# Relativistic fluid dynamics of spin-polarized systems and spin-vorticity coupling

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*Prog. Part. Nucl. Phys.* 108 (2019) 103709 [1811.04409]

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# Angular momentum and spin in non-central heavy-ion collisions

Non-central heavy-ion collisions create fireballs with large global angular momenta
F. Becattini, F. Piccinini, J. Rizzo, PRC 77 (2008) 024906

Some part of the angular momentum can be transferred from the orbital to the spin part
Barnett, Rev. Mod. Phys. 7, 129 (1935)

$$ J_{\text{init}} = L_{\text{init}} \qquad = \qquad L_{\text{final}} + S_{\text{final}} $$

Should be reflected in the polarization of observed hadrons
(e.g. parton scattering polarizes quarks due to spin-orbit coupling)
Z.-T. Liang, X.-N. Wang, PRL 94 (2005) 102301; PLB 629 (2005) 20–26
J.-H. Gao, et al., PRC 77 (2008) 044902
S.-W. Chen, J. Deng, J.-H. Gao, Q. Wang, Front. Phys. China 4 (2009) 509–516
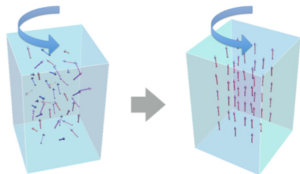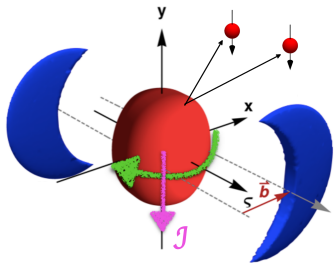B. Betz, M. Gyulassy, G. Torrieri, PRC 76 (2007) 044901

First HI experiments that measured spin polarization in Dubna, CERN and BNL reported negative results
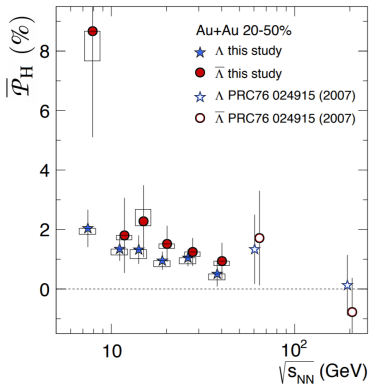M. K. Anikina, et al., Z. Phys. C25 (1984) 1–11
J. Bartke, et al., Z. Phys. C48 (1990) 191–200
B. I. Abelev, et al., PRC 76 (2007) 024915

# First positive measurements of global spin polarization of $\Lambda$ hyperons by STAR



thermal approach $\longrightarrow$ $P_\Lambda \approx \frac{1}{2}\frac{\omega}{T} + \frac{\mu_\Lambda B}{T}$ $\qquad P_{\overline{\Lambda}} \approx \frac{1}{2}\frac{\omega}{T} - \frac{\mu_\Lambda B}{T}$

Becattini, F., Karpenko, I., Lisa, M., Upsal, I. , Voloshin, S. , PRC 95, 054902 (2017)

*. . . the hottest, least viscous – and now, most vortical – fluid produced in the laboratory . . .*

$$\omega = \left(P_\Lambda + P_{\overline{\Lambda}}\right)k_B T/\hbar \sim 0.6 - 2.7 \times 10^{22}\mathrm{s}^{-1}$$

L. Adamczyk et al. (STAR) (2017), Nature 548 (2017) 62-65

# Theory succees - global polarization

**Spin DoF are locally equilibrated**
F.Becattini, F.Piccinini,Ann.Phys.323,2452(2008)
F.Becattini, F.Piccinini, J. Rizzo, PRC 77, 024906 (2008)

**Polarization** given by **thermal vorticity** (GEQ)
F. Becattini, V. Chandra, L. Del Zanna, E. Grossi, Ann. Phys. 338, 32 (2013)

$$\omega^{\mu\nu} \quad \Leftrightarrow \quad \varpi^{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right)$$
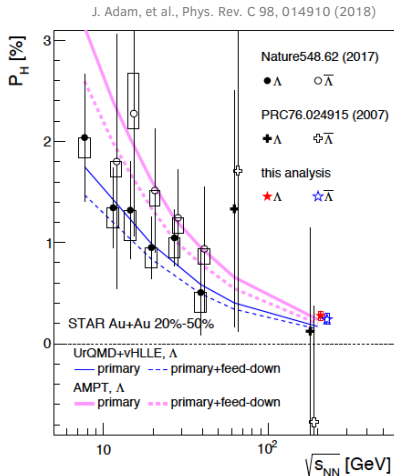
$$\beta_\nu = \frac{u_\nu}{T}$$

**Frozen at the hadronization stage using hydrodynamics/transport without spin**

Possible 15%-20% dilution of primary $\Lambda$ polarization due to **feed-down effect**
F. Becattini, I. Karpenko, M. Lisa, I. Upsal, S. Voloshin, PRC 95 (2017) no.5, 054902
X-L Xia, H. Li, X-G Huang, H. Z. Huang [1905.03120]
F. Becattini, G. Cao, E. Speranza [1905.03123]

# Phenomenological prescription used to describe the data

**Statistical approach quite appealing!**

**Algorithm is:**

1) Run any type of hydro, perfect or viscous, or transport, or whatsoever, without spin

2) Find $\beta_\mu(x) = u_\mu(x)/T(x)$ on the freeze-out hypersurface (defined often by the condition $T$=const)

3) Calculate thermal vorticity $\varpi_{\alpha\beta}(x) \neq$ const

4) Identify thermal vorticity with the spin polarization tensor $\omega_{\mu\nu}$

5) Make predictions about spin polarization

**As we have seen such a method describes very well the global polarization of $\Lambda$
but let us look differential ...**

# Theory failures - azimuthal angle dependence of $P_y$
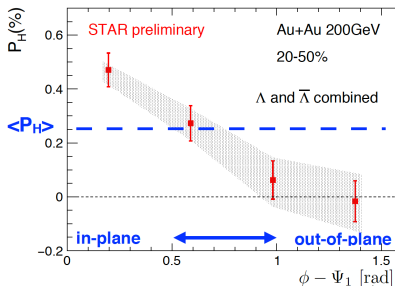


Credit: T.Niida, The 5th Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions, 2019
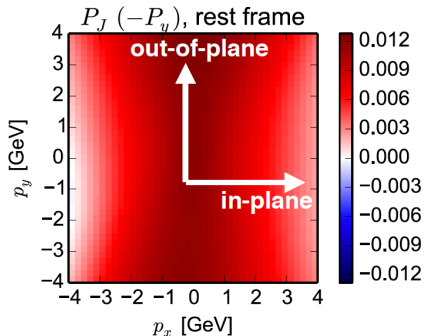
I. Karpenko, F. Becattini, EPJC 77, 213 (2017)

