The quantum Carnot engine and its quantum signature

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Quantum Thermodynamics

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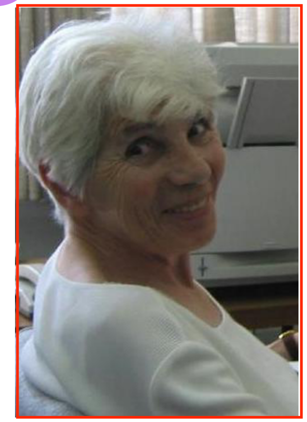

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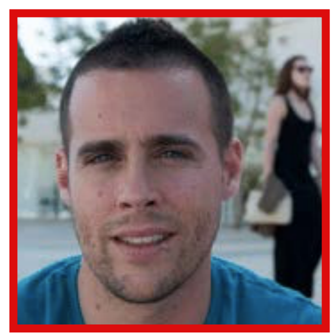

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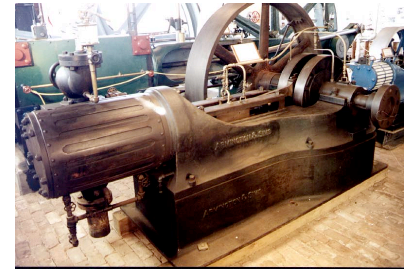
Quantum Thermodynamics



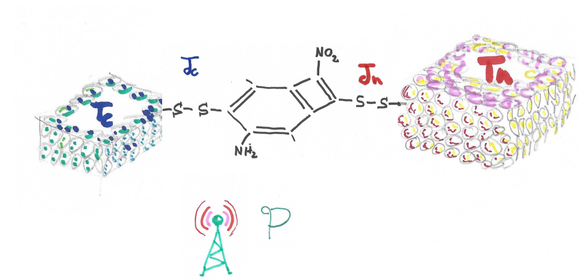
Consistency

Emergence of Thermodynamics from quantum mechanics

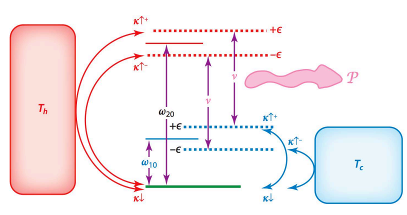
Learning from example | Thermodynamic ideals

Can a Thermodynamical viwpoint be relavant to a single device at the quantum limit ?

What is the limit of minaturization of a quantum heat engine ?

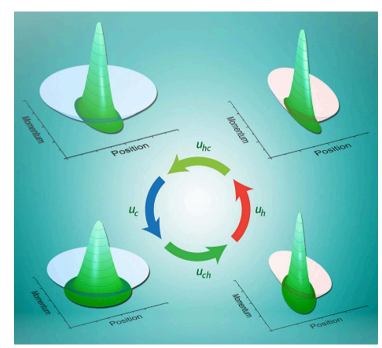


single molecular refrigerator



What is quantum in a quantum heat engine?

Is there quantum supremacy ?

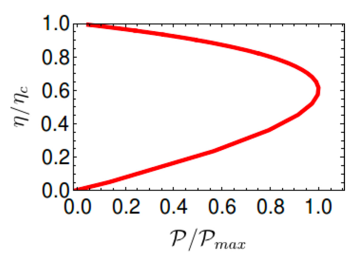


Is a small quantum engine usefull?

Inserting Dynamics into Thermodynamics

Power or efficiency?

$$\eta_{CA} = 1 - \sqrt{\frac{T_c}{T_h}}$$



$$\eta_c = 1 - \frac{T_c}{T_h}$$

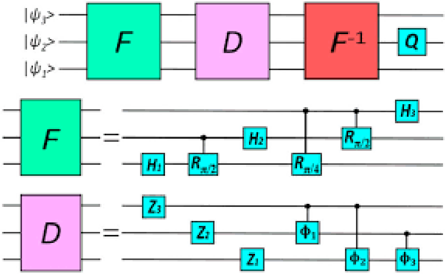
Efficiency at maximum power

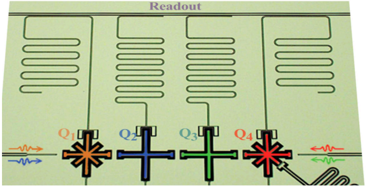
$$\Delta S^u > 0$$

Maximum efficiency

$$\Delta S^u = 0$$

Reciprocating heat engines





Learning from example

How small can an engine be?

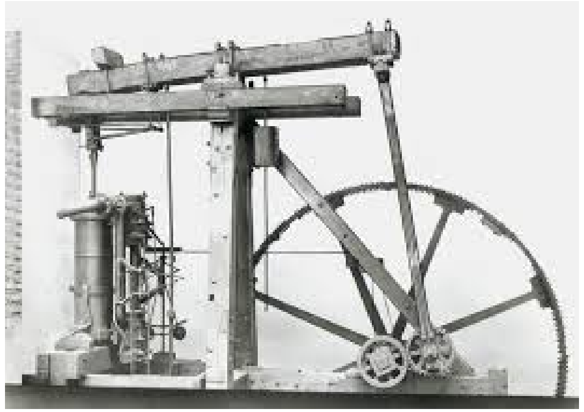
What is the role of coherence?

Coherent contol by interference of pahthways?

Carnot cycle

1. Hot to cold adiabatic stroke Λ_{hc}
2. Cold isotherm Λ_c
3. Cold to hot adiabatic stroke Λ_{ch}
4. Hot isotherm Λ_h

$$\eta_C = 1 - \frac{T_c}{T_h}$$

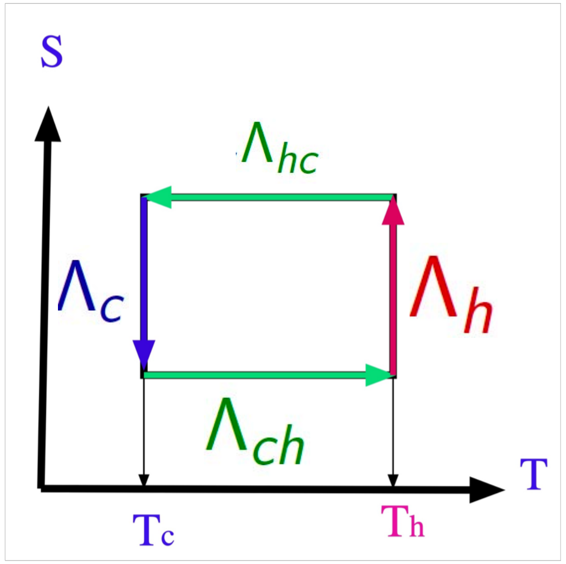


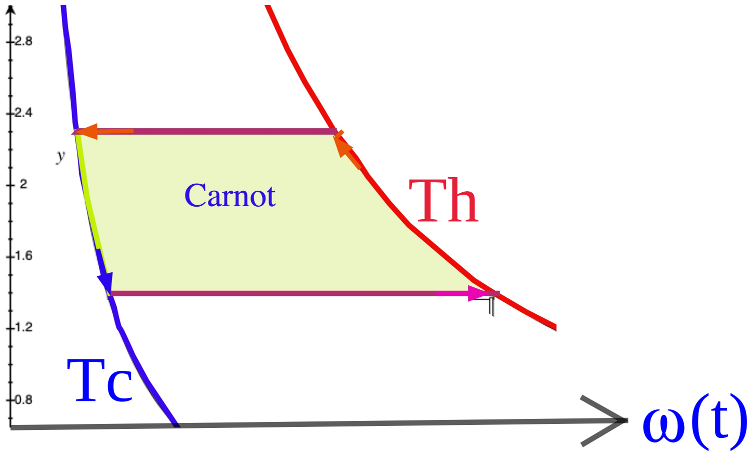
Operating conditions
fixed point of CPTP map

$$\Lambda_{cyc}\hat{\rho}_S = 1\hat{\rho}_S$$

Carnot cycle:

$$\Lambda_{cyc} = \Lambda_h \Lambda_{ch} \Lambda_c \Lambda_{hc}$$





Carnot cycle: The isotherms

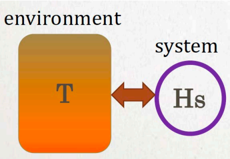
The Problem:

Derive a dynamical description for a driven system coupled to a bath beyond the adiabatic limit:

$$\hat{H} = \hat{H}_S(t) + \hat{H}_B + \hat{H}_{SB}$$

An open system quantum control problem:

State to state control:



$$\hat{\rho}_i \quad \rightarrow \quad \hat{\rho}_f$$

where we have control only on the system Hamiltonian $\hat{H}_S(t)$:

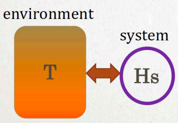
$$\hat{H} = \hat{H}_S(t) + \hat{H}_B + \hat{H}_{SB}$$

The system dynamics is governed by:

$$\frac{d}{dt}\hat{\rho}_S = \mathscr{L}_S \hat{\rho}_S$$

where $\mathscr{L}_S(t)$ depends on the bath implicitly and $\hat{H}_S(t)$.

The theory of open quantum systems



The quantum Markovian Master Equation .

A completely positive map: Kraus 1971


G. Lindblad

$$\Lambda \hat{\rho} = \sum_j \hat{W}_j^\dagger \hat{\rho} \hat{W}_j,$$

where $\sum_j \hat{W}_j^\dagger \hat{W}_j = \hat{I}$

The Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) quantum Master equation 1975

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \frac{1}{2}\sum_j ([\hat{V}_j \hat{\rho}, \hat{V}_j^\dagger] + [\hat{V}_j, \hat{\rho} \hat{V}_j^\dagger]) \equiv -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \mathscr{L}\hat{\rho} .$$

System and bath are in tensor product form in all times Lindblad 1996

Thermodynamical properties.

