Bi-s^*-Concave Distributions

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Shape Constrained Methods:
Inference, Applications, and Practice

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Outline

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1. A Mysterious condition: quantile process theory

- Let $X_1, \ldots, X_n$ be i.i.d. F,
  absolutely continuous with density f.
- