The Poisson frog model on Galton-Watson trees

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Contents

Contents

Definition

The frog model refers to a system of interacting random walks on a rooted graph.

Definition (continued)

Specific properties associated with the model include the following:

- Begins with one active particle (i.e. frog) at the root and some distribution of sleeping particles among the non-root vertices.
- The active particle performs a discrete time nearest neighbor random walk (biased or unbiased) on the graph.
- Any time an active particle lands on a vertex containing a sleeping particle(s), the sleeping particle(s) wakes up and begins performing its own discrete time nearest neighbor random walk (independent of those of other active particles).

Versions previously examined

Versions of the frog model which have already been looked at include the following:

- The infinite d-ary tree $\mathbb{T}_d$ with one sleeping particle at every non-root vertex (Hoffman, Johnson, and Junge, 2017).
- The d-dimensional Euclidean lattice $\mathbb{Z}^d$ with one sleeping particle at every non-root vertex (Telcs and Wormald, 1999).
- $\mathbb{T}_d$ with i.i.d. Poisson many sleeping particles at the non-root vertices (Hoffman, Johnson, and Junge, 2016).
- $\mathbb{Z}$ with i.i.d. particles per non-root vertex where particles perform random walk with drift (Gantert and Schmidt, 2009).

Existing results

Existing results on the frog model include

- Theorems related to cover times.
- Shape theorems.
- Results pertaining to questions of recurrence vs. transience (i.e. whether the probability that the root is visited by infinitely many particles is equal to 1 or 0 respectively).

Contents

Setup of the model

Define a version of the frog model on $\mathbb{T}_d$ (the d-ary tree) with the following properties:

- Each non-root vertex begins with Poiss(λ) sleeping particles.
- Upon activation, particles perform unbiased nearest-neighbor random walks on the tree.

Recurrence and transience on $\mathbb{T}_d$

In 2016 Hoffman, Johnson, and Junge proved that for every $d \geq 2$, there exists $\lambda_c(d) \in (0, \infty)$ such that this version of the frog model on $\mathbb{T}_d$ is recurrent for $\lambda > \lambda_c(d)$ and transient for $\lambda < \lambda_c(d)$.

Dominating the model

- Define a BRW model on $\mathbb{T}_d$ where an active particle gives birth to Poiss(λ) active particles *every* time it steps away from the root.
- Show, via a coupling, that this model dominates the frog model w.r.t. the number of returns to the root.
- To establish transience for the frog model, it suffices to do so for the branching model.

Setting up a weight function

Select a constant $\alpha \in (0,1)$ and define

$$X_n := \sum_{f_i} \alpha^{|f_i|},$$

where the f_i's range over all active particles at time n, and $|f|$ denotes the level at which the active particle f resides.

Completing the proof of transience

Show that for λ sufficiently small, and a suitable choice of α, X_n is a supermartingale. This implies convergence of X_n, which implies transience of the branching model, and thus the frog model as well.

The non-backtracking frog model

Define the non-backtracking frog model, where activated particles perform non-backtracking (i.e. loop-erased) random walks on $\mathbb{T}_d$ that are stopped at the root.

The self-similar frog model

The self-similar frog model is defined by making the following modifications to the non-backtracking model.

- After the first time a vertex v is landed on by a particle, all other particles that jump from $\overleftarrow{v}$ to v are stopped upon hitting v.
- If v is hit for the first time by more than one particle simultaneously, then all but one (chosen randomly) are stopped.

Coupling the models

Let Z represent the number of times the root is hit in the ordinary frog model on $\mathbb{T}_d$, and let Z' represent the number of times it is hit in the self-similar frog model. The two models can be coupled so that $Z \geq Z'$.

Illustration of the self-similar model using star graph on $\mathbb{T}_d$

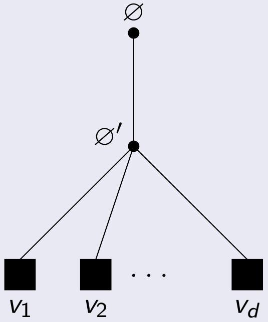

Figure: The number of particles from the subtree rooted at v_i that hit $\varnothing'$ (conditioned on v_i being hit) is treated as a *black box* random variable. Has same distribution as number of active particles that hit $\varnothing$.

Constructing the recursion

We can think of the system as an operator $\mathcal{A}$ acting on the distribution π.

- Let π represent the distribution of the number of active particles from the box v_i that hit $\varnothing$ (conditioned on v_i being activated).
- Let $\mathcal{A}\pi$ represent the distribution of the number of active particles that hit $\varnothing$.

Completing the proof

A bootstrapping argument in order to establish recurrence of the self-similar model can be constructed as follows.

- Show that for λ sufficiently large, we have $\mathcal{A}\big(\text{Poiss}(\mu)\big) \succeq \text{Poiss}(\mu+\epsilon)$ for every $\mu \geq 0$ (for some $\epsilon > 0$).
- Establish monotonicity of $\mathcal{A}$ i.e. $\pi_1 \succeq \pi_2 \implies \mathcal{A}\pi_1 \succeq \mathcal{A}\pi_2$.
- Conclude, based on the fact that $\eta = \mathcal{A}^n\eta \succeq \mathcal{A}^n\big(\text{Poiss}(0)\big) \to \infty$ as $n \to \infty$, that the distribution η is concentrated at ∞.

Questions posed by Hoffman, Johnson, and Junge

Question 1: Does there also exist a critical value $\lambda_c \in (0, \infty)$ separating recurrent and transient regimes for the Poisson frog model on Galton-Watson trees?

Question 2: If so, does the value of λ_c depend only on the maximum value attainable by the offspring distribution Z, or on the entire distribution?

Contents

λ_c

Theorem. Let GW be the measure on Galton-Watson trees induced by an offspring distribution Z for which $(Z \geq 2) = 1$ and $[Z^{4+\epsilon}] < \infty$ for some $\epsilon > 0$. Then there exists a constant $\lambda_c \in (0, \infty)$ such that, for GW-a.s. every **T**, the frog model with i.i.d. Poiss(λ) particles per non-root vertex is transient for every $\lambda < \lambda_c$, and recurrent for every $\lambda > \lambda_c$.

No intermediate regime

To prove the theorem we need to rule out the possibility of an intermediate regime. This means showing that

$$\mathrm{FM}_\lambda(\text{recurrence}) > 0 \implies \mathrm{FM}_\lambda(\text{recurrence}) = 1.$$

Proof involves establishing ergodicity of the random shift operator.

Transience

The proof of transience on non-regular trees of degree 3 or higher is nearly the same as the proof for regular trees.

The truncated frog model

Define the truncated frog model to be a model where activated particles perform loop-erased walks that are stopped at the root, and where only one particle can jump to a vertex v from its parent without being stopped.

Bootstrapping

Let $\mathrm{TFM}_T^{(\lambda)}$ denote law of truncated frog model on T with Poiss(λ) particles per non-root vertex. To prove recurrence, bootstrap is performed on the following quantity:

$$\mathbb{E}_{\mathrm{AGW}\times\mathrm{HARM}_{\mathbf{T}}}\left[\mathrm{TFM}^{(\lambda)}_{\mathbf{T}(v_n)}(v_{n+1} \text{ is activated})\right]$$

Illustration of random path

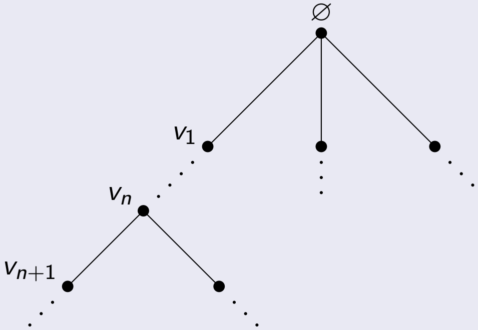

Figure: We choose a non-backtracking path from the root $v_0, v_1, \ldots$. Look at probability v_{n+1} is activated, conditioned on v_n being activated. Then take expectation w.r.t. $\text{AGW} \times \text{HARM}_{\mathbf{T}}$.

Main step

Proposition. There exist constants q, α_o, and λ_o in $(0, \infty)$ such that, for all $\alpha > \alpha_o$ and $\lambda > \lambda_o$, the inequality

$$\mathbb{E}_{\mathsf{AGW} \times \mathsf{HARM}_{\mathbf{T}}} \left[\mathsf{TFM}^{(\lambda)}_{\mathbf{T}(v_n)}(v_{n+1} \text{ is activated}) \right] \geq 1 - e^{-\alpha}$$

holding for all $n \geq 1$ in fact implies that the inequality

$$\mathbb{E}_{\mathsf{AGW} \times \mathsf{HARM}_{\mathbf{T}}} \left[\mathsf{TFM}^{(\lambda)}_{\mathbf{T}(v_n)}(v_{n+1} \text{ is activated}) \right] \geq 1 - e^{-(\alpha+q)}$$

holds for every $n \geq 1$.

Steps in proving the proposition

- If no particles from v_n hit v_{n+1}, then let v^* be sibling of v_{n+1} hit by one of these particles.
- Using induction hypothesis, we know large portion (w.r.t. harmonic measure) of mth generation descendants of v^* are activated with high probability.
- Use fact that $\text{HARM}_T(v) \approx p(v, \varnothing)$ to estimate number of activated particles from descendants of v^* that come back and hit v^*.
- With high likelihood, one of these will then go from v^* to v_n, and then to v_{n+1}, thus activating it. This allows us to bootstrap quantity

$$\mathbb{E}_{\text{AGW}\times\text{HARM}_{\mathbf{T}}}\left[\text{TFM}^{(\lambda)}_{\mathbf{T}(v_n)}(v_{n+1} \text{ is activated})\right].$$

Completing the proof of recurrence

- Combining proposition with bootstrap implies that for all n, and λ large enough,

$$\mathbb{E}_{\mathrm{AGW}\times\mathrm{HARM}_{\mathbf{T}}}\left[\mathrm{TFM}^{(\lambda)}_{\mathbf{T}(v_n)}(v_{n+1}\text{ is activated})\right] = 1.$$

- This then implies all vertices of $\mathbf{T}$ are activated almost surely.
- Then once again use fact that $\mathrm{HARM}_{\mathbf{T}}(v) \approx p(v,\varnothing)$ to complete proof.

Contents

Bibliography

- Nina Gantert and Phillip Schmidt, *Recurrence for the frog model with drift on* $\mathbb{Z}$. Markov Process. Related Fields **15**(1) (2009), 51-58.
- Christopher Hoffman, Tobias Johnson, and Matthew Junge, *From transience to recurrence with Poisson tree frogs*, Ann. Appl. Probab. **26**(3) (2016), 1620-1635.
- Christopher Hoffman, Tobias Johnson, and Matthew Junge, *Recurrence and transience for the frog model on trees*, Ann. Probab. **45**(5) (2017), 2826-2854.
- András Telcs and Nicholas C. Wormald, *Branching and tree indexed random walks on fractals*, J. Appl. Probab. **36**(4) (1999), 999-1011.