Davies construction: The weak coupling limit: Davies 1974

$$\hat{H} = \hat{H}_S + \hat{H}_B + \hat{H}_{int}$$

The system-bath interaction as $\hat{H}_{int} = \sum_k \hat{S}_k \otimes \hat{R}_k$
One obtains the following structure of MME which is in the GKLS form

$$\frac{d}{dt}\hat{\rho} = -i[\hat{H}, \hat{\rho}] + \mathscr{L}\hat{\rho}, \quad \mathscr{L}\hat{\rho} = \sum_{k,l} \sum_{\{\omega\}} \mathscr{L}^{\omega}_{lk}\hat{\rho}$$

where

$$\mathscr{L}^{\omega}_{lk}\hat{\rho} = \frac{1}{2\hbar^2}\tilde{R}_{kl}(\omega)\left\{[\hat{S}_l(\omega)\hat{\rho}, \hat{S}^{\dagger}_k(\omega)] + [\hat{S}_l(\omega), \hat{\rho}\hat{S}^{\dagger}_k(\omega)]\right\}.$$

Here, the operators $\hat{S}_k(\omega)$ originate from the Fourier decomposition

Thermodynamical properties.

ω- denotes the set of Bohr frequencies of $\hat{H}$.

$$e^{i/\hbar \hat{H} t} \hat{S}_k e^{-i/\hbar \hat{H} t} = \sum_{\{\omega\}} e^{-i\omega t} \hat{S}_k(\omega),$$

$\tilde{R}_{kl}(\omega)$ is the Fourier transform of the bath correlation function $\langle \hat{R}_k(t) \hat{R}_l \rangle_{bath}$ computed in the thermodynamic limit

$$\tilde{R}_{kl}(\omega) = \int_{-\infty}^{+\infty} e^{i\omega t} \langle \hat{R}_k(t) \hat{R}_l \rangle_{bath} dt.$$

The derivation of Davis makes sense for a generic stationary state of the bath and implies two properties:

1) the Hamiltonian part $[\hat{H}, \cdot]$ commutes with the dissipative part $\mathscr{L}$,

2) the diagonal (in $\hat{H}$-basis) matrix elements of $\hat{\rho}$ evolve independently of the off-diagonal ones according to the Pauli Master Equation with transition rates given by the Fermi Golden Rule.

No mixing of energy and coherence

Thermodynamical properties.

If additionaly the bath is a heat bath, i.e. an infinite system in a KMS state the additional relation implies that:

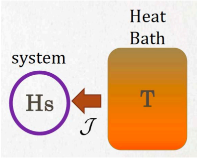
3) Gibbs state $\hat{\rho}_\beta = Z^{-1} \exp -\beta \hat{H}$ is a stationary solution.

4) Under the condition that only scalar operators commute with all $\{\hat{S}_k(\omega), \hat{S}_k^\dagger(\omega)\}$.

Any initial state relaxes asymptotically to the Gibbs state: *The 0-Law of Thermodynamics*.

The bath is able to **"measure"** the energy level structure of the system and transfer heat according to the detailed balance conditions

Quantum conditions of isothermal partition



Thermodynamical properties.

The derivation can be extended to slowly varying time-dependent Hamiltonian within the range of validity of the **adiabatic theorem**

and an open system coupled to several heat baths at the inverse temperatures $\{\beta_k = 1/k_B T_k\}$.

The MME in Heisenberg form:

$$\frac{d}{dt}\hat{Y}(t) = -i[\hat{H}(t), \hat{Y}(t)] + \mathscr{L}^*(t)\hat{Y}(t) + \frac{\partial}{\partial t}\hat{Y}$$

$$, \quad \mathscr{L}^*(t) = \sum_k \mathscr{L}^*{}_k(t).$$

Each $\mathscr{L}_k(t)$ is derived using a temporal Hamiltonian $\hat{H}(t)$, $\mathscr{L}_k(t)\hat{\rho}_j(t) = 0$ with a temporary Gibbs state $\hat{\rho}_j(t) = Z_j^{-1}(t)\exp\{-\beta_j \hat{H}(t)\}$.

Inserting Dynamics into Thermodynamics

Dynamical I-law of thermodynamics

The Heisenberg equations of motion:

$$\frac{d}{dt}\hat{\mathbf{X}} = \frac{i}{\hbar}[\hat{\mathbf{H}}, \hat{\mathbf{X}}] + \mathcal{L}_D(\hat{\mathbf{X}}) + \frac{\partial}{\partial t}\hat{\mathbf{X}}$$

$$\mathcal{L}_D(\hat{\mathbf{X}}) = \sum_n \hat{\mathbf{V}}_n \hat{\mathbf{X}} \hat{\mathbf{V}}_n^* - \frac{1}{2}\{\hat{\mathbf{V}}_n \hat{\mathbf{V}}_n^*, \hat{\mathbf{X}}\}$$

If we choose $\hat{\mathbf{X}} = \hat{\mathbf{H}}$ then:

$$[\mathcal{L}_H, \mathcal{L}_D] = 0$$

Adiabatic limit

$$\frac{d}{dt}\mathbf{E} = \langle \frac{\partial}{\partial t}\hat{\mathbf{H}} \rangle + \langle \mathcal{L}_D(\hat{\mathbf{H}}) \rangle$$

$$\frac{d}{dt}\mathbf{E} = \mathcal{P} + \dot{\mathbf{Q}}$$

Power + Heat current

R. Alicki , J.phys. A Math. Gen. 12 L103 (1979)

Spohn, Herbert, and Joel L. Lebowitz. Adv. Chem. Phys 38 (1978): 109-142

Carnot cycle: The isotherms

$$\left[\hat{H}_S(t), \hat{H}_S(t')\right] \neq 0$$

The task: Isothermal Dynamics

Starting from a thermal initial state $\hat{\rho}_i = e^{-\beta \hat{H}_i}$

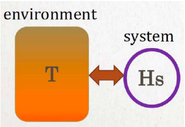
Transform as fast and accurate to the state: $\hat{\rho}_f = e^{-\beta \hat{H}_f}$

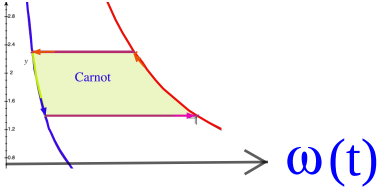
while the system is in contact with a bath of temperature $T = 1/k\beta$

The protocol: $\hat{H}_S(t)$ with $\hat{H}_S(0) = \hat{H}_i$ and $\hat{H}_S(t_f) = \hat{H}_f$

The Problem

We can control directly $\hat{H}_S(t)$ but only indirectly the relaxation rate.
We need the dissipative equation of motion with a time dependent $\hat{H}_S(t)$ with a time dependent protocol.





How can we obtain the Master equation?

Analytic tools:

$$\hat{H} = \hat{H}_S(t) + \hat{H}_B + \hat{H}_{SB}$$

Non-Adiabatic Master Equation (NAME)

Inertial theorem

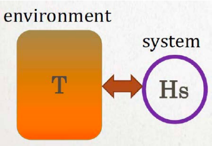
Non-adiabatic open system dynamics

Time-dependent Markovian Master Eq., R. Dann, A. Levy, and R. Kosloff, *Phys. Rev. A* 98, 052129 (2018).
The Inertial Theorem, R. Dann and R. Kosloff, *arXiv*:1810.12094 (2018).

NAME - The driven Non-Adiabatic Master Equation

.

$$\frac{d}{dt}\hat{\rho}_S(t) = \mathscr{L}_s\hat{\rho}_S$$



We change $\hat{\mathbf{H}}_{\mathbf{S}}(t)$ from $\hat{\mathbf{H}}_{\mathbf{S}}(0)$ to $\hat{\mathbf{H}}_{\mathbf{S}}(t_f)$ while coupled to the bath.

Then we expect:

$$\frac{d}{dt}\hat{\rho}_S(t) = -i[\hat{\mathbf{H}}_{\mathbf{S}}(t) + \hat{\mathbf{H}}_{\mathbf{LS}}(t), \hat{\rho}_S] + \sum_k c_k(t)\left(\hat{L}_k(t)\hat{\rho}_S\hat{L}_k^\dagger(t) - \frac{1}{2}\{\hat{L}_k^\dagger(t)\hat{L}_k(t), \rho_S\}\right).$$

$\hat{L}_k$ are the Lindblad jump operators.

Here $\hat{H}_{LS}(t)$ is the time dependent Lamb shift Hamiltonian, $\hat{H}_{LS}(t) = \sum_k S_{kk'}(\alpha(t)\hat{F}_j^\dagger(t)\hat{F}_j(t))$.

The Non-Adiabatic Master Equation (NAME)

$$\hat{\mathbf{H}}(t) = \hat{\mathbf{H}}_S(t) + \hat{\mathbf{H}}_B + \hat{\mathbf{H}}_{SB} \quad . \quad \hat{\mathbf{H}}_{SB} = \sum_k g_k \hat{\mathbf{A}}_k \otimes \hat{\mathbf{B}}_k \quad .$$

Following Davies's derivation, **Consistancy with thermodynamics**

1) Transformation to the interaction picture:

$$\hat{\mathbf{U}}_B{}^\dagger(t,0)\hat{\mathbf{U}}_S^\dagger \hat{\mathbf{H}}(t)(t,0)\hat{\mathbf{U}}_S(t,0)\hat{\mathbf{U}}_B(t,0) = \tilde{\hat{\mathbf{H}}}_{SB}(t) \quad ,$$

Where the system evolution operator

$$i\frac{d}{dt}\hat{\mathbf{U}}_S(t) = \hat{\mathbf{H}}_S(t)\hat{\mathbf{U}}_S(t) \quad , \quad \hat{\mathbf{U}}_S(0) = \hat{\mathbf{I}} \quad .$$

.

2) Second order perturbation theory lead to the Markovian quantum master equation

$$\frac{d}{dt}\tilde{\rho}_S(t) = -\int_0^\infty ds \; \mathrm{tr}_B\{\tilde{\mathbf{H}}_{SB}(t), [\tilde{\mathbf{H}}_{SB}(t-s), [\tilde{\rho}_S(t) \otimes \tilde{\rho}_B]]\} \quad .$$

The Lindblad Jump operators:

For free evolution of a static Hamiltonian:

The propagator in Heisenberg form is:

$$\mathscr{U}(t) = e^{\frac{it}{\hbar}[\hat{\mathbf{H}}_{\mathbf{S}}, \bullet]}$$

The eigenoperators of $\mathscr{U}(t)$ are:

Excitations

$$\mathscr{U}(t)|m\rangle\langle n| = e^{i\omega_{nm}t}|m\rangle\langle n|$$

where $\hat{\mathbf{H}}_{\mathbf{S}}|n\rangle = \varepsilon_n|n\rangle$ and $\omega_{nm} = \frac{1}{\hbar}(\varepsilon_n - \varepsilon_m)$ is the Bohr frequency.

