Hecke algebra traces, Kazhdan-Lusztig basis elements, and chromatic (quasi-) symmetric functions

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⓪ Generating functions

Q: Where do we put a function? (especially if we don't know an explicit formula for it)

- Put $f: \mathbb{N} \times \mathbb{N} \to \mathbb{Z}$ in $\sum_{m \geq 0} \sum_{n \geq 0} f(m,n) x^m y^n$

  $\in \mathbb{Z}[[x,y]]$

- Put $f: S_n \to \mathbb{Z}$ in

  $$\mathrm{Imm}_f(x) := \sum_{w \in S_n} f(w) x_{1,w_1} \cdots x_{n,w_n} \in \mathbb{Z}[x_{11}, \ldots, x_{nn}]$$

- Put $f_q: H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ in

  $$\mathrm{Imm}_{f_q}(x) := \sum_{w \in S_n} f(\tilde{T}_w) x_{1,w_1} \cdots x_{n,w_n} \in A_n(q)$$

$H_n(q)$ = Hecke algebra, an $n!$-dim'l q-extension of $\mathbb{Z}[S_n]$. Natural basis $\{\tilde{T}_w \mid w \in S_n\}$

Kazhdan-Lusztig basis $\{\tilde{C}_w \mid w \in S_n\}$

$H_n(1) = \mathbb{Z}[S_n]$

$A_n(q)$ = "quantum matrix bialgebra"
= a quotient of $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]\langle x_{11}, \ldots, x_{nn}\rangle$

non-commuting vars

Ex: ε_q^λ = induced sign character of $H_n(q)$
η_q^λ = induced trivial character of $H_n(q)$
(from Young subalgebra of type λ)
don't have nice formulas, but
$\mathrm{Imm}_{\varepsilon_q^\lambda}(x)$, $\mathrm{Imm}_{\eta_q^\lambda}(x) \in A_n(q)$ are nice.
(Konvalinka, Skandera 2011)

- Put $f: \mathrm{Par}(n) \to R$ in $\sum_\lambda f(\lambda) m_\lambda \in \Lambda_n$.

Ex: $f(\lambda) = \sum_{\substack{\nu \text{ a chain decomposition} \\ \text{of unit interval order } P \\ \text{of type } \lambda}} q^{\mathrm{asc}(\nu)}$

gives $X_{P,q}$ = Shareshian-Wachs q-chromatic symmetric function.

$\sum_{\lambda \vdash n} \Theta(\lambda)\, m_\lambda$ is really six generating functions in one:

Expanding in other bases, it is

$$\sum_\lambda \Theta_e(\lambda)\, e_\lambda = \sum_\lambda \Theta_h(\lambda)\, h_\lambda = \sum_\lambda \Theta_s(\lambda)\, s_\lambda$$
$$= \sum_\lambda \Theta_p(\lambda)\, p_\lambda = \sum_\lambda \Theta_f(\lambda)\, f_\lambda$$

for some functions $\Theta_e, \Theta_h, \Theta_s, \Theta_p, \Theta_f : \mathrm{Par}(n) \to R$.

① Hecke algebra traces

Call linear functional $\theta_q : H_n(q) \to \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$

a trace if it satisfies $\theta_q(gh) = \theta_q(hg)$
for all $h, g \in H_n(q)$.

The $q=1$ specialization $\theta = \theta_1$ is an S_n
class function, $\theta(vwv^{-1}) = \theta(w)$.

$H_n(q)$ traces form a space ($\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module)
of dimension $|\mathrm{Par}(n)|$.

trace bases		Symm fn. bases
$\{\chi_q^\lambda \mid \lambda \vdash n\}$	irr chars	s_λ
$\{\eta_q^\lambda \mid \lambda \vdash n\}$	induced trivial chars	h_λ
$\{\varepsilon_q^\lambda \mid \lambda \vdash n\}$	Induced sign chars	e_λ
$\{\psi_q^\lambda \mid \lambda \vdash n\}$	power sum traces	p_λ
$\{\phi_q^\lambda \mid \lambda \vdash n\}$	monomial traces	m_λ

are related to one another just like

Claim: For all $g \in H_n(q)$ we have

$$\sum_\lambda \varepsilon_q^\lambda(g)\, m_\lambda = \sum_\lambda \chi_q^{\lambda^T}(g)\, s_\lambda = \sum_\lambda \eta_q^\lambda(g)\, f_\lambda = \sum_\lambda \psi_q^\lambda(g)\, \frac{(-1)^{\ell(\lambda)-n}}{z_\lambda}\, p_\lambda$$

$$= \sum_\lambda \phi_q^\lambda(g)\, e_\lambda.$$

Pf: Vectors $w, x, y, z \in \mathbb{R}^m$ satisfy $w^T x = y^T z$ if $\exists\, A \in Mat_{m\times m}(\mathbb{R})$ s.t. $x = Az$ and $y = A^T w$, since

$$w^T x = w^T A z = (A^T w)^T z = y^T z.$$

Transition matrices of symm fn bases are related this way.

