An introduction to Hessenberg varieties

Hiraku Abe
Osaka Prefecture University

Hessenberg varieties in
Combinatorics, Geometry and Representation Theory

BIRS

2018/10/22

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$$Fl(\mathbb{C}^n) = \{ \ (V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = \mathbb{C}^n) \mid \dim V_i = i, \ \ 1 \le i \le n \}.$$

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Examples:
Flag variety, Springer fibers, Peterson variety, permutohedral variety, etc

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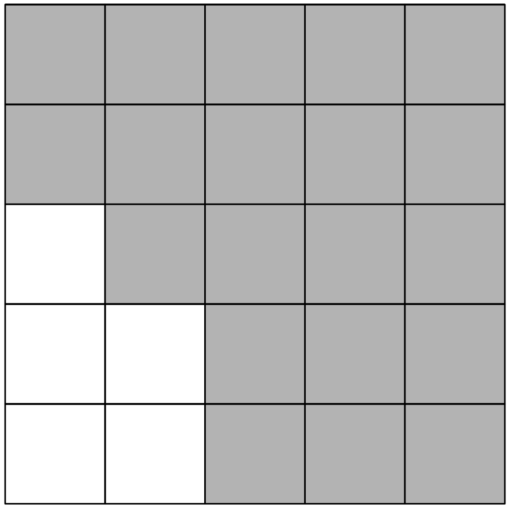
e.g. $h = (2, 3, 5, 5, 5)$



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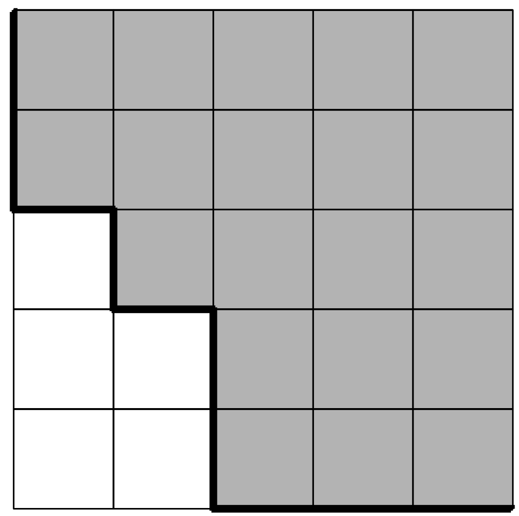

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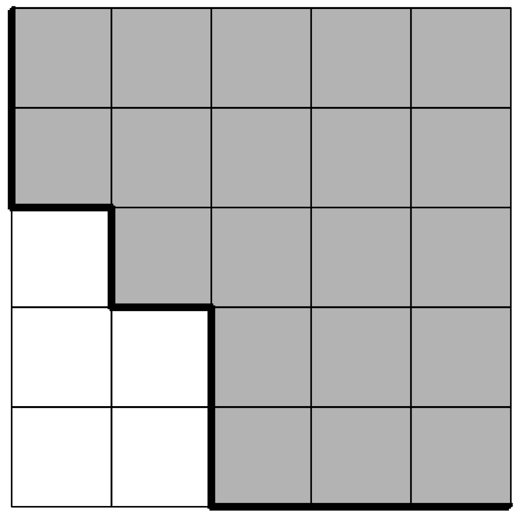

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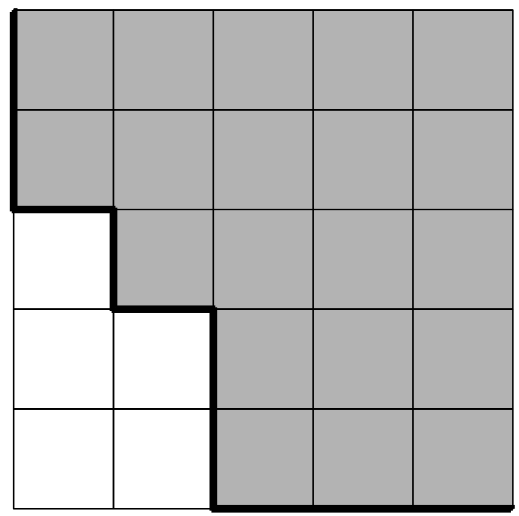