.

The $\hat{\mathbf{H}}_{SB}$ can be expanded: with $\hat{\mathbf{F}}_k = |m\rangle\langle n|$

$$\tilde{\mathbf{H}}_{\mathbf{SB}} = \sum_k g_k e^{i\omega_k t}\hat{\mathbf{F}}_k \otimes \tilde{\mathbf{B}}_{\mathbf{k}}$$

leading to $\hat{\mathbf{F}}_k \equiv \hat{\mathbf{L}}_k$ the Lindblad jump operators.

The Lindblad Jump operators:

For a driven evolution: $\hat{\mathbf{H}}_S(t)$

We choose a time dependent operator base: $\{\hat{\mathbf{X}}_j(t)\}$

The propagator in Heisenberg form is:

$$\mathscr{U}(t) = \mathscr{T} e^{\frac{i}{\hbar}\int_0^t ([\hat{\mathbf{H_S}}(\mathbf{t}'), \bullet] + \frac{\partial}{\partial t'})dt'}$$

We find eigenoperators of $\mathscr{U}(t)$:

$$\mathscr{U}(t)\hat{\mathbf{F}}_k = e^{i\theta_k(t)}\hat{\mathbf{F}}_k \quad , \quad \hat{\mathbf{U}}^\dagger(t)\hat{\mathbf{F}}_k\hat{\mathbf{U}}(t) = e^{i\theta_k(t)}\hat{\mathbf{F}}_k$$

where $\hat{\mathbf{F}}_k$ are time independent.

.

$\hat{\mathbf{H}}_{SB}$ can be expanded in interaction frame with $\hat{\mathbf{F}}_k$

$$\tilde{\mathbf{H}}_{\mathbf{SB}} = \sum_k g_k e^{i\theta_k(t)}\hat{\mathbf{F}}_k \otimes \tilde{\mathbf{B}}_{\mathbf{k}}$$

leading to $\hat{\mathbf{F}}_k \equiv \hat{\mathbf{L}}_k$ the Lindblad jump operators.

Non Adiabatic Master Equation (NAME)

$$\tau_S = \left(\frac{1}{\omega_i(t)}\right) \qquad \tau_B \sim \frac{1}{\Delta\nu} \qquad \tau_R \propto \left(g^2\right)^{-1} \qquad \tau_d$$

1. Weak coupling
2. Born- Markov approximation $\tilde{\rho}(t) = \tilde{\rho}_S(t) \otimes \tilde{\rho}_B$
3. Fast bath dynamics relative to the external driving

1. $\tau_B \ll \tau_R$ 2. $\tau_B \ll \tau_S$ 3. $\tau_B \ll \tau_d$

$$\frac{d}{dt}\tilde{\rho}_S(t) = -i\left[\tilde{H}_{LS}(t), \tilde{\rho}_S(t)\right]$$
$$+\sum_{k,j}\left(\xi_j^k(t)\right)^2 g_k^2 \gamma_{kk}\left(\alpha_j^k(t)\right)\left(\hat{F}_j \tilde{\rho}_S(t)\hat{F}_j^\dagger - \frac{1}{2}\{\hat{F}_j^\dagger \hat{F}_j, \tilde{\rho}_S(t)\}\right)$$

$$\hat{F}_j \equiv \hat{F}_j(0)$$

Lamb-shift $\tilde{H}_{LS}(t) = \sum_{k,j} \hbar S_{kk}\left(\alpha_j^k(t)\right)\hat{F}_j^\dagger \hat{F}_j$

R. Dann, A. Levy, and R. Kosloff, *Phys. Rev. A 98*, 052129 (2018).

Example Parametric harmonic oscillator

$$\hat{\mathbf{H}}(\mathbf{t}) = \frac{\hat{\mathbf{p}}^2}{2\mathbf{m}} + \frac{1}{2}\mathbf{m}\omega^2(\mathbf{t})\hat{\mathbf{q}}^2 \quad , \tag{8}$$

The set
$\hat{\mathbf{H}}(\mathbf{t})$, $\hat{\mathbf{L}}(\mathbf{t}) = \frac{\hat{\mathbf{p}}^2}{2\mathbf{m}} - \frac{1}{2}\omega^2(\mathbf{t})\hat{\mathbf{q}}^2$, $\hat{\mathbf{C}}(\mathbf{t}) = \frac{\omega(\mathbf{t})}{2}(\hat{\mathbf{q}}\hat{\mathbf{p}} + \hat{\mathbf{p}}\hat{\mathbf{q}})$, $\hat{\mathbf{K}}(\mathbf{t}) = \sqrt{\omega(\mathbf{t})}\hat{\mathbf{q}}$, $\hat{\mathbf{J}}(\mathbf{t}) = \frac{\hat{\mathbf{p}}}{\mathbf{m}\sqrt{\omega(\mathbf{t})}}$

is a Lie algebra
The **free dynamics** in terms of the vector $\vec{v} = \{\hat{\mathbf{H}}, \hat{\mathbf{L}}, \hat{\mathbf{C}}, \hat{\mathbf{K}}, \hat{\mathbf{J}}, \hat{I}\}^T$

$$\frac{d}{d\theta}\vec{v}(\theta) = -i\mathscr{B}\vec{v}(\theta) \tag{9}$$

with,

$$\mathscr{B} = i\begin{bmatrix} \chi & -\chi & 0 & 0 & 0 & 0 \\ -\chi & \chi & -2 & 0 & 0 & 0 \\ 0 & 2 & \chi & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\chi}{2} & 1 & 0 \\ 0 & 0 & 0 & -1 & -\frac{\chi}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} . \tag{10}$$

Here, $\chi = \mu = \frac{\dot{\omega}}{\omega^2}$, where μ is the adiabatic parameter, $\theta = \int_0^t dt'\,\omega(t)'$.

The Non-Adiabatic Master Equation (NAME) for the Harmonic oscillator

(in the interaction picture)

$$\frac{d}{dt}\tilde{\rho}_S(t) = -i\left[\tilde{H}_{LS}(t), \tilde{\rho}_S\right] +$$
$$k\uparrow(t)\left(\tilde{b}^\dagger\tilde{\rho}_S\tilde{b} - \frac{1}{2}\left\{\tilde{b}\tilde{b}_\dagger, \tilde{\rho}_S\right\}\right) + k\downarrow(t)\left(\tilde{b}\tilde{\rho}_S\hat{b}^\dagger - \frac{1}{2}\left\{\tilde{b}^\dagger\tilde{b}, \tilde{\rho}_S\right\}\right),$$

$$\tilde{b} \equiv \hat{b}(0) = \sqrt{\frac{m\omega(0)}{2\hbar}}\frac{(\kappa + i\mu)}{\kappa}\left(\hat{Q} + \frac{\mu + i\kappa}{2m\omega(0)}\hat{P}\right)$$

$$[\tilde{b}, \tilde{b}^\dagger] = 1$$

$$\mu = \frac{\dot{\omega}}{\omega^2} \quad , \kappa = \sqrt{4 - \mu^2}. \quad \alpha(t) = \frac{\kappa}{2}\omega(t)$$

$$\frac{k\uparrow(t)}{k\downarrow(t)} = e^{-\frac{\hbar\alpha(t)}{k_B T}}$$

$$\gamma_{na} = k\downarrow(\alpha(t)) = \pi m\alpha(t)J(\alpha(t))(N(\alpha(t)) + 1)$$

Comparing adiabatic to nonadiabatic rates

Nonadiabatic rate:

$$\gamma_{na}(\alpha(t)) = \pi m \frac{\kappa}{2}\omega(t) J(\frac{\kappa}{2}\omega(t))(N(\frac{\kappa}{2}\omega(t)) + 1)$$

$$\kappa = \sqrt{4 - \mu^2}$$

.

Adiabatic rare:

$$\gamma_{ad}(\omega(t)) = \pi m \omega(t) J(\omega(t))(N(\omega(t)) + 1)$$

for $J(\omega) \propto \omega^2$ and low temperature:

$$\gamma_{na}(\alpha(t)) = \frac{1}{8}\kappa^3 \gamma_{ad}(\omega(t))$$

Slowing down the relaxation rate

Comparing non-adiabatic to adiabatic equation

Adiabatic: Lidar 2012

Both equations have time dependent Lindblad form. This guarantee's complete positivity and consistency with thermodynamics

- Doppler like change in frequency:

$$\kappa = \sqrt{4-\mu^2} \quad , \quad \alpha(t) = \frac{\kappa}{2}\omega(t)$$

  slowing down the relaxation rate and changing the instantaneous target of relaxation.

- Mixing Energy and coherence: generating squeezing.

Instantaneous attractor

How to solve the free dynamics with driving ?

Inertial Theorem

Non-adiabatic driving → $\left[\hat{H}_S(t), \hat{H}_S(t')\right] \neq 0$

→ Time-ordering problem → We want to solve this!

The inertial theorem approximates the evolution of a quantum system, driven by an external field. The theorem is valid for fast driving provided the acceleration rate is small.

Liouville space representation: Elements $\{\hat{X}\}$ with inner product $\left(\hat{X}_i, \hat{X}_j\right) \equiv \mathrm{tr}\left(\hat{X}_i^{\dagger}\hat{X}_j\right)$

Operator basis: $\vec{v}(t) = \{\hat{X}_1(t), ..., \hat{X}_N(t)\}$

The Inertial Theorem, R. Dann and R. Kosloff, *arXiv*:1810.12094 (2018).

The Inertial theorem.

For a closed Lie algebra $[\hat{A}_i, \hat{A}_j] = \sum_k c_{ij}^k A_k$
the Heisenberg equation of motion, for the set $\{\hat{A}\} = \vec{v}$ are

$$\frac{d}{dt}\vec{v}(t) = \left(i\left[\hat{H}(t), \bullet\right] + \frac{\partial}{\partial t} \right) \vec{v}(t) \quad , \tag{1}$$

In a vector notation (1) becomes

$$\frac{d}{dt}\vec{v}(t) = -i\mathscr{M}(t)\vec{v}(t) \quad , \tag{2}$$

where $\mathscr{M}$ is a N by N matrix with time-dependent elements and $\vec{v}$ is a vector of size N.