**"thermal vorticity–based" approach fails to describe the azimuthal dependence of $P_y$ component**

# Theory failures - azimuthal angle dependence of $P_z$



$\langle \cos(\theta_p^*) \rangle$ vs $\phi - \Psi_2$ [rad]

Au+Au $\sqrt{s_{NN}}$ = 200 GeV
10%-60%

★ $\Lambda$
☆ $\overline{\Lambda}$

STAR Preliminary
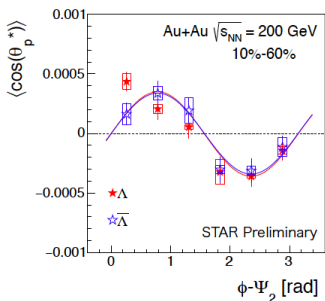
T. Niida, NPA 982 (2019) 511514



$P^z$, $\sqrt{s_{NN}}$ = 200 GeV RHIC
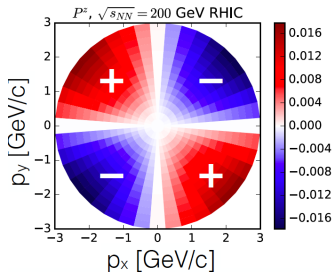
UrQMD+vHLLE: F. Becattini, I. Karpenko, PRL 120 (2018) no.1, 012302,



$\langle P_z \rangle$ vs $\phi_p$
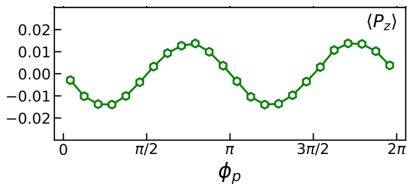
AMPT: X. Xia, H. Li, Z. Tang, Q. Wang, PRC98.024905 (2018)

**"thermal vorticity–based" approach fails to describe the quadrupole
structure of $P_z$ component**

# Fluid dynamics with spin?

**space-time dynamics of spin polarization?**

**relativistic hydrodynamics forms the basis of HIC models**
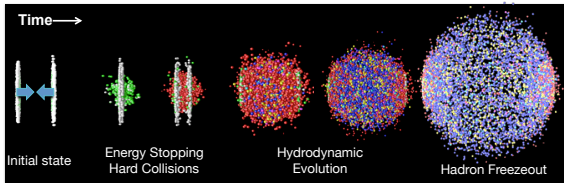


T. K. Nayak, Lepton-Photon 2011 Conference

**perfect fluid dynamics = local equilibrium + conservation laws**

**for particles with spin the conservation of angular momentum is not trivial
new hydrodynamic variables should be introduced**

# Conservation laws of canonical currents

**Constructions of the hydrodynamic frameworks rely on the local conservation laws**

**Noether's theorem:**
(continuous symmetry of the action  $\longleftrightarrow$  conserved current)

**Conservation of charge (baryon number, electric charge, . . . )**

$$\partial_\mu \widehat{N}^\mu(x) = 0 \qquad \text{(1 equation/charge)}$$

**Conservation of energy and momentum**

$$\partial_\mu \widehat{T}_C^{\mu\alpha}(x) = 0 \qquad \text{(4 equations)}$$

**Conservation of total angular momentum**

$$\partial_\mu \widehat{J}_C^{\mu,\alpha\beta}(x) = 0 \qquad \widehat{J}_C^{\mu,\alpha\beta}(x) = -\widehat{J}_C^{\mu,\beta\alpha}(x) \quad \text{(6 equations)}$$

# Orbital and spin parts of angular momentum

**Total angular momentum** $\widehat{J}_C^{\mu,\alpha\beta}(x)$ may be decomposed into **orbital** and **spin** parts

$$\widehat{J}_C^{\mu,\alpha\beta}(x) = \underbrace{x^\alpha \widehat{T}_C^{\mu\beta}(x) - x^\beta \widehat{T}_C^{\mu\alpha}(x)}_{\widehat{L}_C^{\mu,\alpha\beta}(x)} + \widehat{S}_C^{\mu,\alpha\beta}(x)$$

**Conservation of total angular momentum** gives

$$\partial_\mu \widehat{J}_C^{\mu,\alpha\beta}(x) = 0 \qquad \Rightarrow \qquad \partial_\mu \widehat{S}_C^{\mu,\alpha\beta}(x) = \widehat{T}_C^{\beta\alpha}(x) - \widehat{T}_C^{\alpha\beta}(x)$$

$\widehat{T}_C^{\mu\alpha}(x) \neq \widehat{T}_C^{\alpha\mu}(x)$ – **canonical energy-momentum tensor is not symmetric**
$$\Downarrow$$
**canonical spin tensor is not conserved separately**

# Pseudo-gauge transformations

**Inclusion of the conservation of angular momentum is connected with the problem of the localization of energy and spin densities**

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**Pseudo-gauge transformation**
W. Hehl, Rept. Math. Phys. 9 (1976) 55–82; F. Becattini, L. Tinti, Phys. Rev. D 84 (2011) 025013; Phys. Rev. D 87(2) (2013) 025029

$$\widehat{T}'^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\lambda\left(\widehat{\Phi}^{\lambda,\mu\nu} - \widehat{\Phi}^{\mu,\lambda\nu} - \widehat{\Phi}^{\nu,\lambda\mu}\right)$$

$$\widehat{S}'^{\lambda,\mu\nu} = \widehat{S}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu}$$

$\rightsquigarrow$ preserve $\widehat{P}^\mu = \int d^3\Sigma_\lambda \, \widehat{T}^{\lambda\mu}(x)$   $\widehat{J}^{\mu\nu} = \int d^3\Sigma_\lambda \, \widehat{J}^{\lambda,\mu\nu}(x)$

$\rightsquigarrow$ conservation laws unchanged

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**Energy-momentum and spin tensors are not uniquely defined**

**Belinfante-Rosenfeld construction** (choosing superpotential $\widehat{\Phi} = \widehat{S}_C^{\lambda,\mu\nu}$)
Belinfante, F. J. (1939): Physica 6. 887-898, (1940); Rosenfeld, L. (1940): Mem. Acad. Roy. Belgique, cl. SC., tome 18, fasc. 6

$$\widehat{T}_B^{\mu\nu} = \widehat{T}_C^{\mu\nu} + \frac{1}{2}\partial_\lambda\left(\widehat{S}_C^{\lambda,\mu\nu} + \widehat{S}_C^{\mu,\nu\lambda} - \widehat{S}_C^{\nu,\lambda\mu}\right) \qquad \widehat{S}_B^{\lambda,\mu\nu} = 0$$

$\rightsquigarrow$ gives exactly symmetric Hilbert $T^{\mu\nu}$ acting as the source of gravity in GR

$\rightsquigarrow$ long-standing problem of physical significance of the spin tensor

$\rightsquigarrow$ spin tensor is used by the community that studies the spin of proton
X.S. Chen, X.F. Lu, W.M. Sun, F. Wang, T. Goldman, Phys. Rev. Lett. 100 (2008) 232002;
E. Leader, C. Lorce, Phys. Rep. 541 (2014) 163.