QED

Define $\Psi(g)$ = "trace generating fn. for g" $= \sum_\lambda \epsilon_q^\lambda(g)\, m_\lambda$.

(2) Connection to Chromatic Symm Fns

Theorem (Clearman, Hyatt, Shelton, Skandera 2016)
Let $w \in S_n$ avoid the pattern 312.
Let P be the corresponding unit interval order.
Then we have

$$\Psi(\tilde{C}_w) = X_{P,q}.$$

trace gen. fn. for K-L basis elt of $H_n(q)$ indexed by w

Shareshian Wachs q-chromatic symm fn for P.

Proof outline:

1. Represent $\tilde{C}_w$ by a graph G called a descending star network. (Skandera 2008, extending earlier work of Deodhar).

2. Use gen. fn. $\mathrm{Imm}_{\epsilon_q^\lambda}(x) \in A_n(q)$ to show that $\epsilon_q^\lambda(\tilde{C}_w) = \sum_u q^{\mathrm{INV}(u)}$.
   u ← objects called column strict G-tableaux of shape λ^T

3. Show that i-to-i paths in G form a unit interval order P. (This is the bijection $w \longleftrightarrow P$.)

4. Show that tableaux U correspond to colorings / chain decompositions γ of P; $\mathrm{INV}(u) = \mathrm{asc}(\gamma)$.

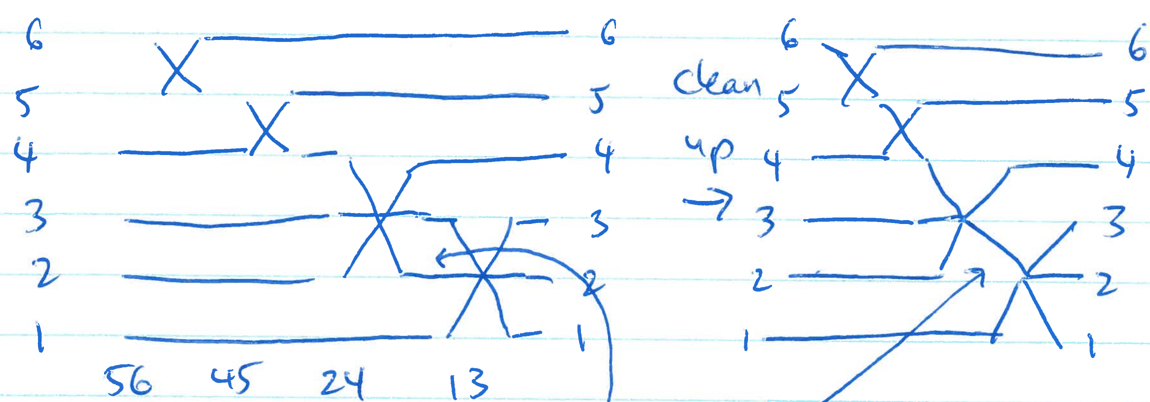
Some details, by example:

- From w to descending star network.

(Ex) Let $w = 3\ 4\ 2\ 5\ 6\ 1$

Record exceedance values, posns: $\frac{3}{1}\ \frac{4}{2}\ \cdots\ \frac{5}{4}\ \frac{6}{5}$

Write stars from Right to left.

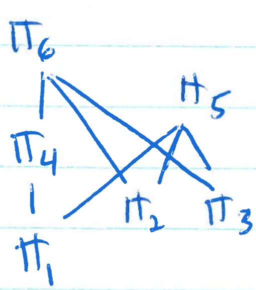


collapse ~~delete~~ double path

- From DSN to poset P.

Define $\pi_i <_P \pi_j$ if $\pi_i \cap \pi_j = \emptyset$ and $i<j$. (can't share even one vertex)

In ex,



- P to Hessenberg matrix / Dyck path

Define antiadjacency matrix $A = (a_{ij})$ of P by

$$a_{ij} = \begin{cases} 1 & \pi_i \not<_P \pi_j \\ 0 & \pi_i <_P \pi_j \end{cases}$$

In ex,

$$\begin{matrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix}$$

③ Conjectures, Theorems of Haiman, Stanley, Stembridge, Shareshian, Wachs.