If we can factor:

$$\mathscr{M}(t) = \Omega(t)\mathscr{B}(\vec{\chi}) \quad . \tag{3}$$

Here, $\Omega(t)$ is a time-dependent real function, and $\mathscr{B}(\vec{\chi})$ is a function of the constant parameters $\{\chi\}$.

For this decomposition, the dynamics becomes

$$\frac{d}{d\theta}\vec{v}(\theta) = -i\mathscr{B}(\vec{\chi})\vec{v}(\theta) \quad , \qquad (4)$$

$\theta \equiv \theta(t) = \int_0^t dt' \Omega(t')$ is scaled time.

The solution

$$\vec{v}(\theta) = \sum_k^N c_k \vec{F}_k(\vec{\chi}) e^{-i\lambda_k \theta} \quad , \qquad (5)$$

where $\vec{F}_k$ and λ_k are eigenvectors and eigenvalues of $\mathscr{B}$ and c_k are constant coefficients. Each eigenvector $\vec{F}_k$ corresponds to the eigenoperator $\hat{F}_k$.

Inertial Theorem

$$\mathcal{M}(t) = \Omega(t)\,\mathcal{B}(\vec{\chi})$$

$$\theta(t) = \int_0^t \Omega(t')\,dt'$$

Inertial solution

$$\vec{v}(\chi,\theta) = \sum_k c_k e^{-i\int_{\theta_0}^{\theta} d\theta' \lambda_k} e^{i\phi_k} \vec{F}_k(\vec{\chi}(\theta))$$

Geometric phase

$$\phi_k(\theta) = i\int_{\vec{\chi}(\theta_0)}^{\vec{\chi}(\theta)} d\vec{\chi}\left(\vec{G}_k|\nabla_{\vec{\chi}}\vec{F}_k\right)$$

Inertial parameter

$$\Upsilon = \sum_{n,m} \left| \frac{\left(\vec{G}_k|\nabla_{\vec{\chi}}\mathcal{B}|\vec{F}_n\right)}{(\lambda_n-\lambda_k)^2} \cdot \frac{d\vec{\chi}}{d\theta} \right|$$

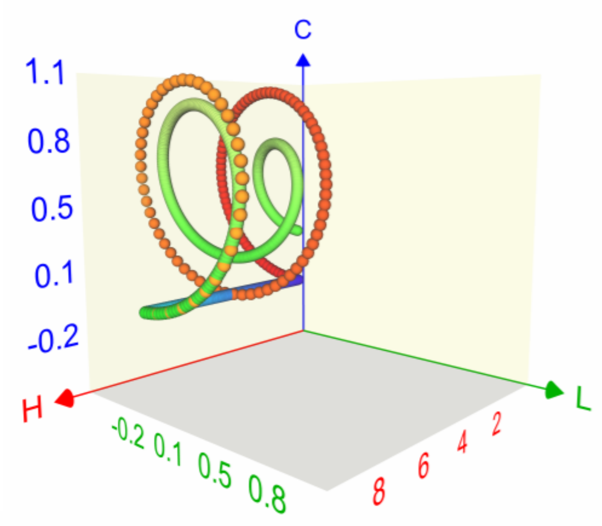
$\mathscr{B}(\vec{\chi})$ **can very slowly in time**

Inertial condition $\Upsilon \ll 1$

R. Dann and R. Kosloff, arXiv preprint arXiv:1810.12094 (2018).

Protocol: $\mu(t) = \mu(0) + a \cdot t$ $a = -5 \cdot 10^{-3}$

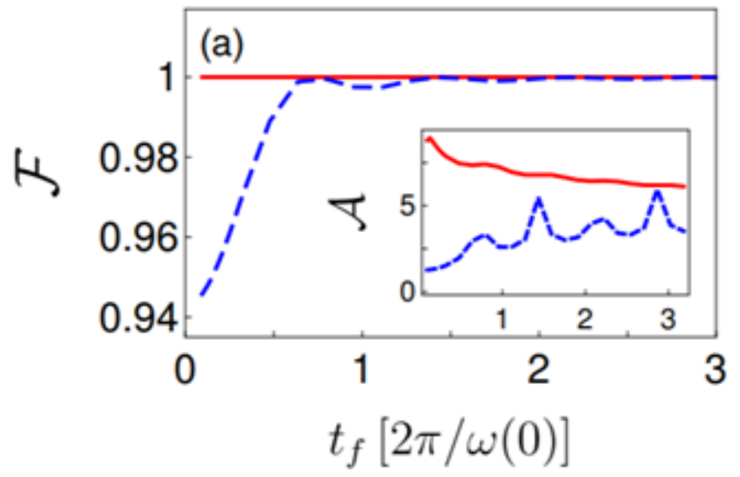
Illustration

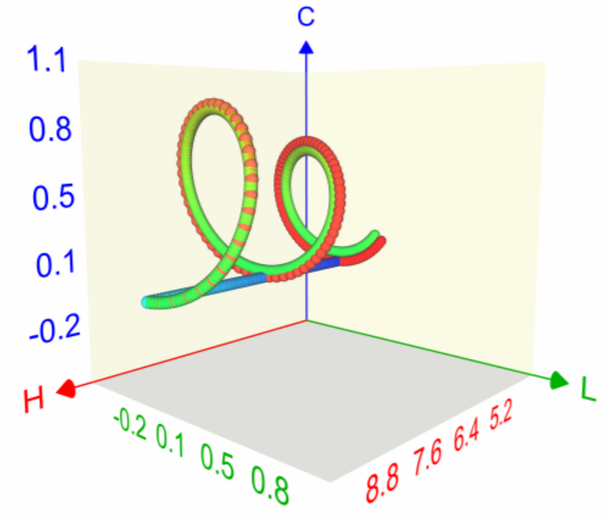


$$\hat{H}_S = \frac{\hat{P}^2}{2m} + \frac{1}{2} m \omega^2(t) \hat{Q}^2$$

$$\vec{\chi} = \mu = \frac{\dot{\omega}}{\omega^2}$$

$\omega(t_f) = 10$ $\omega(0) = 20$

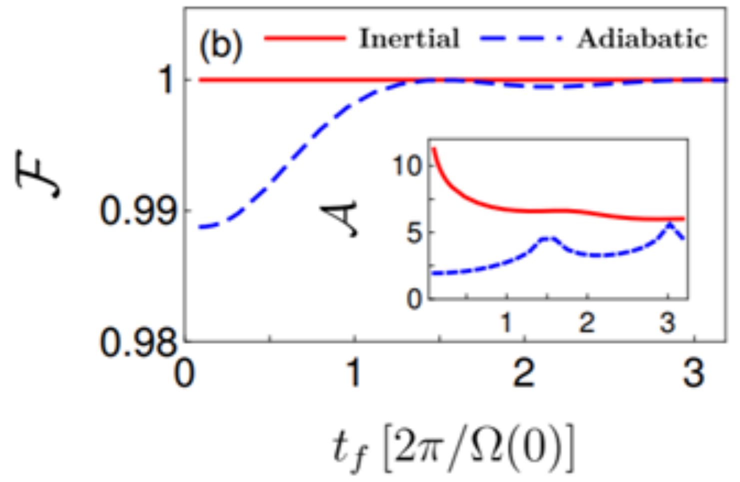




$$\hat{H}_S = \frac{1}{2}\left(\omega(t)\hat{\sigma}_z + \epsilon(t)\hat{\sigma}_x\right)$$

$$\vec{\chi} = \bar{\mu} = \frac{\dot{\omega}\varepsilon - \omega\dot{\varepsilon}}{\Omega^3}$$