# Local equilibrium density operator - Belinfante

In quantum statistical mechanics LE can be defined as a maximum of the von Neumann entropy

D.N. Zubarev, Sov. Phys. Dokl. 10 (1966) 850; D.N. Zubarev, A.V. Prozorkevich, S.A. Smolyanskii, Theor. Math. Phys. 40 (1979) 821.
Ch.G. Van Weert, Ann. Phys. 140 (1982) 133;
F. Becattini, Phys. Rev. Lett. 108 (2012) 244502.;F. Becattini, L. Bucciantini, E. Grossi, L. Tinti, Eur. Phys. J. C 75(5) (2015) 191;

$$S = -\text{tr}(\widehat{\rho}_B \log \widehat{\rho}_B)$$

Constraints of given mean densities of conserved currents over some spacelike hypersurface $\Sigma$

$$n_\mu \text{tr}\left[\widehat{\rho}_B \, \widehat{j^\mu}(x)\right] = n_\mu j^\mu(x)$$

$$n_\mu \text{tr}\left[\widehat{\rho}_B \, \widehat{T}^{\mu\nu}(x)\right] = n_\mu T^{\mu\nu}(x)$$

$$n_\mu \text{tr}\left[\widehat{\rho}_B \, \widehat{J}^{\mu,\lambda\nu}(x)\right] = n_\mu \text{tr}\left[\widehat{\rho}_B \left(x^\lambda \widehat{T}_B^{\mu\nu}(x) - x^\nu \widehat{T}_B^{\mu\lambda}(x)\right)\right] = n_\mu J_B^{\mu,\lambda\nu}(x)$$

$n^\mu$ - vector orthogonal to $\Sigma$

$$\widehat{S}_B^{\lambda,\mu\nu} = 0$$
$$\Downarrow$$
**condition on total angular momentum is redundant - follows from the constraint on energy and momentum**

Resulting LE density operator in this case is

$$\widehat{\rho}_B = \frac{1}{Z} \exp\left[ -\int_\Sigma d\Sigma_\mu \left(\widehat{T}_B^{\mu\nu}(x)\beta_\nu(x) - \widehat{j}^\mu(x)\xi(x)\right)\right] \qquad \beta_\nu = \frac{u_\nu}{T} \qquad \xi = \frac{\mu}{T}$$

# Local equilibrium density operator - canonical

Again, use maximization of entropy

$$S = -\mathrm{tr}(\widehat{\rho}_C \log \widehat{\rho}_C)$$

Constraints on charge, energy, momentum and total angular momentum

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_C \widehat{j}^\mu(x)\right] = n_\mu j^\mu(x)$$

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_C \widehat{T}_C^{\mu\nu}(x)\right] = n_\mu T_C^{\mu\nu}(x)$$

$$n_\mu \mathrm{tr}\left[\widehat{\rho}_C \widehat{J}_C^{\mu,\lambda\nu}(x)\right] = n_\mu \mathrm{tr}\left[\widehat{\rho}_C \left(x^\lambda \widehat{T}_C^{\mu\nu}(x) - x^\nu \widehat{T}_C^{\mu\lambda}(x) + \widehat{S}_C^{\mu,\lambda\nu}\right)\right] = n_\mu J_C^{\mu,\lambda\nu}(x)$$

**Spin tensor gives non-trivial constraint on total angular momentum**
⇓
**One has to introduce additional Lagrange multipliers,**
**spin polarization tensor** $\omega_{\lambda\nu}$

Resulting LE density operator is

$$\widehat{\rho}_C = \frac{1}{Z} \exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}_C^{\mu\nu}(x)\beta_\nu(x) - \widehat{j}^\mu(x)\xi(x) - \frac{1}{2}\widehat{S}_C^{\mu,\lambda\nu}(x)\omega_{\lambda\nu}(x)\right)\right]$$

$$\widehat{\rho}_C = \frac{1}{Z} \exp\left[-\int_\Sigma d\Sigma_\mu \left(\widehat{T}_B^{\mu\nu}(x)\beta_\nu(x) - \widehat{j}^\mu(x)\xi(x) - \frac{1}{2}\widehat{S}_C^{\mu,\lambda\nu}(\omega_{\lambda\nu} - \varpi_{\lambda\nu})\right)\right]$$

$$\omega_{\lambda\nu} = \varpi_{\lambda\nu} = -\frac{1}{2}\left(\partial_\lambda \beta_\nu - \partial_\nu \beta_\lambda\right) \quad \Rightarrow \quad \widehat{\rho}_C = \widehat{\rho}_B$$

**Description based on the Belinfante tensors is reduced compared to the description employing the canonical tensors**

# Global equilibrium conditions - canonical

**In global equilibrium (GE) the density operator becomes stationary - divergence of the integrand must vanish**

$$\widehat{\rho}_{\text{LEQ}} = \frac{1}{Z} \exp\left[ -\int_{\Sigma} d\Sigma_{\mu} \left( \widehat{T}_C^{\mu\nu} \left( \beta_{\nu} - \omega_{\nu\lambda} x^{\lambda} \right) - \frac{1}{2} \omega_{\lambda\nu} \widehat{J}_C^{\mu,\lambda\nu} - \widehat{\xi} \widehat{j}^{\mu} \right) \right]$$

Using conservation laws it leads to the conditions

$$\partial_{\mu} \omega_{\nu\lambda} = 0, \qquad \partial_{\mu} \beta_{\nu} = \omega_{\nu\mu}, \qquad \partial_{\mu} \xi = 0$$

and

$$\partial_{\mu} \beta_{\nu} + \partial_{\nu} \beta_{\mu} = 0 \quad \text{(Killing equation)} \quad \beta_{\nu} = b_{\nu} + \omega_{\nu\lambda} x^{\lambda}$$

which means that

$$
\begin{aligned}
\xi &= \text{const.} \\
\omega_{\lambda\nu} &= \varpi_{\lambda\nu} = -\frac{1}{2} \left( \partial_{\lambda} \beta_{\nu} - \partial_{\nu} \beta_{\lambda} \right) = \text{const}
\end{aligned}
$$

**The spin polarization tensor is equal to thermal vorticity**

**This is not the case for a symmetric energy-momentum tensor**

# General concept of perfect-fluid hydrodynamics with spin

**If GE conditions are not satisfied, the integral over $\Sigma$ depends on its choice. The integrals over hypersurfaces $\Sigma_1$ and $\Sigma_2$ differ by the volume integral which describes dissipative phenomena**

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**If we neglect dissipation we may treat the LE operator $\widehat{\rho}_{\mathrm{LEQ}}$ as constant.**
**In this case we define the expectation values of the conserved currents through the expressions**

$$T^{\mu\nu} = \mathrm{tr}\left(\widehat{\rho}_{\mathrm{LEQ}}\,\widehat{T}^{\mu\nu}\right), \quad S^{\mu,\lambda\nu} = \mathrm{tr}\left(\widehat{\rho}_{\mathrm{LEQ}}\,\widehat{S}^{\mu,\lambda\nu}\right), \quad j^{\mu} = \mathrm{tr}\left(\widehat{\rho}_{\mathrm{LEQ}}\,\widehat{j}^{\mu}\right)$$

