Stability of nonlinear gravity waves in the atmosphere

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Horizontally homogeneous modulation equations for nonlinear inviscid Boussinesq waves in uniformly stratified atmosphere

$$\begin{aligned}
\partial_T k_z + \partial_Z\big(\hat{\omega}(k_z) + K_x u\big) &= 0 \\
\partial_T a + \partial_Z\big(\hat{\omega}'(k_z) a\big) &= 0 \\
\partial_T u + \partial_Z\big(\hat{\omega}'(k_z) K_x a\big) &= 0
\end{aligned} \tag{1}$$

where

$$\hat{\omega}(k_z) = \frac{N K_x}{\sqrt{K_x^2 + k_z^2}} \tag{2}$$

is the non-hydrostatic intrinsic frequency (Muraschko et al., 2015).

- Modulation equations in vector form for $y = (k_z, a, u)^T \in \mathbb{R}^3$

$$\partial_T y + \partial_Z F(y) = 0 \qquad (3)$$

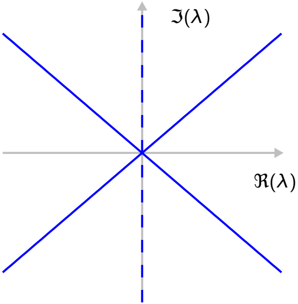

Figure: Stable ($\hat{\omega}''(K_z) \geq 0$) and unstable ($\hat{\omega}''(K_z) < 0$) spectrum of operator $\mathcal{L}_Y$. This is known as modulational instability.

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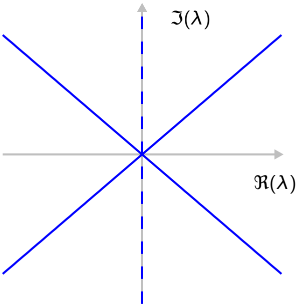

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- Linearize for stability

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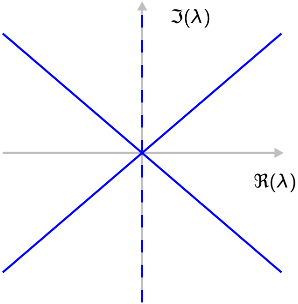

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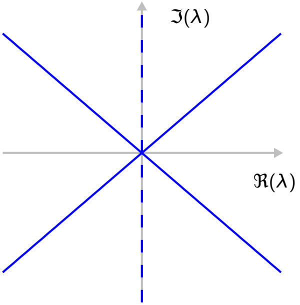
$$y(Z, T) = y(Z) e^{\lambda T} \qquad (5)$$



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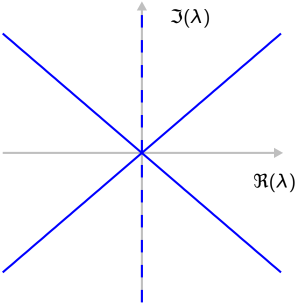
- Translates (4) to eigenvalue problem, $\mathcal{L}_Y y = \lambda y$, for operator $\mathcal{L}_Y = -DF(Y)\partial_Z$ on L^2.



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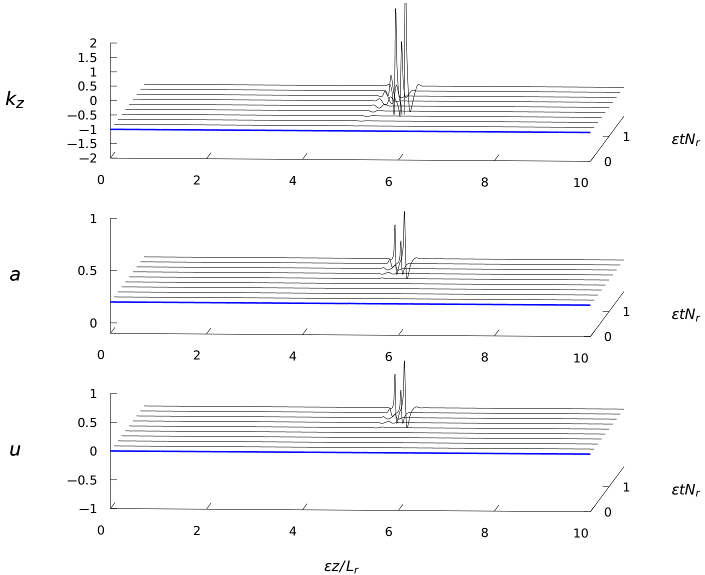
Absolute instability of the inviscid Boussinesq plane wave



Boussinesq does not account for varying background density but pseudo-incompressible (Durran, 1989) can:

$$\begin{aligned}
\partial_T k_z + \partial_Z\big(\hat{\omega}(k_z) + K_x u\big) &= 0 \\
\partial_T a + \partial_Z\big(\hat{\omega}'(k_z)a\big) &= -\eta\hat{\omega}'(k_z)a \\
\partial_T u + \partial_Z\big(\hat{\omega}'(k_z)K_x a\big) &= -\eta\hat{\omega}'(k_z)K_x a
\end{aligned} \tag{6}$$

where the background density is

$$\rho(Z) = \rho_0 e^{\eta Z} \tag{7}$$

in the isothermal atmosphere.