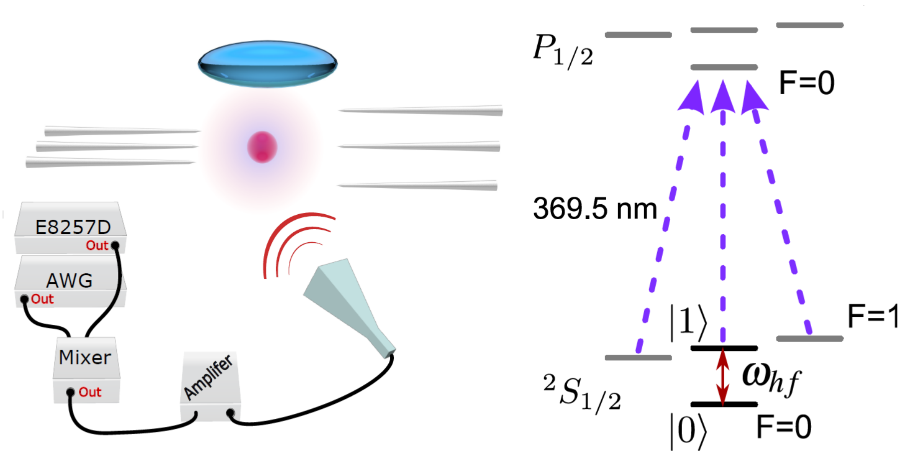
$\Omega(0) = 20$ $\Omega(t_f) = 10$

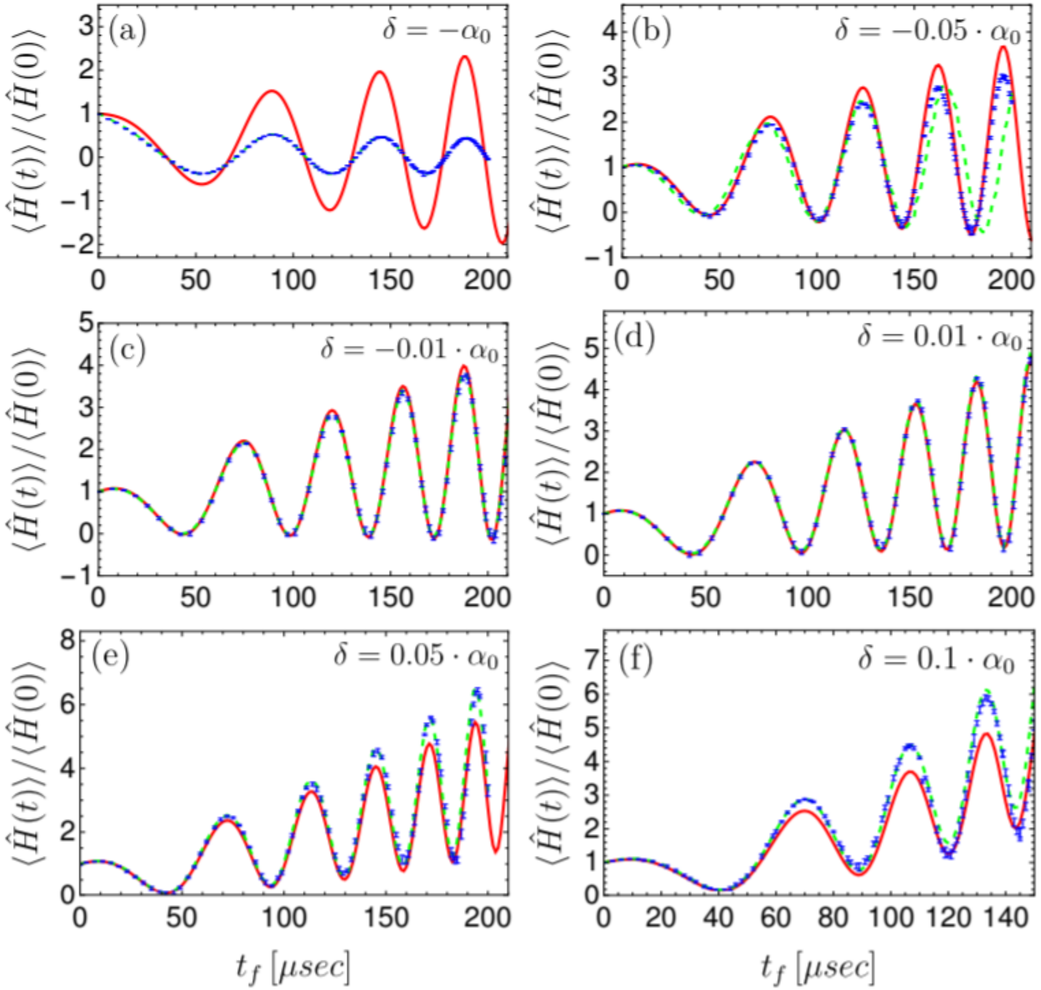


R. Dann and R. Kosloff, arXiv preprint arXiv:1810.12094 (2018).

Experimental verification of the Inertial Theorem

$$\hat{H}(t) = \frac{1}{2}\left(\omega(t)\hat{\sigma}_z + \varepsilon(t)\hat{\sigma}_x\right)$$





Inertial solution
Experimental
Numerical

$$\vec{\chi} = \bar{\mu} = \frac{\dot{\omega}\varepsilon - \omega\dot{\varepsilon}}{\Omega^3}$$

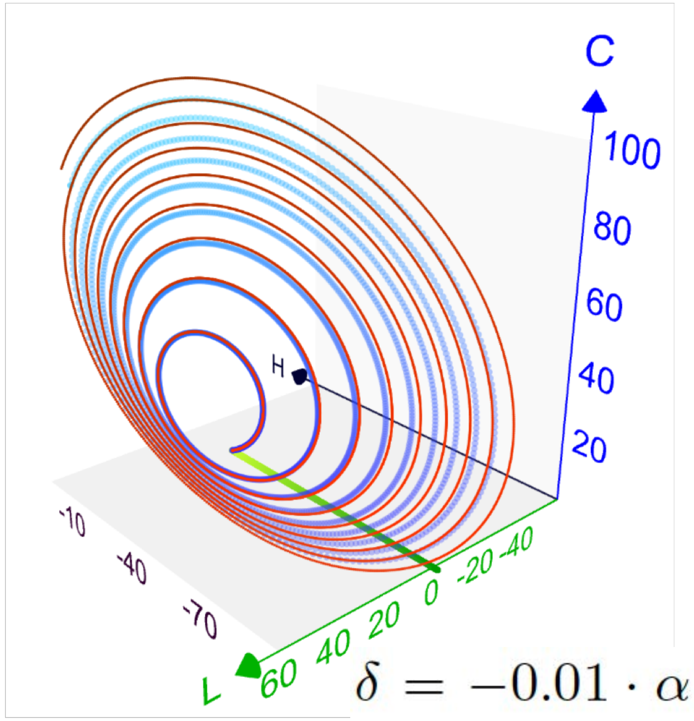
$$\mu(t) = \mu(0) + \delta \cdot t$$

$$\omega(t) = -\frac{(\alpha(0)+2\gamma\cdot t)}{\mu(0)+\delta\cdot t} \cdot \cos\left((\alpha(0)+\gamma t)\cdot t\right)$$

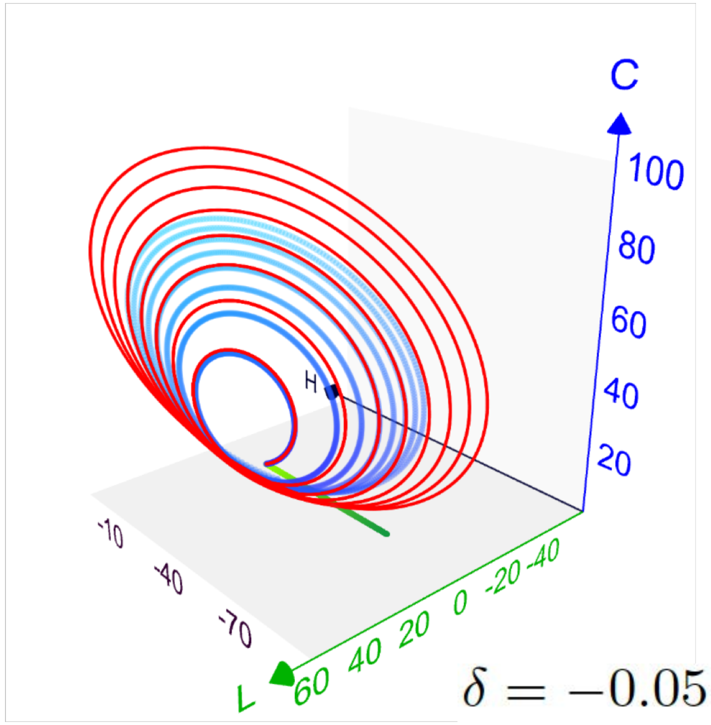
$$\varepsilon(t) = -\frac{(\alpha(0)+2\gamma\cdot t)}{\mu(0)+\delta\cdot t} \cdot \sin\left((\alpha(0)+\gamma t)\cdot t\right)$$

The Inertial Theorem, R. Dann and R. Kosloff, *arXiv*:1810.12094 (2018).
Experimental Verification of the Inertial Theorem, C.K.Hu, R.Dann, *et al.* , *arXiv*:1903.00404

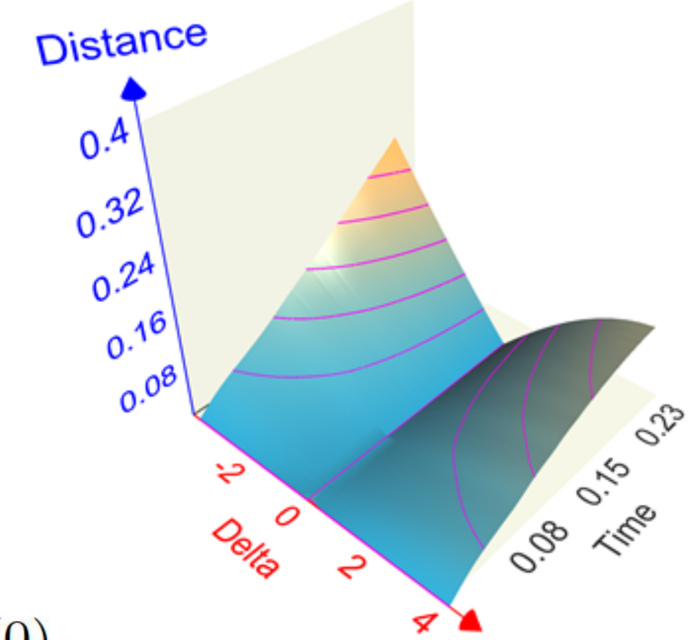
Experimental verification of the Inertial Theorem



$\delta = -0.01 \cdot \alpha(0)$



$\delta = -0.05 \cdot \alpha(0).$

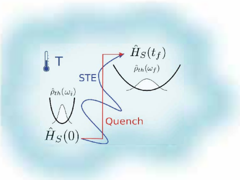


$$\hat{\mathbf{H}}(t) = \frac{1}{2}(\omega(t)\hat{\sigma}_z + \varepsilon(t)\hat{\sigma}_x) \qquad \hat{\mathbf{L}}(t) = \frac{1}{2}(\varepsilon(t)\hat{\sigma}_z - \omega(t)\hat{\sigma}_x) \qquad \hat{\mathbf{C}}(t) = \frac{1}{2}\Omega(t)\hat{\sigma}_y$$

Experimental Verification of the Inertial Theorem, C.K.Hu, R.Dann, *et al.* , *arXiv*:1903.00404

Shortcut to Equilibration (STE)

Carnot cycle: The isotherms



The task: Isothermal Dynamics

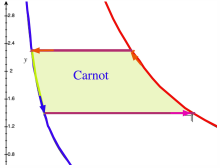
Starting from a thermal initial state $\hat{\rho}_i = e^{-\beta \hat{H}_i}$

Transform as fast and accurate to the state: $\hat{\rho}_f = e^{-\beta \hat{H}_f}$

while the system is in contact with a bath of temperature $T = 1/k\beta$

The protocol: $\hat{H}_S(t)$ with $\hat{H}_S(0) = \hat{H}_i$ and $\hat{H}_S(t_f) = \hat{H}_f$

Entropy change



Shortcut to Equilibration of an Open Quantum System,
R. Dann, A. Tobalina, and R. Kosloff, *PRL* 122, 250402 (2019)

Solving NAME for fast isothermal strokes

Change of variable in interaction representation:

$$\frac{d}{dt}\tilde{\rho}_S(t) = \tilde{\gamma}\downarrow\left(\hat{b}\tilde{\rho}_S\hat{b}^{\dagger} - \frac{1}{2}\left\{\hat{b}^{\dagger}\hat{b},\tilde{\rho}_S\right\}\right)$$
$$\tilde{\gamma}\uparrow\left(\hat{b}^{\dagger}\tilde{\rho}_S\hat{b} - \frac{1}{2}\{\hat{b}\hat{b}^{\dagger},\tilde{\rho}_S\}\right) \quad (2)$$

we can try a solution in a generalized canonical form:

$$\tilde{\rho}_S(t) = \frac{1}{Z(t)}e^{\gamma(t)\hat{b}^2}e^{\beta(t)\hat{b}^{\dagger}\hat{b}}e^{\gamma^*(t)\hat{b}^{\dagger 2}}$$

Maximum entropy subject to constraints: $\langle\hat{b}^{\dagger}\hat{b}\rangle$, $\langle\hat{b}^{\dagger}\rangle$, $\langle\hat{b}^2\rangle$

Canonical invariance: Openheim 1964, Alhassid & Levine 1978.