These tensors are all functions of $\beta_\mu$, $\omega_{\mu\nu}$, and $\xi$ which enter constitutive equations

$$T^{\mu\nu} = T^{\mu\nu}[\beta, \omega, \xi], \quad S^{\mu,\lambda\nu} = S^{\mu,\lambda\nu}[\beta, \omega, \xi], \quad j^{\mu} = j^{\mu}[\beta, \omega, \xi]$$

and satisfy the conservation laws

$$\partial_\mu T^{\mu\nu} = 0, \quad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}, \quad \partial_\mu j^{\mu} = 0$$

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**These are 11 equations for 11 unknown functions, which represent a generalization of the standard perfect-fluid hydrodynamics to the case including spin.**

first steps in this direction have been done in:
W. Florkowski, B. Friman, A. Jaiswal, E. Speranza, Phys. Rev. C 97(4) (2018) 041901
W. Florkowski, B. Friman, A. Jaiswal, R. R., E. Speranza, Phys. Rev. D 97 (2018) 116017

**The dissipative processes must drive the system to GE, hence $\omega$ should converge to $\varpi$ eventually**

as argued in:
D. Montenegro, L. Tinti, G. Torrieri, Phys. Rev. D 96 (2017) 076016

# Equilibrium distribution functions for particles with spin-$1/2$

**Alternatively, hydrodynamics may be derived from the underlying microscopic theory**

Phase-space distribution functions for spin-$1/2$ particles (+) and antiparticles (-) ($2\times2$ spin density matrices)

$$f_{rs}^+(x,p) = \frac{1}{2m}\bar{u}_r(p)X^+u_s(p), \quad f_{rs}^-(x,p) = -\frac{1}{2m}\bar{v}_s(p)X^-v_r(p)$$

$\beta^\mu = u^\mu/T$

$\xi = \mu/T$
$m$ - mass of particles
$T$ - temperature
$\mu$ - chemical potential
$u^\mu$ - four velocity ($u^2 = 1$)
$\omega_{\mu\nu}$ - spin-polarization tensor
$\hat{\Sigma}^{\mu\nu} = (i/4)[\gamma^\mu, \gamma^\nu]$ - spin operator

where

$$X^\pm = \exp\left[\pm\xi(x) - \beta_\mu(x)p^\mu \pm \frac{1}{2}\omega_{\mu\nu}(x)\hat{\Sigma}^{\mu\nu}\right]$$

**However, keep in mind that it has not been derived from any underlying microscopic model or theory**

# Equilibrium Wigner functions

**Phase-space distribution functions á la Becattini et al. can be used to determine explicit expressions for the corresponding equilibrium (particle and antiparticle) Wigner functions**

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**Equilibrium Wigner functions - De Groot, van Leeuwen, van Weert (GLW) framework**

De Groot, van Leeuwen, van Weert: *Relativistic Kinetic Theory. Principles and Applications*, 1980.

$$\mathcal{W}_{\text{eq}}^{+}(x,k) = \frac{1}{2} \sum_{r,s=1}^{2} \int dP\, \delta^{(4)}(k-p) u^r(p) \bar{u}^s(p) f_{rs}^{+}(x,p)$$

$$\mathcal{W}_{\text{eq}}^{-}(x,k) = -\frac{1}{2} \sum_{r,s=1}^{2} \int dP\, \delta^{(4)}(k+p) v^s(p) \bar{v}^r(p) f_{rs}^{-}(x,p)$$

$$\mathcal{W}_{\text{eq}}(x,k) = \mathcal{W}_{\text{eq}}^{+}(x,k) + \mathcal{W}_{\text{eq}}^{-}(x,k).$$

---

$$\mathcal{W}_{\text{eq}}^{\pm}(x,k) = \frac{e^{\pm\xi}}{4m} \int dP\, e^{-\beta \cdot p}\, \delta^{(4)}(k \mp p) \left[ 2m(m \pm \not{p}) \cosh(\zeta) \pm \frac{\sinh(\zeta)}{2\zeta} \omega_{\mu\nu} (\not{p} \pm m)\Sigma^{\mu\nu}(\not{p} \pm m) \right].$$

**The presence of the Dirac delta functions indicates that, to large extent, $\mathcal{W}_{\text{eq}}^{\pm}(x,k)$ describe classical motion - they cannot be regarded as complete, quantum equilibrium distributions.**

**Nevertheless, the functions $\mathcal{W}_{\text{eq}}^{\pm}(x,k)$ incorporate spin DOF's and may serve to construct the formalism of hydrodynamics with spin (in the LO in $\hbar$).**

# Clifford-algebra expansion

<div style="border:1px solid;">

**Clifford-algebra expansion**
(expansion in terms of 16 independent generators of the Clifford algebra with real coefficients )
C. Itzykson, J. B. Zuber, Quantum Field Theory, McGraw-Hill, New York, 1980.

$$\mathcal{W}_{\text{eq}}^{\pm}(x,k) = \frac{1}{4}\left[\mathcal{F}_{\text{eq}}^{\pm}(x,k) + i\gamma_5\mathcal{P}_{\text{eq}}^{\pm}(x,k) + \gamma^{\mu}\mathcal{V}_{\text{eq},\mu}^{\pm}(x,k) + \gamma_5\gamma^{\mu}\mathcal{A}_{\text{eq},\mu}^{\pm}(x,k) + \Sigma^{\mu\nu}\mathcal{S}_{\text{eq},\mu\nu}^{\pm}(x,k)\right].$$

</div>

The coefficient functions $\mathcal{X} \in \{\mathcal{F}, \mathcal{P}, \mathcal{V}_{\mu}, \mathcal{A}_{\mu}, \mathcal{S}_{\nu\mu}\}$ in the expansion can be obtained by calculating the trace of $\mathcal{W}_{\text{eq}}^{\pm}(x,k)$ multiplied first by the matrices: $\mathbf{1}, -i\gamma_5, \gamma_{\mu}, \gamma_{\mu}\gamma_5$ and $2\Sigma_{\mu\nu}$.

D. Vasak, M. Gyulassy, H. T. Elze, Annals Phys. 173 (1987) 462–492
P. Zhuang, U. W. Heinz, Annals Phys. 245 (1996) 311–338.
W. Florkowski, J. Hufner, S. P. Klevansky, L. Neise, Annals Phys. 245 (1996) 445–463
...