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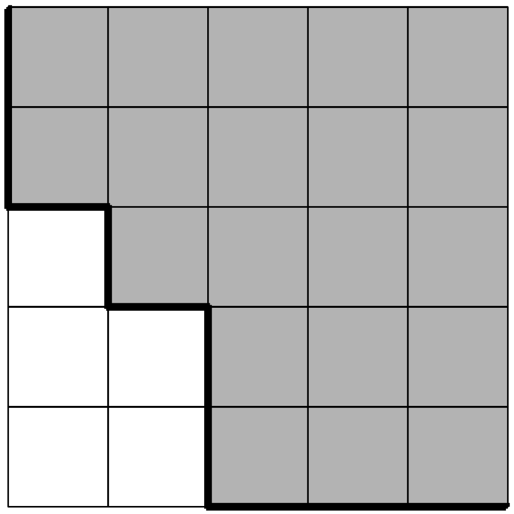

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To share the idea that we can study Hessenberg varieties from several different perspectives.

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§0. Paving by affines

Theorem(Tymoczko '06)

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For any X and any h, Hess(X, h) is paved by complex affine spaces.

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- $H^{\text{odd}}(\text{Hess}(X, h); \mathbb{Z}) = 0$

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- other Lie types ?

§1. cohomology, $\mathfrak{S}_n$-reps, hyperplane arr.

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e.g. $h = (n, n, \cdots, n)$: $\text{Hess}(S,h) = Fl(\mathbb{C}^n)$,

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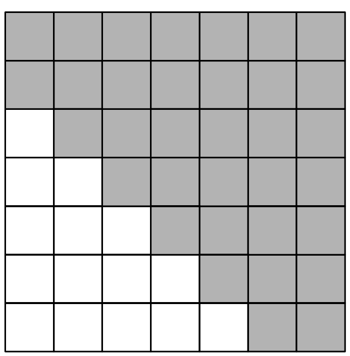

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$\mathfrak{S}_n \curvearrowright H^*(\mathrm{Hess}(S,h); \mathbb{C})$: representation of symmetric group

(monodromy action (geometry), or
torus equiv cohomology (combinatorics))

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e.g. $h = (n,n,\cdots,n)$: $\mathrm{Hess}(N,h) = Fl(\mathbb{C}^n)$,

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	Hess(N, h)	Hess(S, h)

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dimension	$\sum_{i=1}^{n}(h(i) - i)$	$\sum_{i=1}^{n}(h(i) - i)$

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$h = (2, 3, 4, \ldots, n, n)$	Peterson variety	permutohedral variety

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Theorem(A-Harada-Horiguchi-Masuda '15)

$$H^*(\text{Hess}(N, h); \mathbb{C}) \qquad H^*(\text{Hess}(S, h); \mathbb{C})$$

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Theorem(A-Harada-Horiguchi-Masuda '15)

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There is a natural ring isomorphism

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- Hyperplane arrangements
- Shareshian-Wachs conjecture
- $H^*(\text{Hess}(S,h);\mathbb{C})$
- Schubert polynomials and $H^*(\text{Hess}(N,h);\mathbb{C})$
- (non-regular) semisimple Hessenberg varieties

§2. Algebro-geometric aspects

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$$\Gamma(t) = \begin{pmatrix} tc_1 & 1 & & \\ & tc_2 & 1 & \\ & & tc_3 & 1 \\ & & & tc_4 \end{pmatrix} \quad (t \in \mathbb{C})$$

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After the work of Insko-Yong and Anderson-Tymoczko,

Theorem(A-DeDieu-Galetto-Harada '16)

(1)

(2)

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After the work of Insko-Yong and Anderson-Tymoczko,

Theorem(A-DeDieu-Galetto-Harada '16)

(1) $X = \mathrm{Hess}(N, h)$ is a local complete intersection.