Theorem: (Haiman '93) $\chi_q^\lambda(\tilde{C}_w) \in \mathbb{N}[q]$ $\forall w \in S_n$

Cor: $\epsilon_q^\lambda(\tilde{C}_w) \in \mathbb{N}[q]$, $\eta_q^\lambda(\tilde{C}_w) \in \mathbb{N}[q]$.

Combinatorial interps not known in general.

Conj: (Haiman '93) $\phi_q^\lambda(\tilde{C}_w) \in \mathbb{N}[q]$ $\forall w \in S_n$

Consequence (weaker conj.) $\psi_q^\lambda(\tilde{C}_w) \in \mathbb{N}[q]$.

Combinatorial interps not even conjectured in general.

Partial results

We have combinatorial interps, for w avoiding
3412, 4231, of
$\varepsilon_q^\lambda(\tilde{C}_w)$, $\chi_q^\lambda(\tilde{C}_w)$, $\eta_q^\lambda(\tilde{C}_w)$, $f_q^\lambda(\tilde{C}_w)$.

Clearman, Hyatt, Shelton, Skandera
Shareshian, Wachs
Athanasiadis
Gasharov
Ellzey

6 different interps!

We have combinatorial interps, for w avoiding
3412, 4231, of $\phi_q^\lambda(\tilde{C}_w)$
in the special case that $\lambda_1 \leq 2$ (λ has ≤ 2 cols)
(CHSS)
of $\phi^\lambda(\tilde{C}_w(1))$ ($q=1$ case)
in the special case that λ is a rectangular shape.
(Stembridge)
We have a recursive formula for $\phi_q^\lambda(\tilde{C}_w)$
when the unit interval order P corr to w
has no chain of 3 elts.
(Precup)

the conjecture poset: ($\uparrow$ stronger)

$\phi_q^\lambda(\tilde{C}_w) \in \mathbb{N}[q]\ \forall w \in S_n$
(Haiman '93)

*

$\phi_q^\lambda(\tilde{C}_{r_1} \cdots \tilde{C}_{r_k}) \in \mathbb{N}[q]$
where $r_1, \ldots, r_k$ are any reversals,
(because
$\tilde{C}_{r_1} \cdots \tilde{C}_{r_k} \in \mathrm{span}_{\mathbb{N}[q]}\{\tilde{C}_w \mid w \in S_n\}$)

(q=1 case)
$\phi^\lambda(\tilde{C}_{r_1} \cdots \tilde{C}_{r_k}) \in \mathbb{N}$
(Stembridge, '91)

$\mathrm{Imm}_{\phi^\lambda}(A) \geq 0$
for every
totally nonnegative
matrix A
(Stembridge, '91)

$\phi_q^\lambda(\tilde{C}_w) \in \mathbb{N}[q]$
for w avoiding 3412, 4231

$|||$ equiv. (See CHSS)

$\phi_q^\lambda(\tilde{C}_w) \in \mathbb{N}[q]$
for w avoiding 312

$|||$ equiv. by CHSS theorem

$X_{P,q} \in \mathrm{span}_{\mathbb{N}[q]}\{e_\lambda \mid \lambda \vdash n\}$
(Shareshian, Wachs)

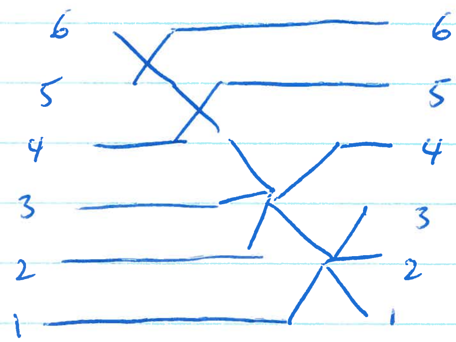
$\downarrow q=1$

$X_P \in \mathrm{span}_{\mathbb{N}}\{e_\lambda \mid \lambda \vdash n\}$
(Stanley, Stembridge '93, '95)

* Haiman '93 conjectures equivalence here, too.

④ Combinatorial interpretations in terms of descending star networks

For example, take $\lambda = 321$, $w = 342561$

$G =$ 

π_i = i-to-i path

- $\varepsilon_q^{\lambda}(\tilde{C}_w) = \sum q^{INV(u)}$

  u, a column strict G-tableau of shape λ^T.