- The pseudo-incompressible modulation equations are solved by traveling wave fronts.

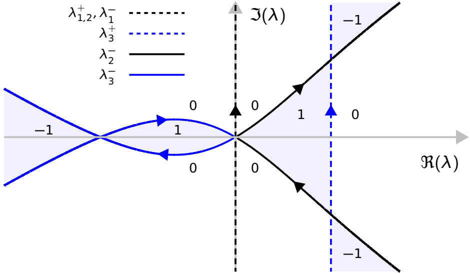


Figure: Unconditionally unstable essential spectrum of operator $\mathcal{L}_Y$.

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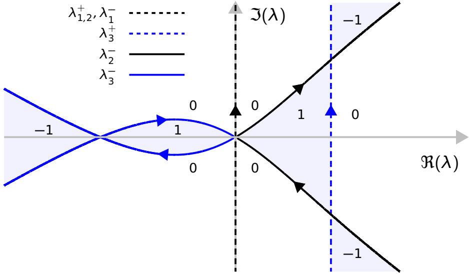
- Assess stability be linearization as before.



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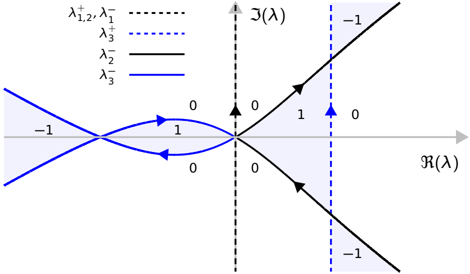
- The pseudo-incompressible modulation equations are solved by traveling wave fronts.
- Assess stability be linearization as before.
- Solve eigenvalue problem for $\mathcal{L}_Y$ in terms of Fredholm operator theory.



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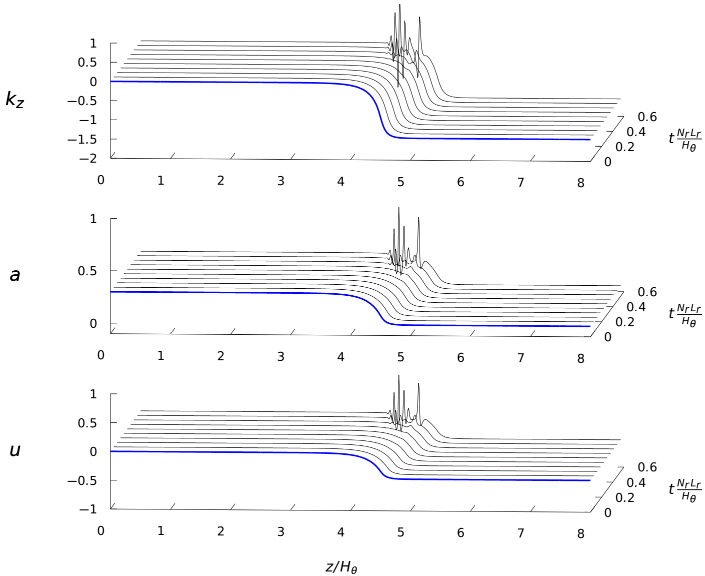
Absolute instability of the upward-traveling wave front



- Inviscid spectra allow for arbitrarily large instability growth rates $\Re(\lambda)$ which is evidence for ill-posedness.

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- Regularization is found by including dissipation.

$$
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&\partial_T k_z + \partial_Z\big(\hat{\omega}(k_z) + K_x u\big) = 0 \\
&\partial_T a + \partial_Z\big(\hat{\omega}'(k_z) a\big) = -\eta\hat{\omega}'(k_z) a - \Lambda(K_x^2 + k_z^2) a \\
&\partial_T u + \partial_Z\big(\hat{\omega}'(k_z) K_x a\big) = -\eta\hat{\omega}'(k_z) K_x a
\end{aligned} \tag{8}
$$

- The dissipative Grimshaw modulation equations are solved by up(down)ward-traveling wave packets.

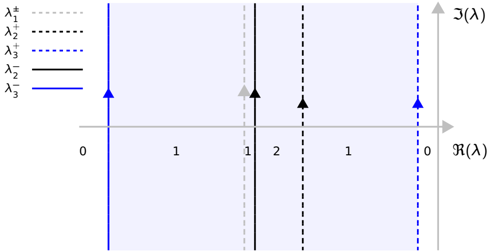
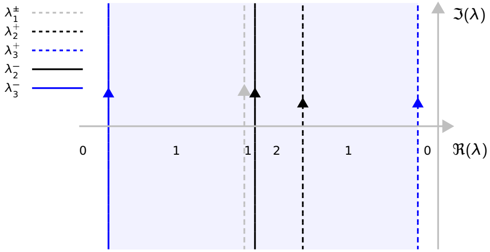


Figure: Stable essential spectrum of operator $\mathcal{L}_Y$ in weighted space L^2_α.

- The dissipative Grimshaw modulation equations are solved by up(down)ward-traveling wave packets.
- Upward-traveling wave packets are transient unstable if

$$C > \hat{\omega}'(K_z^+) > \hat{\omega}'(K_z^-) > 0$$

and absolute unstable otherwise.



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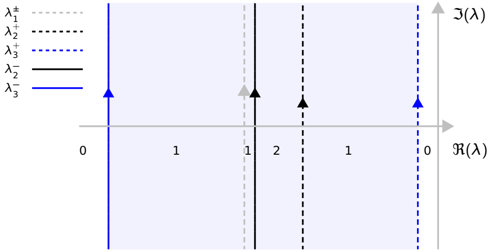
  and absolute unstable otherwise.
- Downward-traveling wave packets are unconditionally transient unstable.



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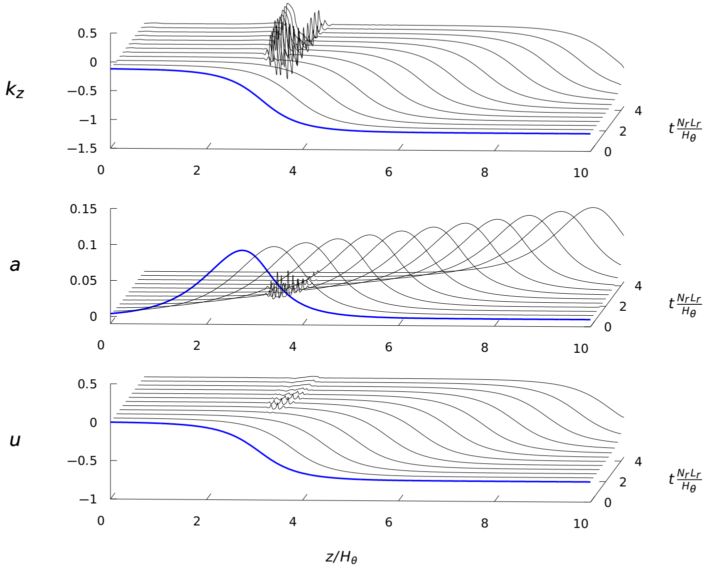
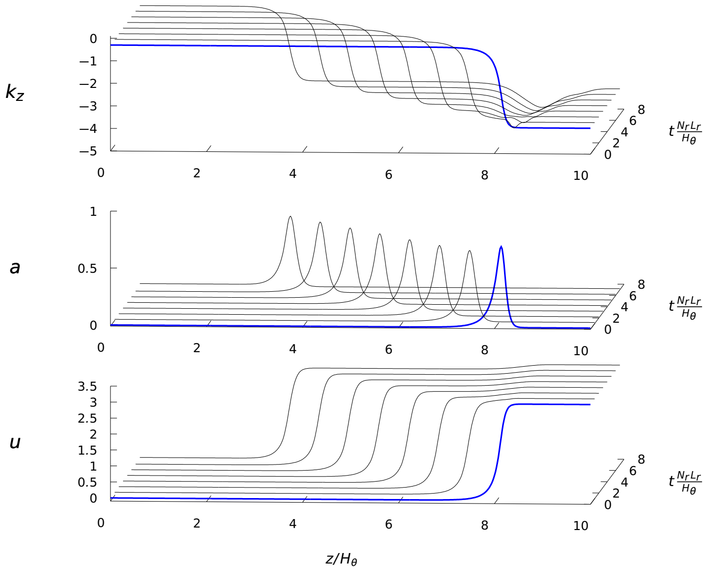
Transient instability of the upward-traveling wave packet



Transient instability of the downward-traveling wave packet

MS-GWaves



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Thank you for your attention!