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e.g. Peterson variety is a flat limit of permutohedral variety

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- formula for the K-class [Hess(R,h)]

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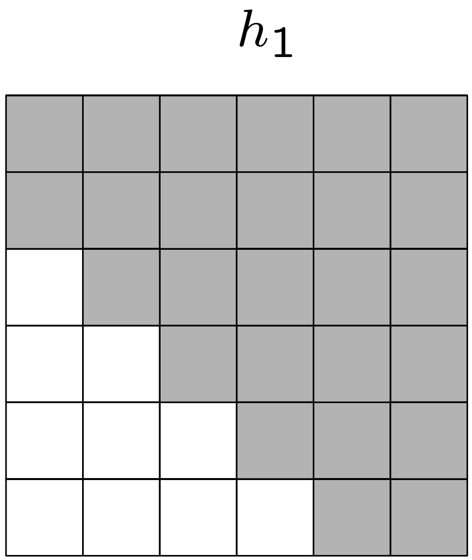
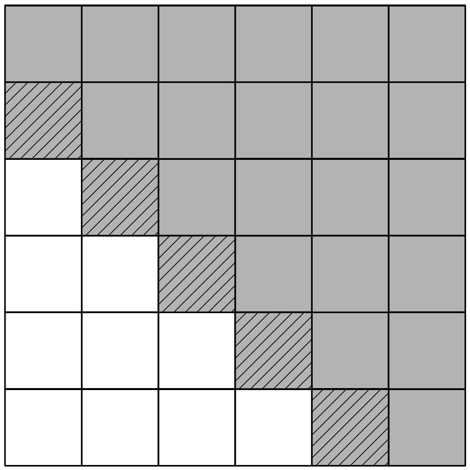


(weak) Fano Hessenberg varieties

h_1



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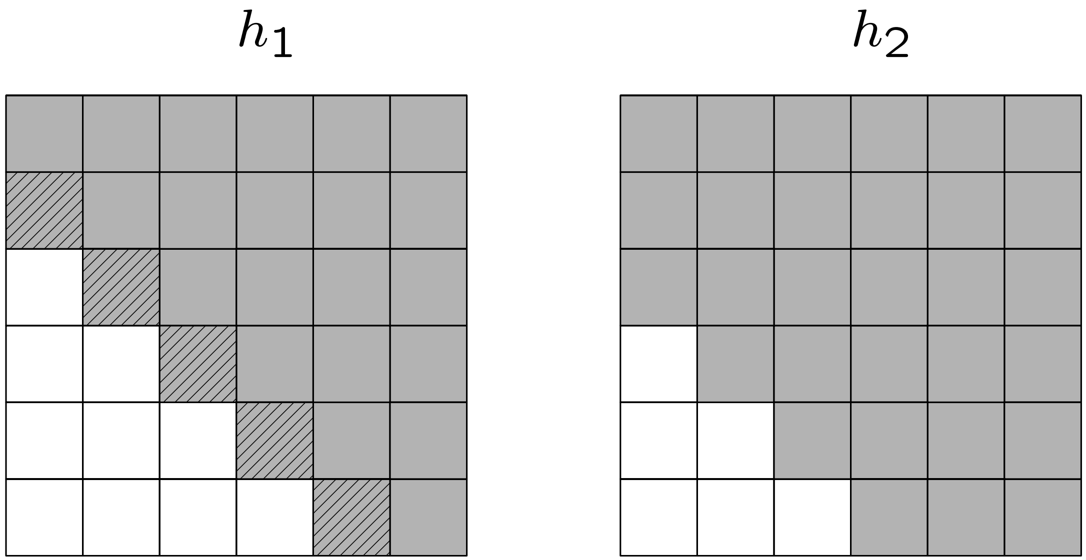

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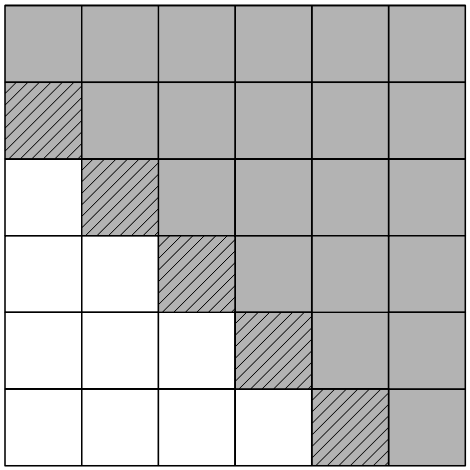
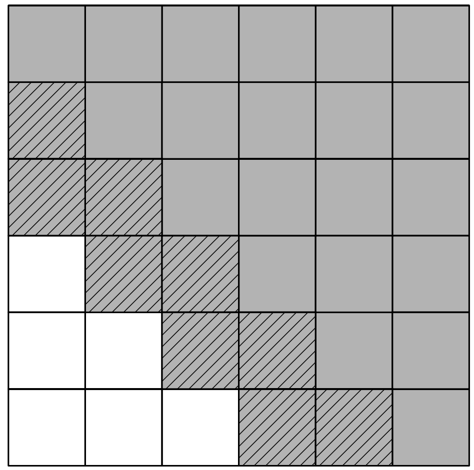


h_2



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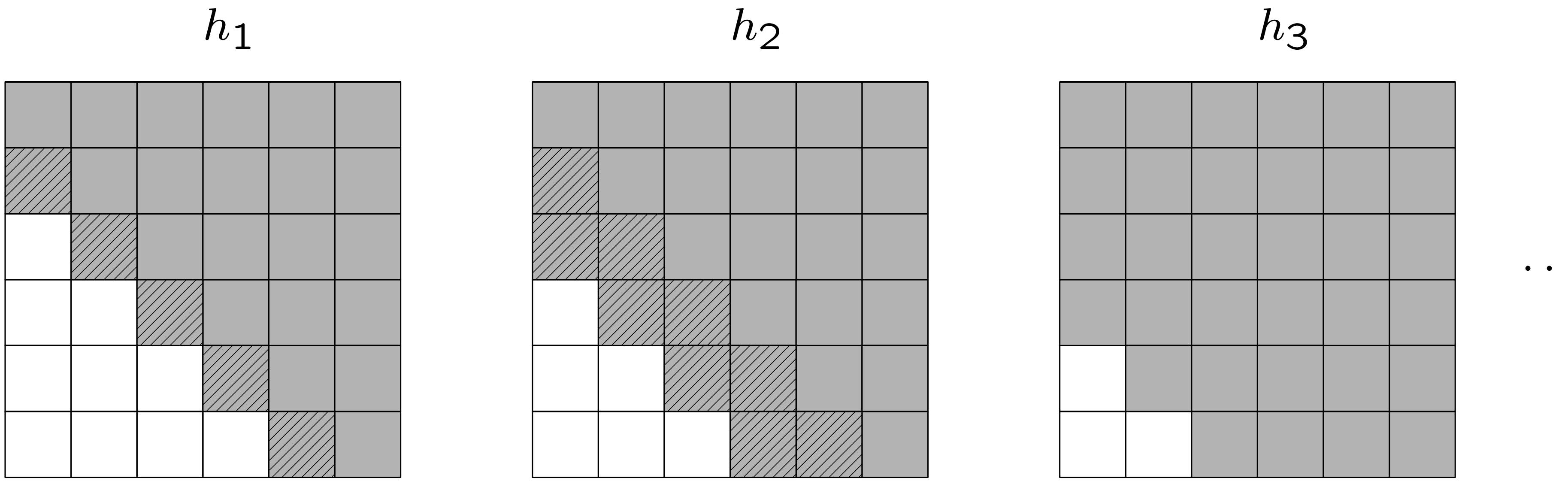

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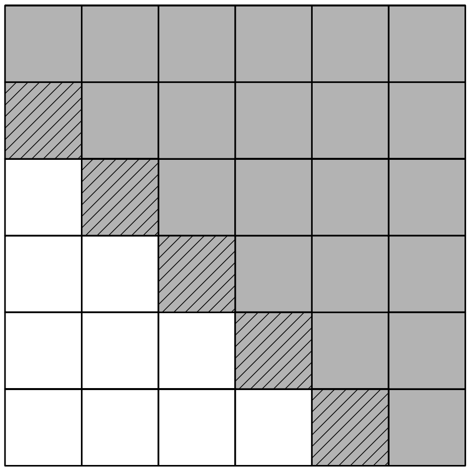

