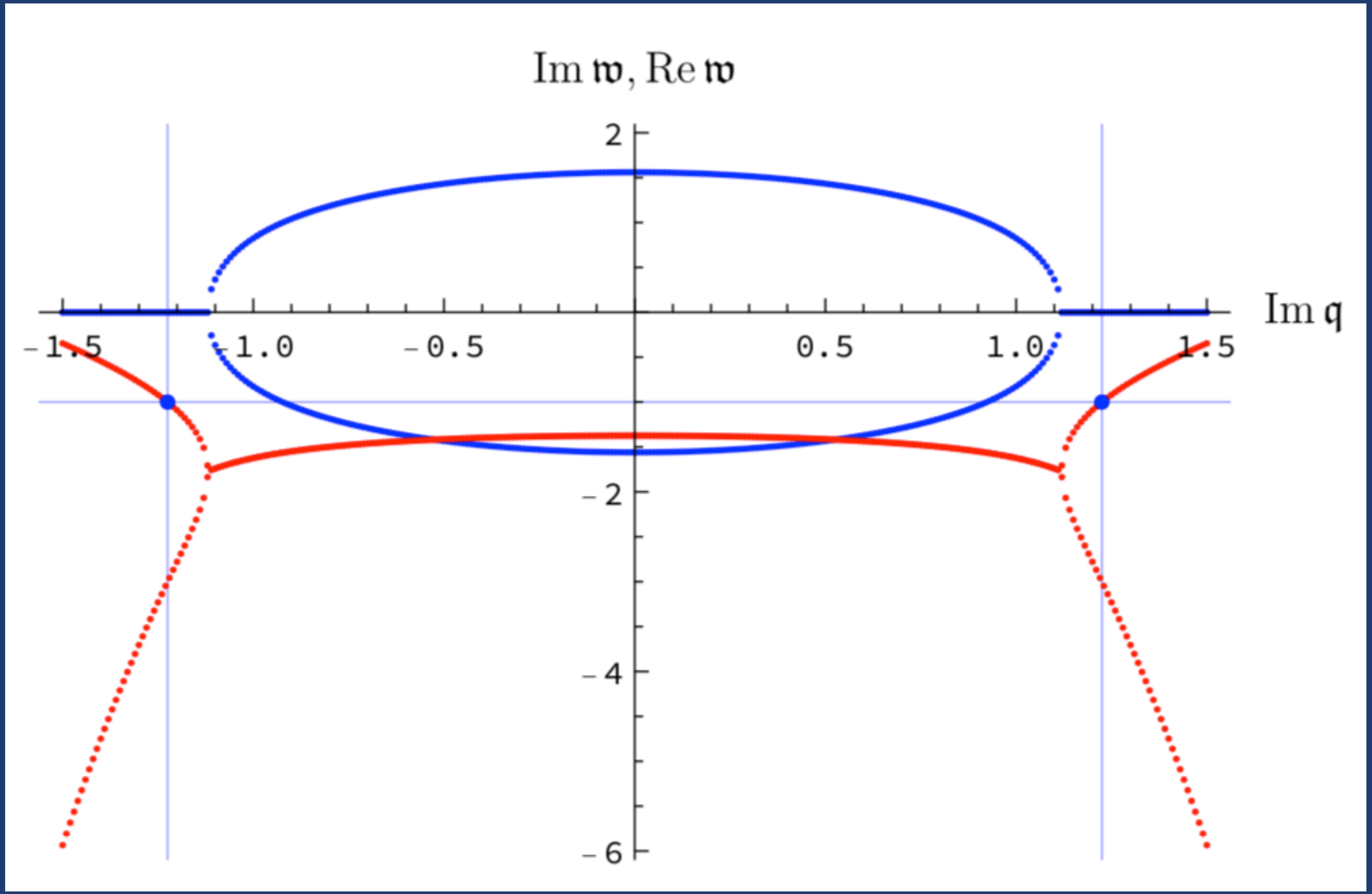
$$\text{scalar}: \ \omega_c = \omega(q_c = iq_0) = -i\lambda_L$$

$$q_0 \in \mathbb{R}$$

$$\text{Lyapunov exponent}: \ \lambda_L = |\omega_c| = 2\pi T$$
$$\text{butterfly velocity}: \ v_B = |\omega_c/q_c|$$

scalar:

$$\langle T_{xy}(-\omega,-q_z), T_{xy}(\omega,q_z)\rangle$$



POLE-SKIPPING AT FINITE COUPLING AND N_C

- pole-skipping exists at finite 't Hooft coupling, and even for leading *1/N*$1/N_c$ [Grozdanov (2019)]

- for generic coupling and generic N_c, the situation is unknown; long-time tails, possible only early-time exponential growth, multiple Lyapunov exponents, …

- at weak coupling, kinetic theory [Grozdanov, Schalm, Scopelliti, PRE (2018)] suggests connection to hydrodynamics; the status of pole-skipping is unknown

- intriguing connection to the KSS bound:

$$v_B = v_B^\infty \left(1+\delta\right), \quad \frac{\eta}{s} = \frac{1}{4\pi}\left(1+\Delta\right)$$

  thus, one may speculate

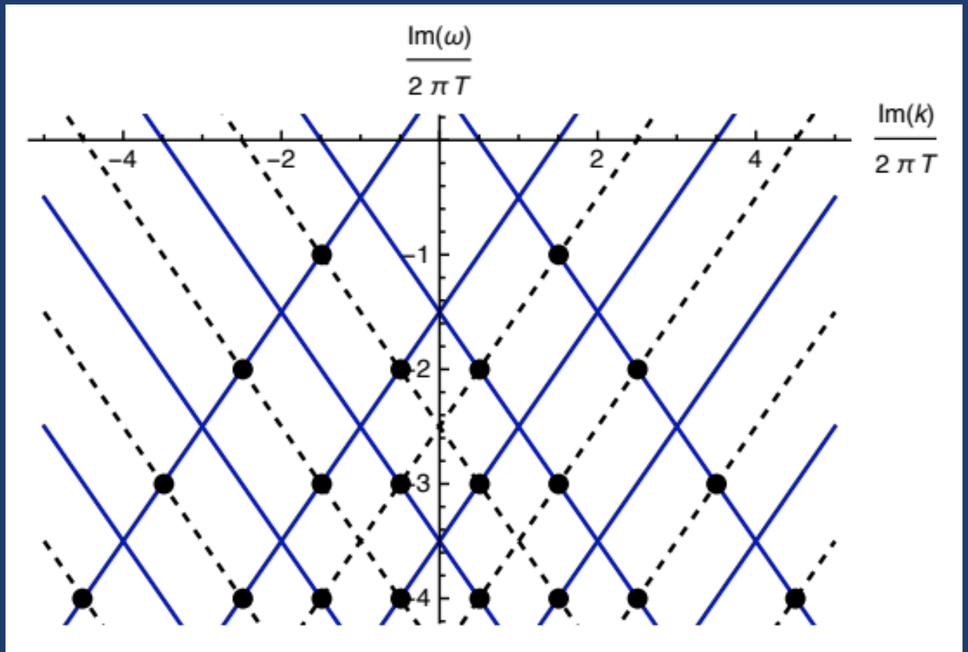
$$v_B(\lambda = g_{YM}^2 N_c \to \infty, N_c) \leq v_B(\lambda, N_c) \leq 1, \text{ with } v_B \to 1 \text{ as } g_{YM} \to 0$$

  as a functions of coupling, then

$$\mathcal{D} \equiv \mathcal{C}\frac{v_B^2}{\lambda_L} \geq \mathcal{C}\frac{v_B^2(\lambda\to\infty)}{2\pi T}$$

HIGHER POLE-SKIPPING POINTS

- (holographic) CFTs exhibit an infinite tower of pole-skipping points (scalars, currents, energy-momentum tensor components, fermions)
- the frequencies are dominated by multiples of Matsubara frequencies
- pole-skipping imposes constraints on the structures of correlators
- example: *1+1* dimensional CFT dual to the BTZ black hole
  [Grozdanov, Kovtun, Starinets, Tadić (2019); Blake, Davison, Vegh (2019)]



[plot from Blake, Davison, Vegh (2019)]

$$\langle \mathcal{O}(\omega,k)\mathcal{O}(-\omega,-k)\rangle_R$$

$$\omega_n = -i2\pi T n, \quad k_{n,q} = \pm 2\pi i T(n-2q+\Delta)$$

$$n \in \{1,2,\ldots\}, \quad q \in \{1,\ldots,n\}$$

$$\Delta = 2.5$$

- similar structure exists even at finite 't Hooft coupling in *N=4* SYM
  [Grozdanov (2018); Natsuume, Okamura (2019); Wu (2019)]

OTHER FUTURE DIRECTIONS

- further applications of complex analysis and algebraic geometry methods to studies of thermal and quantum spectra
- pole-skipping or its generalisation in perturbative, weakly coupled QFTs through our kinetic theory for quantum many-body chaos
  [Grozdanov, Schalm, Scopelliti, PRE (2018)]
- what is the physical meaning of pole-skipping?
- other new (complex) analytic properties of hydrodynamics and thermal spectra
- experimental signatures such pole-skipping or its generalisations?
- towards rigorous proofs of various bounds, monotonicity statements, etc.
- a better understanding of all aspects of physics for generic N_c; quantum gravity, hydrodynamic long-time tails

THANK YOU!