# Global equilibrium Wigner function

**The Wigner function satisfies the kinetic equation**

$$\left(\gamma_\mu K^\mu - m\right)\mathcal{W}(x,k) = C[\mathcal{W}(x,k)] \qquad K^\mu = k^\mu + \frac{i\hbar}{2}\partial^\mu$$

In LE/GE the Wigner function $\mathcal{W}(x,k)$ exactly satisfies the equation

$$\left(\gamma_\mu K^\mu - m\right)\mathcal{W}(x,k) = 0$$

**Semi-classical expansion**
(expansion in powers of $\hbar$ of the coefficient functions of the Clifford algebra expansion)

$$\mathcal{X} = \mathcal{X}^{(0)} + \hbar\mathcal{X}^{(1)} + \hbar^2\mathcal{X}^{(2)} + \cdots . \qquad \mathcal{X} \in \{\mathcal{F}, \mathcal{P}, \mathcal{V}_\mu, \mathcal{A}_\mu, \mathcal{S}_{\nu\mu}\}$$

**LO in $\hbar$ :**

$\mathcal{F}_{(0)}$ and $\mathcal{A}^\mu_{(0)}$ may be treated as the only independent coefficients provided $k_\nu \mathcal{A}^\nu_{(0)}(x,k) = 0$

Algebraic relations imposed by the kinetic equation are satisfied by $\mathcal{W}^\pm_{\text{eq}}(x,k)$

$\mathcal{W}^\pm_{\text{eq}}(x,k)$ **can be identified with the LO terms in $\hbar$ of the "true" equilibrium Wigner functions $\mathcal{W}^\pm(x,k)$ which satisfy the quantum kinetic equation.**

**NLO in $\hbar$:**

Kinetic equations to be satisfied by the scalar and axial-vector coefficient are

$$k^\mu \partial_\mu \mathcal{F}_{(0)}(x,k) = 0 \qquad k^\mu \partial_\mu \mathcal{A}^\nu_{(0)}(x,k) = 0$$

# From global to local equilibrium

**GE**: the following equations are exactly fulfilled

$$k^\mu \partial_\mu \mathcal{F}_{\text{eq}}(x,k) = 0, \quad k^\mu \partial_\mu \mathcal{A}^\nu_{\text{eq}}(x,k) = 0, \quad k_\nu \mathcal{A}^\nu_{\text{eq}}(x,k) = 0.$$

$\beta_\mu$ **field has to be the Killing vector while the parameters $\xi$ and $\omega_{\mu\nu}$ are constant. The kinetic equations do not constrain the spin polarization tensor $\omega_{\mu\nu}$ to be equal to the thermal vorticity $\varpi_{\mu\nu}$.**

**LE**: one requires that only moments of the kinetic equations are satisfied by allowing for space–time dependence of $\beta_\mu$, $\xi$, and $\omega_{\mu\nu}$

G. S. Denicol, E. Molnr, H. Niemi and D. H. Rischke, Eur. Phys. J. A 48, 170 (2012)

G. S. Denicol, J. Phys. G 41, 124004 (2014)

..

$$\partial_\alpha N^\alpha_{\text{eq}}(x) = 0, \quad \partial_\alpha T^{\alpha\beta}_{\text{eq}}(x) = 0, \quad \partial_\lambda S^{\lambda,\mu\nu}_{\text{eq}}(x) = 0$$

$$\widehat{T}^{\beta\alpha}_{\text{eq}}(x) = \widehat{T}^{\alpha\beta}_{\text{eq}}(x) \qquad \Rightarrow \qquad \text{spin tensor is conserved}$$

**Adopting the kinetic-theory framework derived by de Groot, van Leeuwen, and van Weert, one can show that $T^{\mu\nu}_{\text{GLW}}(x) = T^{\mu\nu}_{\text{eq}}(x)$ and $S^{\lambda,\mu\nu}_{\text{GLW}} = S^{\lambda,\mu\nu}_{\text{eq}}$**

# NLO corrections in $\hbar$

**Since the spin tensor enters with an extra power of $\hbar$, it is important to examine the Wigner function up to the next-to-leading order (NLO)**

LO generates corrections in the NLO

$$\mathcal{P}^{(1)} = -\frac{1}{2m}\partial^\mu \mathcal{A}_{\text{eq},\mu}$$

$$\mathcal{V}_\mu^{(1)} = -\frac{1}{2m}\partial^\nu \mathcal{S}_{\text{eq},\nu\mu}$$

$$\mathcal{S}_{\mu\nu}^{(1)} = \frac{1}{2m}\left(\partial_\mu \mathcal{V}_{\text{eq},\nu} - \partial_\nu \mathcal{V}_{\text{eq},\mu}\right)$$

IMPORTANT IF the canonical formalism is used

$$T_{\text{GLW}}^{\mu\nu}(x) = \frac{1}{m}\text{tr}_4 \int d^4k\, k^\mu\, k^\nu \mathcal{W}(x,k) = \frac{1}{m}\int d^4k\, k^\mu\, k^\nu \mathcal{F}(x,k)$$

$$T_{\text{C}}^{\mu\nu}(x) = \int d^4k\, k^\nu \mathcal{V}^\mu(x,k)$$

quantum corrections induce asymmetry $T_{\text{C}}^{\mu\nu}(x) \neq T_{\text{C}}^{\nu\mu}(x)$

# From canonical to GLW case

Including the components of $\mathcal{V}^\mu(x,k)$ up to the first order in the equilibrium case we obtain

$$T_C^{\mu\nu}(x) = T_{GLW}^{\mu\nu}(x) + \delta T_C^{\mu\nu}(x)$$

where

$$\delta T_C^{\mu\nu}(x) = -\frac{\hbar}{2m} \int d^4k \, k^\nu \partial_\lambda \mathcal{S}_{eq}^{\lambda\mu}(x,k) = -\partial_\lambda S_{GLW}^{\nu,\lambda\mu}(x)$$

$$\partial_\alpha T_C^{\alpha\beta}(x) = 0$$

**The canonical energy-momentum tensor is conserved**

$$
\begin{aligned}
S_C^{\lambda,\mu\nu} &= \frac{\hbar}{4} \int d^4k \, \mathrm{tr}_4\left[\left\{\sigma^{\mu\nu}, \gamma^\lambda\right\} \mathcal{W}(x,k)\right] = \frac{\hbar}{2}\epsilon^{\kappa\lambda\mu\nu} \int d^4k \, \mathcal{A}_\kappa(x,k) \\
S_C^{\lambda,\mu\nu} &= S_{GLW}^{\lambda,\mu\nu} + S_{GLW}^{\mu,\nu\lambda} + S_{GLW}^{\nu,\lambda\mu}
\end{aligned}
$$

**The canonical spin tensor is not conserved!**

$$\partial_\lambda S_C^{\lambda,\mu\nu}(x) = T_C^{\nu\mu} - T_C^{\mu\nu} = \partial_\lambda S_{GLW}^{\nu,\lambda\mu}(x) - \partial_\lambda S_{GLW}^{\mu,\lambda\nu}(x)$$

# From canonical to GLW case

**The canonical and GLW frameworks are connected by a pseudo-gauge transformation. Similarly to Belinfante, it leads to a symmetric energy-momentum tensor, however, does not eliminate the spin degrees of freedom**