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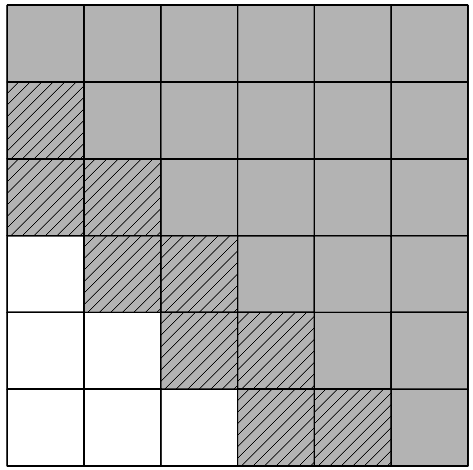
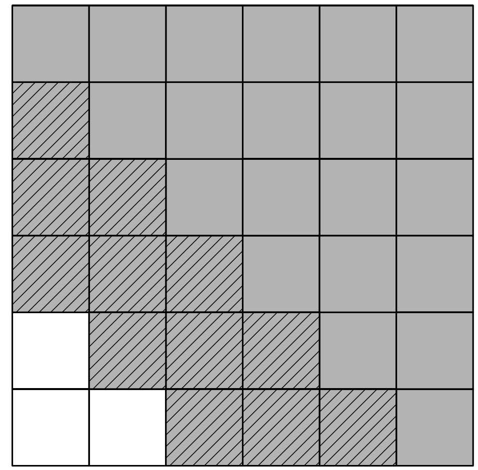


h_3



$\cdots$

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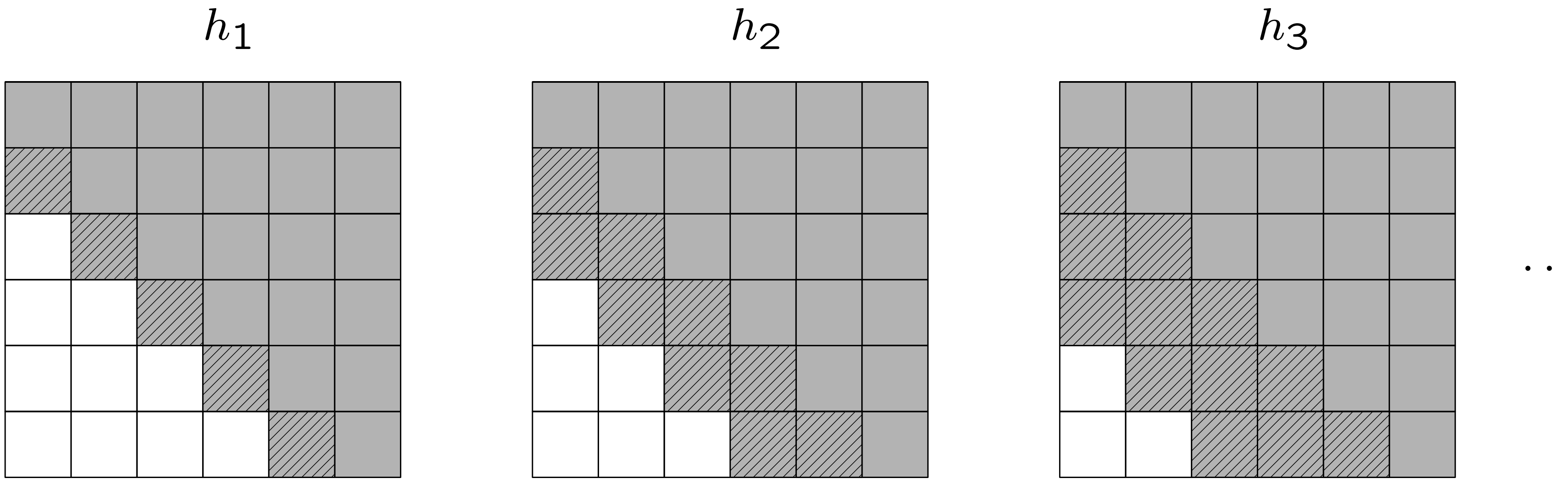

$\rightsquigarrow$ Similarly, we define h_k $(1 \leq k \leq n-1)$.

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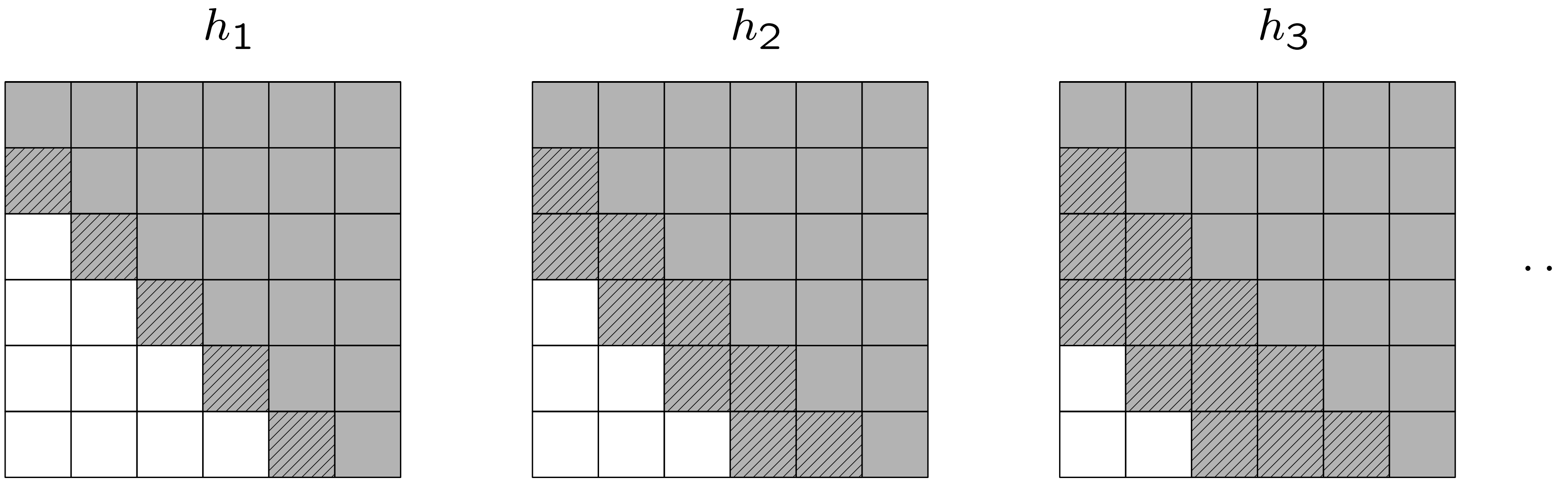

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- Hess(S, h_1) = permutohedral variety,
- Hess$(S, h_{n-1}) = Fl(\mathbb{C}^n)$.

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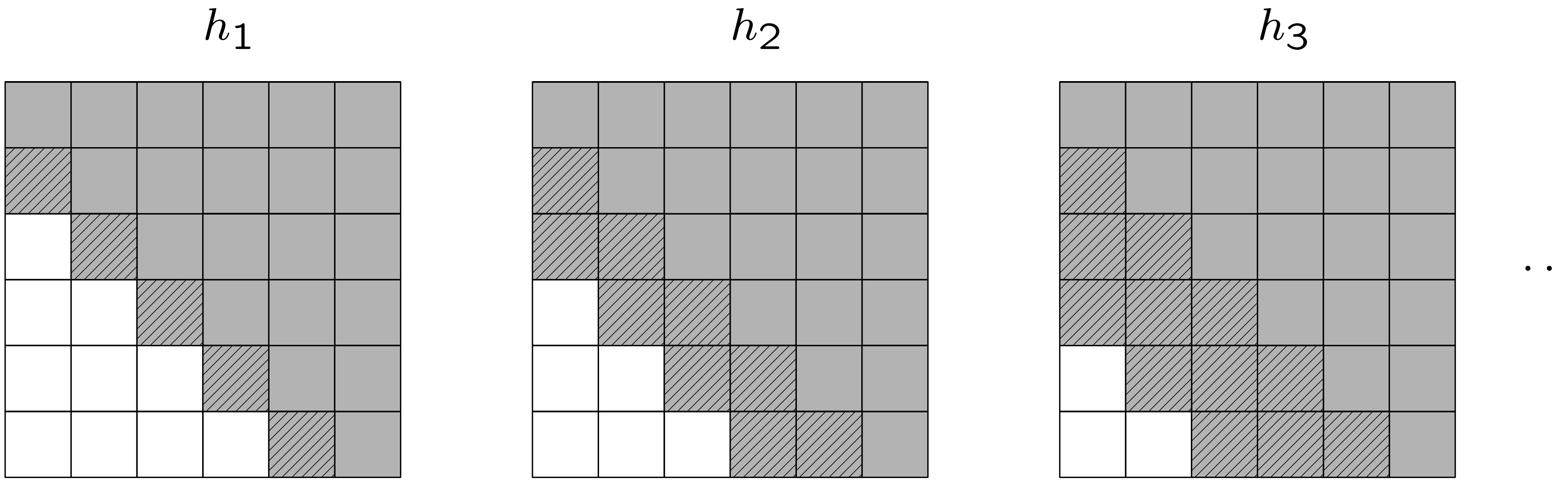