Andersen, H. C., Oppenheim, I., Shuler, K. E., & Weiss, G. H., *Jour. of Math. Phys.* (1964)

Y. Alhassid and R. D. Levine, Phys. Rev. A 18, 89 (1978

Dynamics for any squeezed thermal state.

$$\dot{\beta} = k_{\downarrow}\left(e^{\beta} - 1\right) + k_{\uparrow}\left(e^{-\beta} - 1 + 4e^{\beta}|\gamma|^2\right),$$
$$\dot{\gamma} = \left(k_{\downarrow} + k_{\uparrow}\right)\gamma - 2k_{\downarrow}\gamma e^{-\beta} \quad , \tag{11}$$

We assume that the system is in a thermal state at initial time, which infers $\gamma(0) = 0$. This simplifies to

$$\tilde{\rho}_S\left(\beta\left(t\right), \mu\left(t\right)\right) = \frac{1}{Z} e^{\beta \hat{b}^{\dagger} \hat{b}(\mu)} \quad . \tag{12}$$

The system dynamics are described by

$$\dot{\beta} = k_{\downarrow}\left(t\right)\left(e^{\beta} - 1\right) + k_{\uparrow}\left(t\right)\left(e^{-\beta} - 1\right) \quad , \tag{13}$$

with initial conditions $\beta\left(0\right) = \frac{\hbar\omega(0)}{k_B T}$ and $\mu\left(0\right) = 0$.

Engineering the **shortcut to equilibration** protocol

Guessing a **solution** in the form of Generalized Canonical form

$$\tilde{\rho}_S(t) = (Z(t))^{-1} e^{\gamma(t)\tilde{b}^2} e^{\beta(t)\tilde{b}^\dagger\tilde{b}} e^{\gamma^*(t)\left(\tilde{b}^\dagger\right)^2}$$

$$\dot{\beta} = k_\downarrow\left(e^\beta - 1\right) + k_\uparrow\left(e^{-\beta} - 1 + 4e^\beta|\gamma|^2\right)$$

$$\dot{\gamma} = (k_\downarrow + k_\uparrow)\gamma - 2k_\uparrow\gamma e^{-\beta}$$

$$\dot{\beta} = k_\downarrow(\alpha(t))\left(e^\beta - 1\right) + k_\uparrow(\alpha(t))\left(e^{-\beta} - 1\right)$$

For an initial thermal state

$$\beta(0) = -\frac{\hbar\omega(0)}{k_B T} \qquad \beta(t_f) = -\frac{\hbar\omega(t_f)}{k_B T}$$

$$\mu(0) = \mu(t_f) = 0$$

$$y = e^\beta$$

$$y(s) = y(0) + c_3 s^3 + c_4 s^4 + c_5 s^5$$

$$s = t/t_f$$

$$\alpha(t) = \sqrt{1 - \frac{1}{4}\left(\frac{\dot{\omega}(t)}{\omega^2(t)}\right)^2}\,\omega(t) \leftarrow \beta(t) \leftarrow y(s)$$

$$\frac{d}{dt}\tilde{\rho}_S(t) \rightarrow \dot{\beta} \rightarrow \beta(t) \rightarrow \alpha(t) \rightarrow \omega(t)$$

$$\hat{H}_S = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2(t)\hat{Q}^2$$

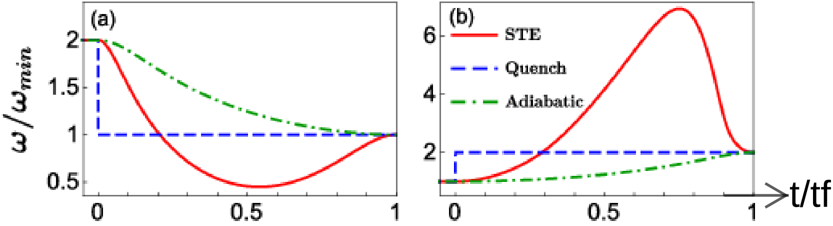
Control

R. Dann, A. Tobalina, and R. Kosloff, PRL (2019).

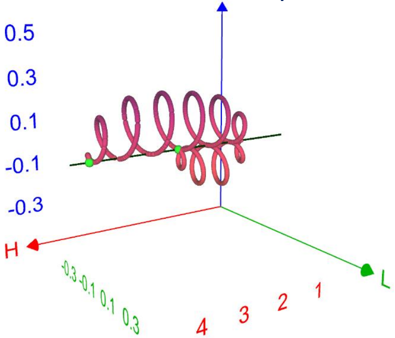
Shortcuts to Equilibrium (STE)

The shortcut protocol $\hat{H}_S(t) \rightarrow \omega(t)$:

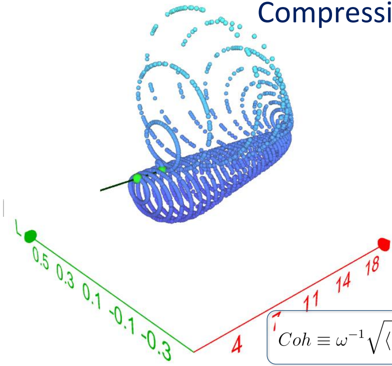
Overshoot



Expansion



Compression

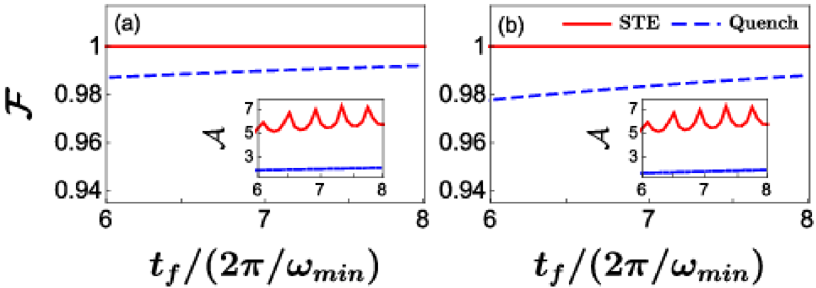


$$Coh \equiv \omega^{-1}\sqrt{\langle\hat{L}\rangle^2 + \langle\hat{C}\rangle^2}$$

Shortcuts to Equilibrium (STE)

The fidelity $\mathscr{F}$ and $\mathscr{A} = -\log_{10}(1-\mathscr{F})$:

Expansion   Compression



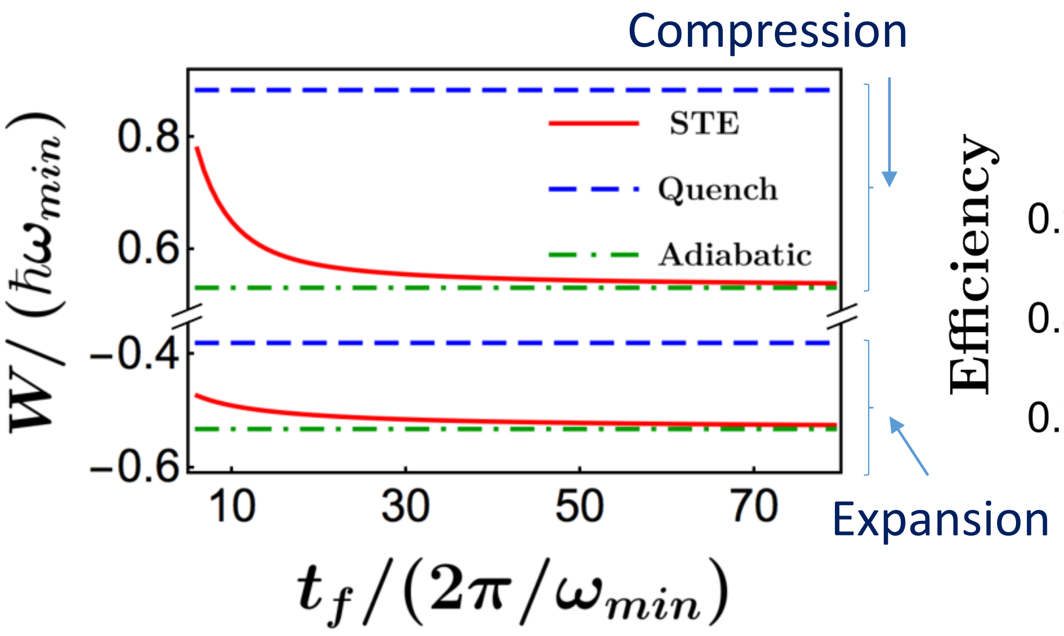
3 fold improvement in time

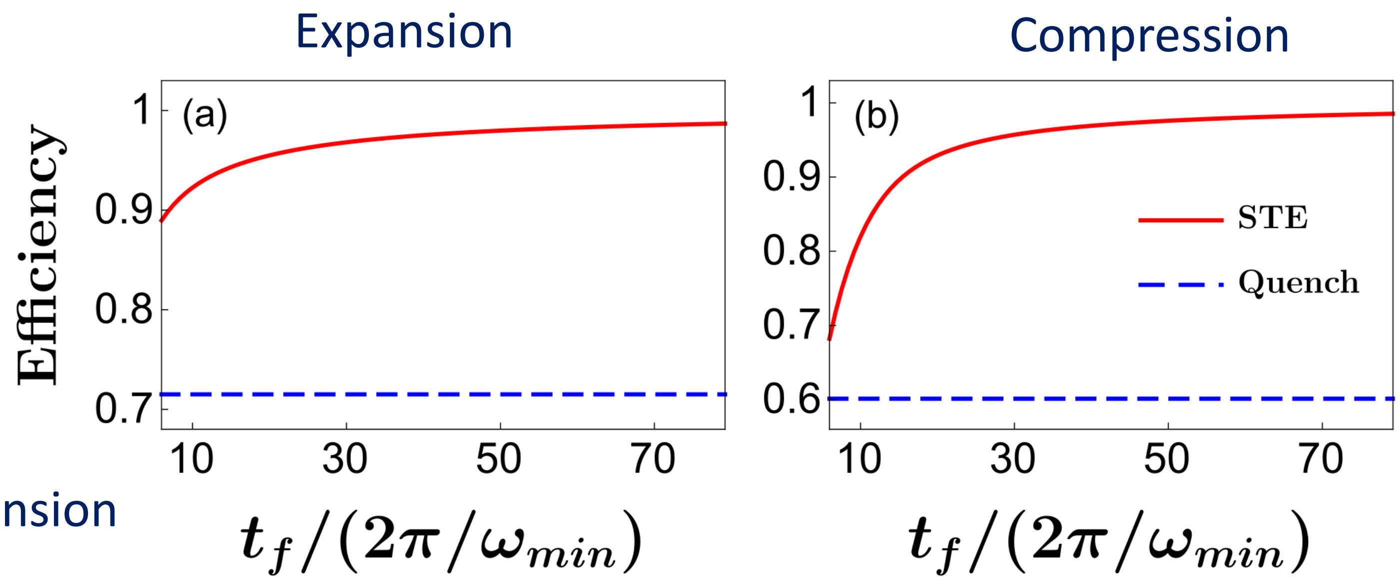
R. Dann, A. Tobalina, and R. Kosloff, *PRL* **122**, 250402 (2019)

STE- How much does it cost?

STE Quench Adiabatic

Rapid driving costs!

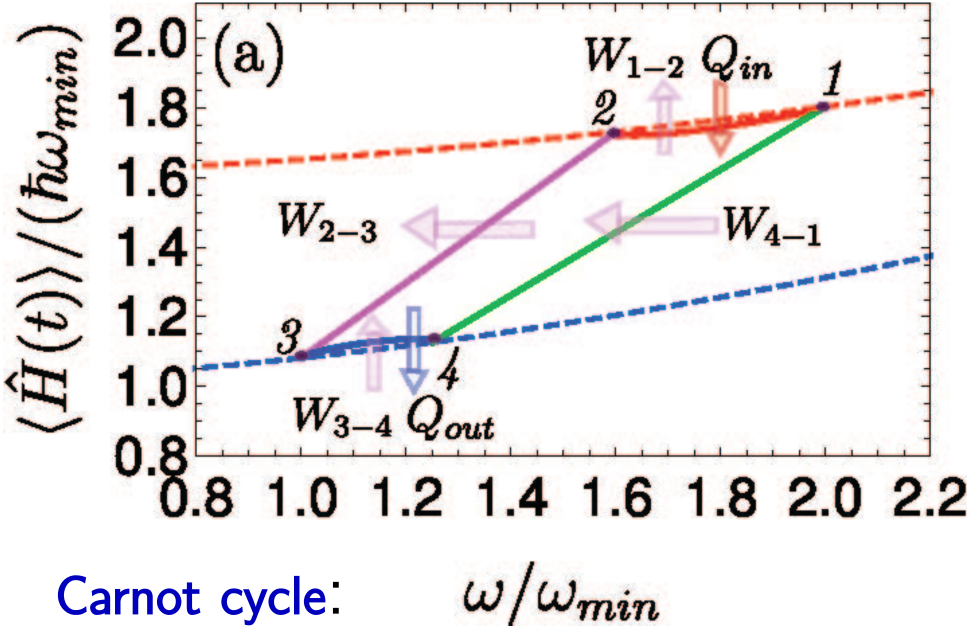




$$W = \int_0^t dt' \, \mathrm{tr}\left(\frac{\partial \hat{H}(t')}{\partial t'} \hat{\rho}_S(t')\right)$$

Efficiency: $\frac{W}{W_{ideal}}$

At last: Shortcut to four stroke Carnot cycle



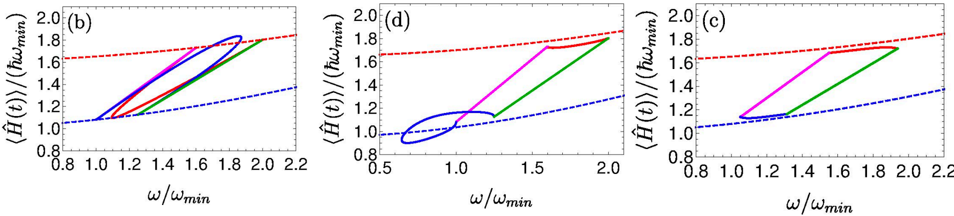
Carnot cycle:

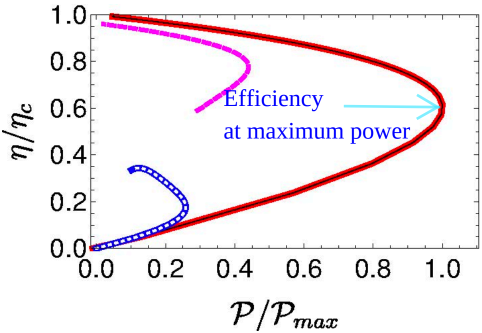
$$\Lambda_{cyc} = \Lambda_h \Lambda_{ch} \Lambda_c \Lambda_{hc}$$

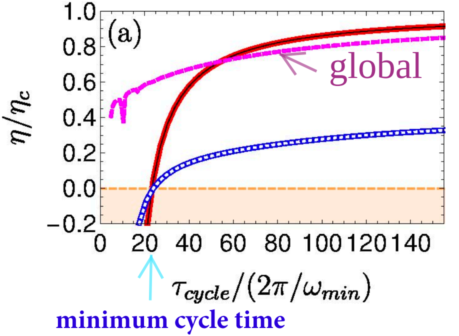
Quantum Signatures in the Quantum Carnot Cycle
Roie Dann, Ronnie Kosloff arXiv:1906.06946

Performance of Shortcut to Carnot

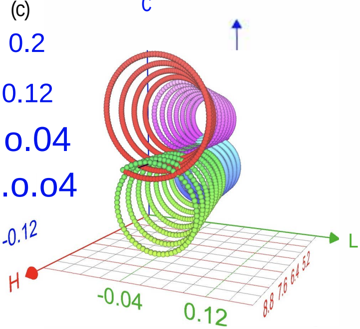
Shortcut fast | Shortcut Endo | Endo slow global

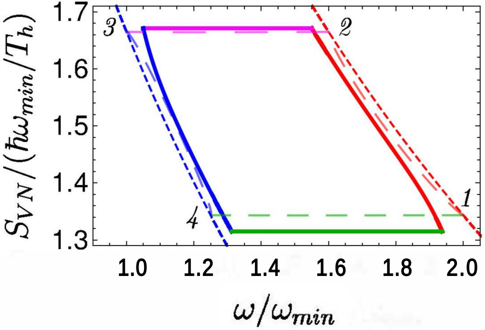


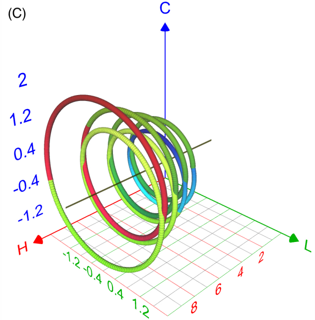




Endo Global

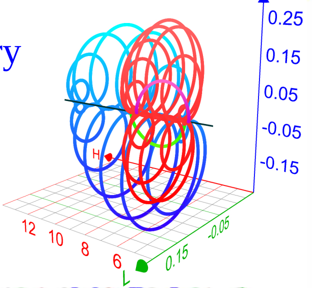






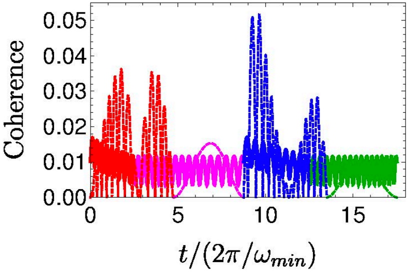
Cycle trajectory

shortcut



Quantum eqivalence

The propagator: $\mathscr{U} = e^{\mathscr{L} t}$



Four stroke cycle propagator:

$$\mathscr{U}_{cyc} = \mathscr{U}_c \mathscr{U}_{hc} \mathscr{U}_h \mathscr{U}_{ch} = e^{\mathscr{L}_c t} e^{\mathscr{L}_{hc} t} e^{\mathscr{L}_h t} e^{\mathscr{L}_{ch} t}$$

In the limit of small action: $s = ||\mathscr{L} t|| \ll \hbar$

$$\mathscr{U}_{cyc} = e^{\mathscr{L}_c t/2} e^{\mathscr{L}_{hc} t} e^{\mathscr{L}_h t} e^{\mathscr{L}_{ch} t} e^{\mathscr{L}_c t/2}$$

$$\mathscr{U}_{cyc} \approx e^{(\mathscr{L}_c + \mathscr{L}_{hc} + \mathscr{L}_h + \mathscr{L}_{ch}) t} + O(s^3)$$

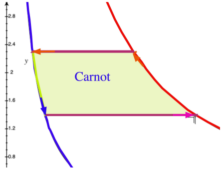
Raam Uzdin, Amikam Levy, and Ronnie Kosloff
Equivalence of Quantum Heat Machines, and Quantum-Thermodynamic
.(Phys. Rev. X 5, 031044 2015

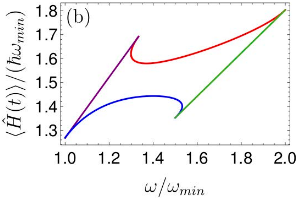
The Voyage:

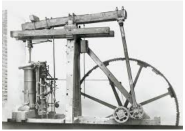
Seeking for quantum open system description of the Carnot cycle

- Non Adiabatic Master Equation **NAME**.
- The **inertial theorem**.
- Shortcuts to non unitary maps with **entropy change**.
- Finite time quantum **Carnot cycle**.

