Low-rank approximations for time-dependent differential equations

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joint work with L. Einkemmer, A. Ostermann, H. Walach

Low-rank approximations

Low-rank approximations $\longleftrightarrow$ reduce computational complexity

$$\dot{X}(t) = F(t, X(t)), \quad X(t) \in \mathbb{R}^{n \times n}, \quad n \text{ large}$$

- Large-scale ODEs: Lyapunov and Riccati differential equations
- Discretized PDEs

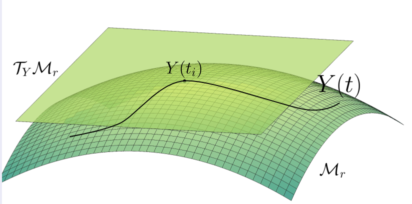
Truncated singular value decomposition:

$$Y(t) = U(t)S(t)V(t)^{\mathsf{T}}$$

$U(t), V(t) \in \mathbb{R}^{n \times r}$, $S(t) \in \mathbb{R}^{r \times r}$, $\quad r \ll n$

Orthogonality constraints: $U^{\mathsf{T}}U = I$, $V^{\mathsf{T}}V = I$.

Dynamical low-rank approximation



Follow the rank-r solution $Y(t)$

$\downarrow$

update the low-rank factors dynamically in time

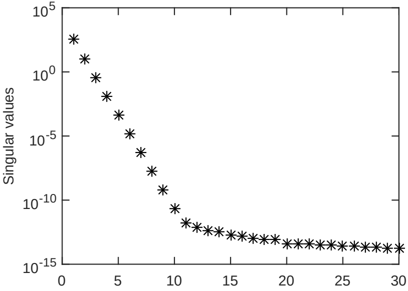
Instead of searching for $Y(t) \approx X(t) \in \mathbb{R}^{n \times n}$ we solve:

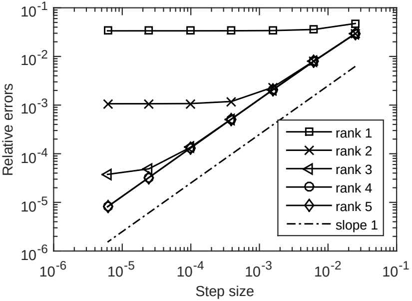
$$\begin{aligned} \dot{U} &= \ldots\ldots \qquad \rightarrow U \in \mathbb{R}^{n \times r}, \\ \dot{S} &= \ldots\ldots \qquad \rightarrow S \in \mathbb{R}^{r \times r}, \\ \dot{V} &= \ldots\ldots \qquad \rightarrow V \in \mathbb{R}^{n \times r}. \end{aligned}$$

O. Koch, C. Lubich, 2007, C. Lubich, I. Oseledets, 2014

Discretized PDE

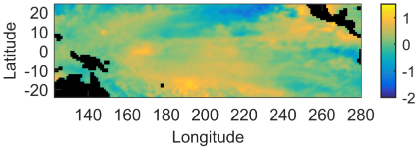
$$\partial_t v = \alpha \Delta v + v^3 \text{ on } \Omega = [0,1]^2$$



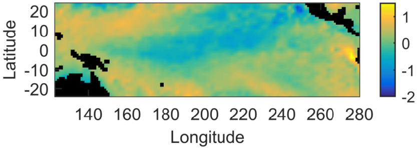


Results at $T = 0.5$, $n = 500$.

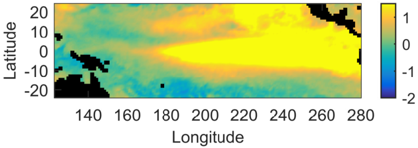
Number of unknowns: 3900^2, approximation rank = 10



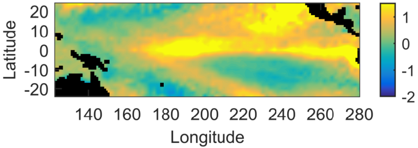
Measured anomalies 2013



Simulated anomalies 2013



Measured anomalies 2015



Simulated anomalies 2015

Joint work with H. Mena, L.-M. Pfurtscheller

Vlasov–Maxwell equations

Kinetic description of plasma interacting with an electromagnetic field

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + F \cdot \nabla_v f(t,x,v) = 0$$
$$F = E + v \times B$$

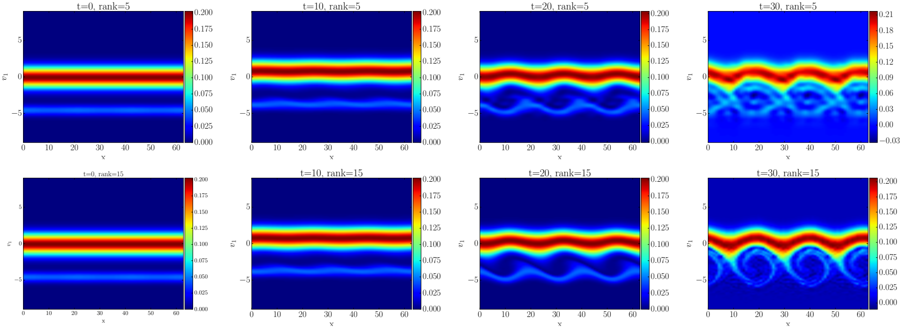
$(x, v) \in \Omega_x \times \Omega_v$: 3+3 dimensional phase-space

Low-rank approximation of f:

$$f(t,x,v) = \sum_{i,j=1}^{r} X_i(t,x) S_{ij}(t) V_j(t,v),$$

Orthogonality constraints: $\langle X_i, X_j \rangle = \delta_{ij}$, $\langle V_i, V_j \rangle = \delta_{ij}$.

Bump-on-tail instability



H. Mena, A. Ostermann, L.-M. Pfurtscheller, and C. Piazzola.
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J. Comput. Appl. Math., 340, 602–614, 2018.

A. Ostermann, C. Piazzola, H. Walach.
Convergence of a low-rank Lie–Trotter splitting for stiff matrix differential equations.
arXiv, 2018. https://arxiv.org/abs/1803.10473.

L. Einkemmer, A. Ostermann, C. Piazzola.
A low-rank projector-splitting integrator for the Vlasov–Maxwell equations with divergence correction.
In preparation.

Thank you for your attention!