If we introduce the superpotential $\Phi_{\mathrm{C}}^{\lambda,\mu\nu}$ defined by the relation

$$\Phi_{\mathrm{C}}^{\lambda,\mu\nu} \equiv S_{\mathrm{GLW}}^{\mu,\lambda\nu} - S_{\mathrm{GLW}}^{\nu,\lambda\mu}$$

we can write

$$S_{\mathrm{C}}^{\lambda,\mu\nu} = S_{\mathrm{GLW}}^{\lambda,\mu\nu} - \Phi_{\mathrm{C}}^{\lambda,\mu\nu}$$

and

$$T_{\mathrm{C}}^{\mu\nu} = T_{\mathrm{GLW}}^{\mu\nu} + \frac{1}{2}\partial_\lambda \left( \Phi_{\mathrm{C}}^{\lambda,\mu\nu} + \Phi_{\mathrm{C}}^{\mu,\nu\lambda} + \Phi_{\mathrm{C}}^{\nu,\mu\lambda} \right)$$

## Conclusions and Summary

- At the moment the differential polarization data is a chalange for theory
- Fails of the current theories suggest that polarization evolves independently of the thermal vorticity and maybe finally converges to it
- GLW forms of the energy-momentum and spin tensors are natural candidates for formulating hydrodynamic equations (symmetric energy-momentum tensor; the spin tensor strictly conserved; follow from kinetic theory)
- Separate conservation of the spin tensor may be traced back to the locality of the collision term. Departure from the locality is necessary to include dissipative effects.
- Independent calculations within the classical approach to spin are leading to the same conclusions

**Thank you for your attention!**

Backup slides

# Hydrodynamics with spin based on entropy-current analysis

**Phenomenological derivation of hydrodynamics based on the conservation laws**
K. Hattori, M. Hongo, X-G Huang, M. Matsuo, H. Taya,
PLB 795 (2019) 100-106

$$\partial_\mu T^{\mu\nu} = 0 \qquad T^{\mu\nu} \equiv T^{\mu\nu}_{(s)} + T^{\mu\nu}_{(a)}$$
$$\partial_\mu J^{\mu\alpha\beta} = 0 \qquad J^{\mu\alpha\beta} = \left( x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} \right) + \Sigma^{\mu\alpha\beta}$$

Dynamical variables near local thermal equilibrium are assumed to satisfy the first law of thermodynamics generalized with finite spin density $S$. (spin potential $\omega$ is conjugate to the spin density $S$)

$$Ts = e + p - \omega_{\mu\nu}S^{\mu\nu}, \quad Tds = de - \omega_{\mu\nu}dS^{\mu\nu}$$

One may organize the constitutive relations on the basis of a derivative expansion $T^{\mu\nu}_{(1)} \sim O(\partial^1)$

$$T^{\mu\nu} = eu^\mu u^\nu + p\Delta^{\mu\nu} + T^{\mu\nu}_{(1)}$$
$$\Sigma^{\mu\alpha\beta} = u^\mu S^{\alpha\beta} + \Sigma^{\mu\alpha\beta}_{(1)}$$

Lowest-order hydrodynamic equations of motion do not conserve the lowest-order entropy current (in contrast to the case of a fluid without spin)

$$\partial_\mu s^\mu_{(0)} = 2\beta\omega_{\alpha\beta} T^{\alpha\beta}_{(1a)}$$

**The entropy production implies that spin density is inherently a dissipative quantity**
⇓
**no counterpart of ideal spin-less hydrodynamics ?**

**new transport coefficients appear that control the relaxation time of spin density (rotational viscosity, boost heat conductivity)**
⇓
**relativistic generalization of a non-relativistic micropolar hydrodynamics**
R. Takahashi, et al. Nat. Phys. 12(1) (2016) 52
M. Matsuo, Y. Ohnuma, S. Maekawa, PRB 96 (2017) 020401
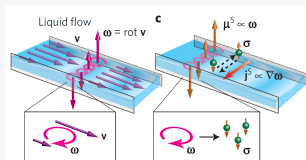


**1st order dissipative corrections**
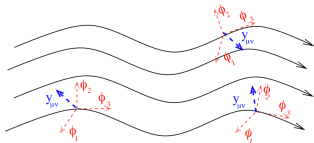⇓
**causality, stability?**

# Lagrangian formulation of relativistic fluid mechanics for spin-polarized systems

**One constructs lagrangian which contains the information of the EoS including an entropy term derived from the fluid coordinate d.o.f.** $b = \left( \det_{IJ} \left[ \partial_\mu \phi_I \partial^\mu \phi_J \right] \right)^{1/2}$ **as well as polarization tensor** $y^{\mu\nu}$

For small polarizations, the EoS reduces to

$$\mathcal{L} = F(b,y) = F\left( b\left( 1 - c y_{\mu\nu} y^{\mu\nu} \right) \right)$$

$c > 0$ - ferromagnet, $c < 0$ - antiferromagnet

**The ideal limit of hydrodynamics with polarization is generally non-causal (due to Ostrogradski theorem)**

$$y_{\mu\nu} = \chi\left( b, \Omega_{\mu\nu}\Omega^{\mu\nu} \right) \Omega_{\mu\nu}, \qquad \Omega_{\mu\nu} = \nabla_{[\mu} u_{\nu]}$$
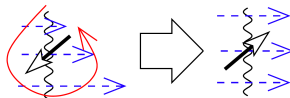
Causality can be cured by using Israel-Stewart type lagrangian, written in doubled coordinates and employing non-equilibrium polarization d.o.f. $Y$ giving Maxwell-Cataneo type relation

$$\tau_Y \partial_\tau \delta Y_{\mu\nu} + \delta Y_{\mu\nu} = y_{\mu\nu}$$

**causality can be fixed by a relaxation type term**
⇓
**any material with a non-zero spin must have a minimum amount of dissipation**

# Measurement of global spin polarization of $\Lambda$ hyperons



$\vec{L} \parallel \vec{B}$

Reaction Plane

particle

antiparticle

$L$ or $B$

## Parity-violating decay of hyperons

Daughter baryon is preferentially emitted in the direction of hyperon's spin (opposite for anti-particle)

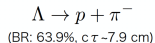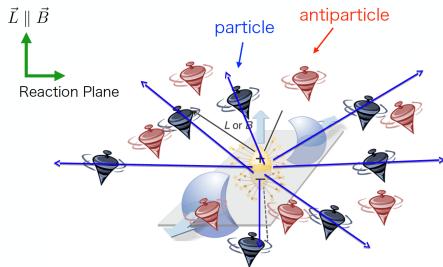$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha_H \mathbf{P}_H \cdot \mathbf{p_p^*})$$

$P_H$: $\Lambda$ polarization
$p_p^*$: proton momentum in the $\Lambda$ rest frame
$\alpha_H$: $\Lambda$ decay parameter
    ($\alpha_\Lambda = -\alpha_{\bar{\Lambda}} = 0.642 \pm 0.013$)