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Hess(S, h_k) is weak Fano $(\forall k)$.

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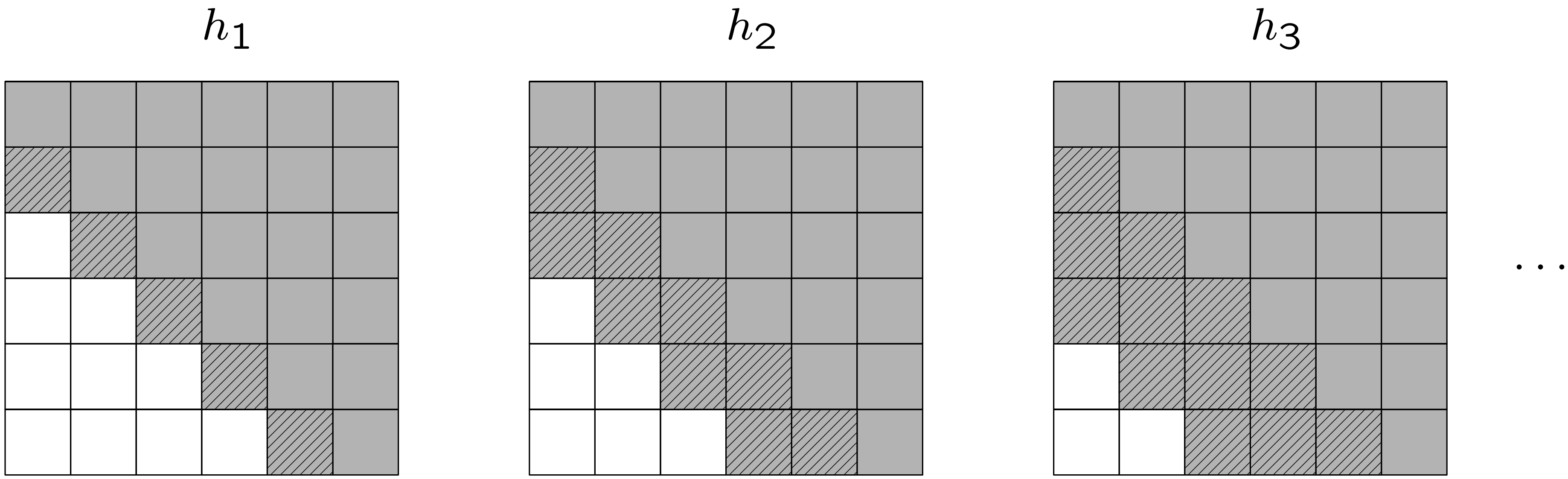

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Q. Can we construct explicit integrable system on Hess(S, h_k)?

§2. More developmemnts

- Harada-Precup : deeper study of $\mathfrak{S}_n$-representation on $H^*(\mathrm{Hess}(S,h))$
  (verifying Stanley-Stembridge conjecture in certain cases)

- Drellich : poset of Hessenberg varieties

- Ayzenberg-Buchstaber : topological twin of $\mathrm{Hess}(S,h)$

- A-Crooks : a holomorphic completely integrable system on
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Thank you for your attention!