Column-strict means $\boxed{\begin{array}{c}\pi_j \\ \hline \pi_i\end{array}} \Rightarrow \pi_i <_P \pi_j$

Ex:

$$\begin{array}{lll} \pi_6 & & \\ \pi_4 & \pi_5 & \\ \pi_1 & \pi_3 & \pi_2 \end{array}, \qquad \begin{array}{lll} \pi_6 & & \\ \pi_4 & \pi_5 & \\ \pi_1 & \pi_2 & \pi_3 \end{array}$$

$INV = \#\{(\pi_i, \pi_j) \mid \pi_j$ appears in earlier column, $j > i$ (in $\mathbb{Z}$), $\pi_i \cap \pi_j \neq \emptyset\}$

Inversions: $\pi_6\pi_5$, $\pi_4\pi_3$, $\pi_4\pi_2$, $\pi_3\pi_2$ $\qquad$ $\pi_6\pi_5$, $\pi_4\pi_3$, $\pi_4\pi_2$

Contribution: q^4 $\qquad$ q^3

$(\varepsilon_q^{\lambda}(\tilde{C}_w) = q^3 + q^4)$

For all traces θ_q^λ, we sum $q^{\mathrm{stat}(u)}$ over appropriate tableaux of shape λ.

Some statistic related to inversions

- $\eta_q^\lambda(\tilde{C}_w) = \sum_{u \text{ row-semistrict of shape } \lambda} q^{\mathrm{stat}(u)}$

Row-semistrict means $\boxed{\pi_i \,|\, \pi_j} \Rightarrow \pi_j \not<_P \pi_i$

Ex:

π_2
$\pi_5 \quad \pi_6$
$\pi_1 \quad \pi_4 \quad \pi_3$

- $\chi_q^\lambda(\tilde{C}_w) = \sum_{u \text{ standard of shape } \lambda} q^{\mathrm{stat}(u)}$

Standard means column-strict and row-semistrict.

Ex: Both ϵ_q^λ tableaux above work.

Note: standard tableaux are a hybrid of column-strict and row-semistrict tableaux.

This makes sense since $\quad S_n = h_n \quad (X_q^n = \eta_q^n)$

$\qquad\qquad\qquad\qquad\quad S_{1^n} = e_n \quad (X_q^{1^n} = \varepsilon_q^n)$

single row standard = single row row-semistrict
single col standard = single col column-strict.

- $\psi_q^\lambda(\hat{C}_w) = \sum_{u \text{ cylindrical of shape } \lambda} q^{\mathrm{stat}(u)}$

cylindrical means paths can be i-to-j ($\neq i$)
Each row is

$$\boxed{(i_1, i_2) \mid (i_2, i_3)} \;\cdots\; \boxed{(i_k, i_1)}$$

same index

same index

Ex:

(4,4)		
(6,5)	(5,6)	
(1,3)	(3,2)	(2,1)

- $\phi_q^\lambda(\hat{C}_w) = ?$

Recall : $\quad m_n = p_n \quad (\phi_q^n = \psi_q^n)$

$\qquad\qquad\quad m_{1^n} = e_n \quad (\phi_q^{1^n} = \varepsilon_q^n)$

Maybe ϕ_q^λ tableau should be a hybrid of cylindrical and column-strict tableaux.

We know

$$\varphi_q^{n}(\tilde{C}_w) = \sum_{u \text{ cylindrical of shape } n} q^{\operatorname{stat}(u)}$$

$$\varphi_q^{1^n}(\tilde{C}_w) = \sum_{u \text{ col strict of shape } 1^n} q^{\operatorname{INV}(u)}$$

$\left(= \#\{u \mid u \text{ col strict of shape } 1^n\}\right)$
(can't have an inversion with only one column)

Idea: Maybe $\varphi_q^{\lambda}(\tilde{C}_w) = \sum_{u \text{ col-strict, cylindrical}} q^{\operatorname{stat}(u)}$.

Problem: this always seems to give coefficients that are too small.

Ex:

(6,6)		
(4,5)	(5,4)	
(2,1)	(1,3)	(3,2)

Q: Can we use transition matrix $L = L_{\lambda\mu}$ defined by $p_\lambda = \sum_\mu L_{\lambda\mu} m_\mu$

and col-strict cylindrical tableaux to show that

$$\sum_\mu L_{\lambda\mu} \cdot (\#\text{ CSCT of shape } \mu) \le \varphi^\lambda(\tilde{C}_{w(1)})$$

and therefore that $\varphi^\lambda(\tilde{C}_{w(1)})$ must be larger than # CSCT of shape λ ?