$\vec{S}_\Lambda^*$

$\theta^*$

$\vec{p}_p^*$

$\vec{p}_\pi^*$

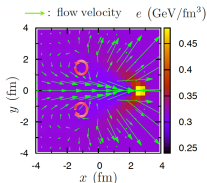$$\Lambda \rightarrow p + \pi^-$$

(BR: 63.9%, c$\tau$ ~7.9 cm)

C. Patrignani et al. (PDG), Chin. Phys. C 40, 100001 (2016)

Credit: T.Niida, The 5th Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions, 2019

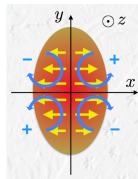# Theory failures - azimuthal angle dependence of $P_z$

Non-trivial flow structure in the transverse plane (jet, ebe fluctuations etc) generates longitudinal polarization
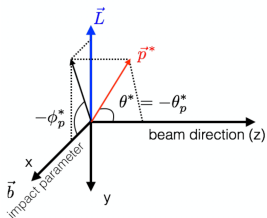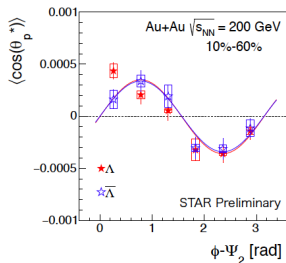


Y. Tachibana and T. Hirano, NPA904-905 (2013) 1023
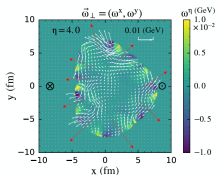
L.-G. Pang, H. Peterson, Q. Wang, and X.-N. Wang, PRL117, 192301 (2016)

F. Becattini and I. Karpenko, PRL120.012302 (2018)
S. Voloshin, EPJ Web Conf.171, 07002 (2018)



$$\frac{dN}{d\Omega^*} = \frac{1}{4\pi}(1 + \alpha_H \mathbf{P_H} \cdot \mathbf{p}_p^*)$$

$$\langle \cos\theta_p^* \rangle = \int \frac{dN}{d\Omega^*} \cos\theta_p^* d\Omega^*$$

$$= \alpha_H P_z \langle (\cos\theta_p^*)^2 \rangle$$

$$\therefore P_z = \frac{\langle \cos\theta_p^* \rangle}{\alpha_H \langle (\cos\theta_p^*)^2 \rangle}$$

$$= \frac{3\langle \cos\theta_p^* \rangle}{\alpha_H} \quad \text{(if perfect detector)}$$

$\alpha_H$: hyperon decay parameter
$\theta_p^*$: $\theta$ of daughter proton in $\Lambda$ rest frame

Credit: T.Niida, The 5th Workshop on Chirality, Vorticity and Magnetic Field in Heavy Ion Collisions, 2019

## 2. Spin polarization – standard QM treatment

Expansion in terms of Pauli matrices

$$f^{\pm}(x,p) = e^{\pm\xi - p\cdot\beta}\left[\cosh(\zeta) - \frac{\sinh(\zeta)}{2\zeta}\,\boldsymbol{P}\cdot\boldsymbol{\sigma}\right]$$

average polarization vector

$$\boldsymbol{\mathcal{P}} = \frac{1}{2}\frac{\mathrm{tr}_2\left[(f^+ + f^-)\boldsymbol{s}\right]}{\mathrm{tr}_2\left[f^+ + f^-\right]} = -\frac{1}{2}\tanh(\zeta)\frac{\boldsymbol{P}}{2\zeta}$$

$$\boldsymbol{\mathcal{P}} = -\frac{1}{2}\tanh\left[\frac{1}{2}\sqrt{\boldsymbol{b}_*\cdot\boldsymbol{b}_* - \boldsymbol{e}_*\cdot\boldsymbol{e}_*}\right]\frac{\boldsymbol{b}_*}{\sqrt{\boldsymbol{b}_*\cdot\boldsymbol{b}_* - \boldsymbol{e}_*\cdot\boldsymbol{e}_*}}$$

where the spin polarization is expressed by the matrix

$$\omega_{\mu\nu} = \begin{bmatrix} 0 & e^1 & e^2 & e^3 \\ -e^1 & 0 & -b^3 & b^2 \\ -e^2 & b^3 & 0 & -b^1 \\ -e^3 & -b^2 & b^1 & 0 \end{bmatrix}$$

∗ denotes the PARTICLE REST FRAME

# 3. Pauli-Lubański four-vector (phase-space density $\Pi_\mu(x, p)$)

$J^{\lambda, \nu\alpha}(x, p)$ is the phase-space density of the angular momentum of particles

$$E_p \frac{d\Delta\Pi_\mu(x, p)}{d^3p} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta} \Delta\Sigma_\lambda(x) E_p \frac{dJ^{\lambda, \nu\alpha}(x, p)}{d^3p} \frac{p^\beta}{m}$$

$$E_p \frac{dJ^{\lambda, \nu\alpha}(x, p)}{d^3p} = \frac{\kappa}{2} p^\lambda (x^\nu p^\alpha - x^\alpha p^\nu) \operatorname{tr}_4(X^+ + X^-) + \frac{\kappa}{2} p^\lambda \operatorname{tr}_4\left[(X^+ - X^-) \Sigma^{\nu\alpha}\right]$$

particle density in the volume $\Delta\Sigma$

$$E_p \frac{d\Delta\mathcal{N}}{d^3p} = \frac{\kappa}{2} \Delta\Sigma \cdot p \operatorname{tr}_4 (X^+ + X^-)$$

$$\pi_\mu(x, p) = \frac{\Delta\Pi_\mu(x, p)}{\Delta\mathcal{N}(x, p)}$$

By applying the Lorentz transformation we find that the PL four-vector calculated in the PRF agrees with the spin polarization (!)

$$\pi_*^0 = 0, \quad \boldsymbol{\pi}_* = \mathcal{P} = -\frac{1}{2}\tanh(\zeta)\bar{\boldsymbol{P}}$$

# Clifford-algebra expansion

The Wigner functions $\mathcal{W}_{\text{eq}}^{\pm}(x,k)$, can be always expanded in terms of the 16 independent generators of the Clifford algebra

D. Vasak, M. Gyulassy, and H. T. Elze, Annals Phys. 173 (1987) 462–492
C. Itzykson and J. B. Zuber, Quantum Field Theory. McGraw-Hill, New York, 1980

$$\mathcal{W}_{\text{eq}}^{\pm}(x,k) = \frac{1}{4}\left[\mathcal{F}_{\text{eq}}^{\pm}(x,k) + i\gamma_5\mathcal{P}_{\text{eq}}^{\pm}(x,k) + \gamma^{\mu}\mathcal{V}_{\text{eq},\mu}^{\pm}(x,k) + \gamma_5\gamma^{\mu}\mathcal{A}_{\text{eq},\mu}^{\pm}(x,k) + \Sigma^{\mu\nu}\mathcal{S}_{\text{eq},\mu\nu}^{\pm}(x,k)\right].$$