Quantum signature!







Thank you

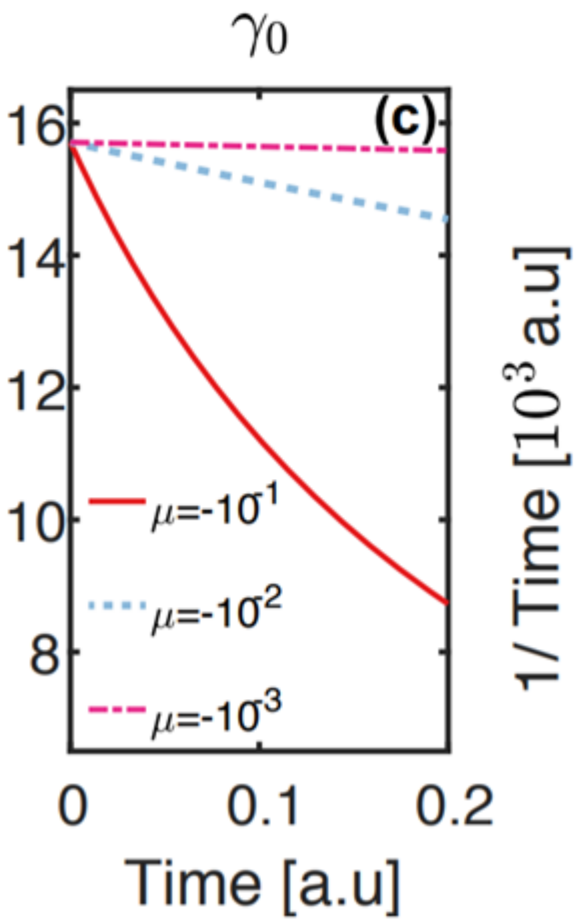
The end

Solution of the NAME for the HO

Protocol

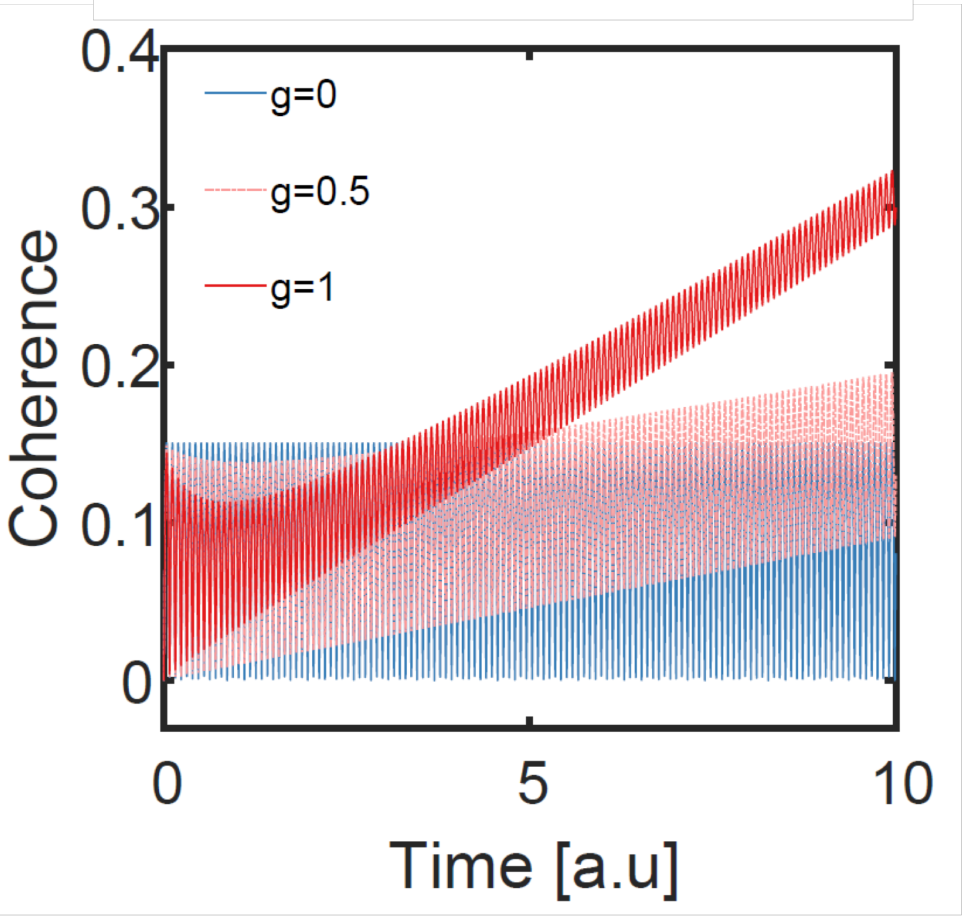
$$\omega(t) = \frac{\omega(0)}{1 - \mu\omega(0)t}$$

$$\mu < 0$$



Slowing down the rate

$$Coh \equiv \omega^{-1}\sqrt{\langle\hat{L}\rangle^2 + \langle\hat{C}\rangle^2}$$

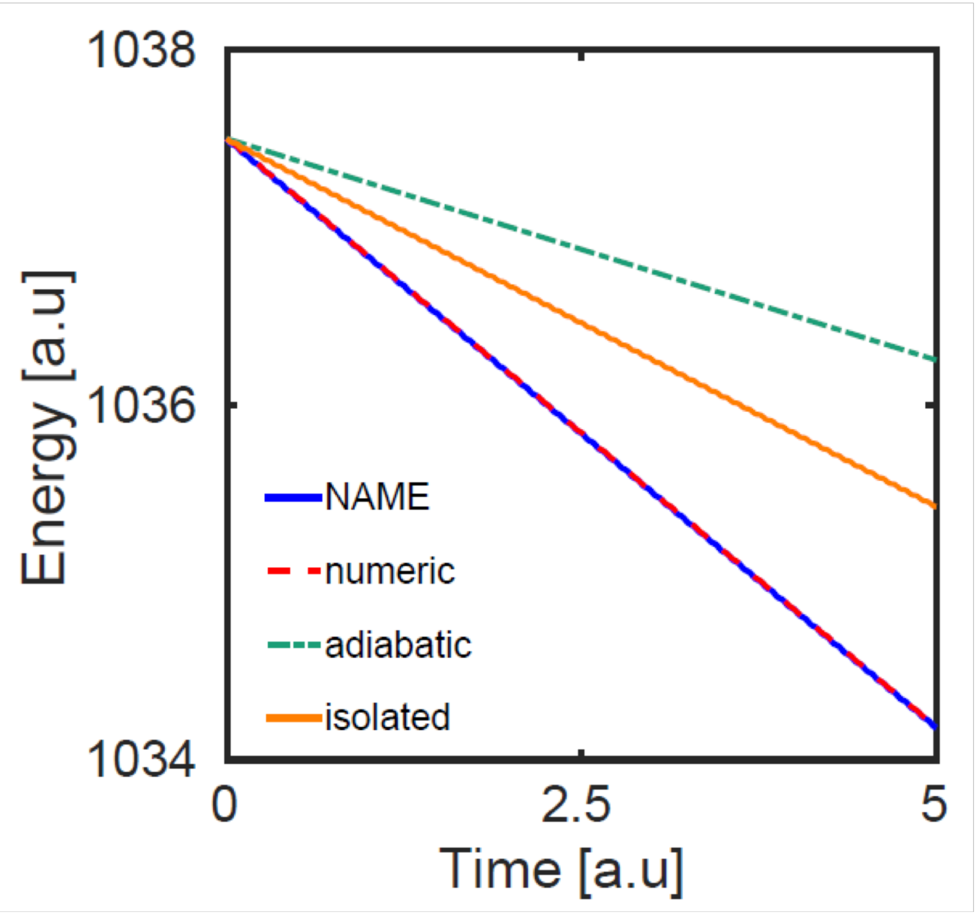


Coherence generated “by” the bath.

R. Dann, A. Levy, and R. Kosloff, Phys. Rev. A 98, 052129 (2018).

Solution of the NAME for HO

Comparison to numerical simulation



Protocol

$$\omega(t) = \frac{\omega(0)}{1 - \mu\omega(0)\, t}$$

$$\mu < 0$$

R. Dann, A. Levy, and R. Kosloff, Phys. Rev. A 98, 052129 (2018).

Solution for the propagator

For $\mu = const$ can be solved in terms of $\{\hat{H}_S, \hat{L}, \hat{C}, \hat{I}\}^T$

$$\mathcal{U}(t,0) = \frac{\omega(t)}{\omega(0)} \frac{1}{\kappa^2} \begin{bmatrix} 4-\mu^2 c & -\mu\kappa s & -2\mu(c-1) & 0 \\ -\mu\kappa s & \kappa^2 c & -2\kappa s & 0 \\ 2\mu(c-1) & 2\kappa s & 4c-\mu^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $s = \sin(\kappa\theta(t))$ and $c = \cos(\kappa\theta(t))$

$$\kappa = \sqrt{4-\mu^2}$$

$$\theta(t) = -\frac{1}{\mu}\log\left(\frac{\omega(t)}{\omega(0)}\right)$$

Kosloff, R., & Rezek, Y. (2017 *Entropy*, *19*(4), 136.

The NAME in Heisenberg form:

.

The Heisenberg picture is given by the equation of motion:

$$\frac{d}{dt}\hat{O} = \mathscr{V}^{\dagger}(t,0)\mathscr{L}^{\dagger}(t)\hat{O} \quad .$$

For such a case the adjoint propagator has the form:

$$\mathscr{V}^{\dagger}(t,t_0) = \mathbf{T}\exp\int_{t_0}^{t} ds \mathscr{L}^{\dagger}(s)$$

where **T** is the anti-chronological time ordering. $\mathscr{V}^{\dagger}(t,t_0)$ is defines by the adjoint generator $\mathscr{L}^{\dagger}$ by the differential equation

$$\frac{\partial}{\partial t}\mathscr{V}^{\dagger}(t,t_0) = \mathscr{V}^{\dagger}(t,t_0)\mathscr{L}^{\dagger}(t)$$

.

Breuer, H. P., & Petruccione, F. (2002). *The theory of open quantum systems*. (pg. 124-125)

The end

Thank you