The coefficient functions in the equilibrium Wigner function expansion can be obtained by the respective traces:

$$
\begin{aligned}
\mathcal{F}_{\text{eq}}^{\pm}(x,k) &= \text{tr}\left[\mathcal{W}_{\text{eq}}^{\pm}(x,k)\right], \\
\mathcal{P}_{\text{eq}}^{\pm}(x,k) &= -i\,\text{tr}\left[\gamma^5\mathcal{W}_{\text{eq}}^{\pm}(x,k)\right], \\
\mathcal{V}_{\text{eq},\mu}^{\pm}(x,k) &= \text{tr}\left[\gamma_{\mu}\mathcal{W}_{\text{eq}}^{\pm}(x,k)\right] \\
\mathcal{A}_{\text{eq},\mu}^{\pm}(x,k) &= \text{tr}\left[\gamma_{\mu}\gamma^5\mathcal{W}_{\text{eq}}^{\pm}(x,k)\right], \\
\mathcal{S}_{\text{eq},\mu\nu}^{\pm}(x,k) &= 2\,\text{tr}\left[\Sigma^{\mu\nu}\mathcal{W}_{\text{eq}}^{\pm}(x,k)\right].
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
\mathcal{F}_{\text{eq}}^{\pm}(x,k) &= 2m\cosh(\zeta)\int dP\, e^{-\beta\cdot p\pm\xi}\,\delta^{(4)}(k\mp p), \\
\mathcal{P}_{\text{eq}}^{\pm}(x,k) &= 0, \\
\mathcal{V}_{\text{eq},\mu}^{\pm}(x,k) &= \pm 2\cosh(\zeta)\int dP\, e^{-\beta\cdot p\pm\xi}\,\delta^{(4)}(k\mp p)\,p_{\mu}, \\
\mathcal{A}_{\text{eq},\mu}^{\pm}(x,k) &= -\frac{\sinh(\zeta)}{\zeta}\int dP\, e^{-\beta\cdot p\pm\xi}\,\delta^{(4)}(k\mp p)\,\tilde{\omega}_{\mu\nu}\,p^{\nu}, \\
\mathcal{S}_{\text{eq},\mu\nu}^{\pm}(x,k) &= \pm\frac{\sinh(\zeta)}{m\zeta}\int dP\, e^{-\beta\cdot p\pm\xi}\,\delta^{(4)}(k\mp p)\left[\left(p_{\mu}\omega_{\nu\alpha} - p_{\nu}\omega_{\mu\alpha}\right)p^{\alpha} + \right.
\end{aligned}
$$

## Relations between equilibrium coefficient functions

One can verify that the equilibrium coefficient functions satisfy the following set of constraints:

$$k^\mu \, \mathcal{V}^\pm_{\text{eq},\mu}(x,k) = m \, \mathcal{F}^\pm_{\text{eq}}(x,k)$$

$$k_\mu \, \mathcal{F}^\pm_{\text{eq}}(x,k) = m \, \mathcal{V}^\pm_{\text{eq},\mu}(x,k)$$

$$\mathcal{P}^\pm_{\text{eq}}(x,k) = 0$$

$$k^\mu \, \mathcal{A}^\pm_{\text{eq},\,\mu}(x,k) = 0$$

$$k^\mu \, \mathcal{S}^\pm_{\text{eq},\,\mu\nu}(x,k) = 0$$

$$k^\beta \, \tilde{\mathcal{S}}^\pm_{\text{eq},\mu\beta}(x,k) + m \, \mathcal{A}^\pm_{\text{eq},\,\mu}(x,k) = 0$$

$$\epsilon_{\mu\nu\alpha\beta} \, k^\alpha \, \mathcal{A}^{\pm\,\beta}_{\text{eq}}(x,k) + m \, \mathcal{S}^\pm_{\text{eq},\,\mu\nu}(x,k) = 0$$

These hold for any form of the fields: $\beta_\mu(x)$, $\xi(x)$, and $\omega_{\mu\nu}(x)$.

## Semi-classical expansion

Lets now use a general form of the Wigner function

$$\mathcal{W}(x,k) = \frac{1}{4}\left[\mathcal{F}(x,k) + i\gamma_5\mathcal{P}(x,k) + \gamma^\mu\mathcal{V}_\mu(x,k) + \gamma_5\gamma^\mu\mathcal{A}_\mu(x,k) + \Sigma^{\mu\nu}\mathcal{S}_{\mu\nu}(x,k)\right]$$

In the case where the effects of both the mean fields and collisions can be neglected, the Wigner function satisfies the equation of the form

$$\left(\gamma_\mu K^\mu - m\right)\mathcal{W}(x,k) = 0 \qquad\qquad K^\mu = k^\mu + \frac{i\hbar}{2}\partial^\mu$$

Real and imaginary parts give

$$k^\mu\mathcal{V}_\mu - m\mathcal{F} = 0$$

$$\frac{\hbar}{2}\partial^\mu\mathcal{A}_\mu + m\mathcal{P} = 0$$

$$k_\mu\mathcal{F} - \frac{\hbar}{2}\partial^\nu\mathcal{S}_{\nu\mu} - m\mathcal{V}_\mu = 0$$

$$-\frac{\hbar}{2}\partial_\mu\mathcal{P} + k^\beta\tilde{\mathcal{S}}_{\mu\beta} + m\mathcal{A}_\mu = 0$$

$$\frac{\hbar}{2}\left(\partial_\mu\mathcal{V}_\nu - \partial_\nu\mathcal{V}_\mu\right) - \epsilon_{\mu\nu\alpha\beta}k^\alpha\mathcal{A}^\beta - m\mathcal{S}_{\mu\nu} = 0$$

$$\hbar\partial^\mu\mathcal{V}_\mu = 0$$

$$k^\mu\mathcal{A}_\mu = 0$$

$$\frac{\hbar}{2}\partial_\mu\mathcal{F} + k^\nu\mathcal{S}_{\nu\mu} = 0$$

$$k_\mu\mathcal{P} + \frac{\hbar}{2}\partial^\beta\tilde{\mathcal{S}}_{\mu\beta} = 0$$

$$\left(k_\mu\mathcal{V}_\nu - k_\nu\mathcal{V}_\mu\right) + \frac{\hbar}{2}\epsilon_{\mu\nu\alpha\beta}\partial^\alpha\mathcal{A}^\beta = 0$$

This suggest to search for solutions in the form

$$\mathcal{X} = \mathcal{X}^{(0)} + \hbar\mathcal{X}^{(1)} + \hbar^2\mathcal{X}^{(2)} + \cdots. \qquad\qquad \mathcal{X} \in \{\mathcal{F},\mathcal{P},\mathcal{V}_\mu,\mathcal{A}_\mu,\mathcal{S}_{\nu\mu}\}$$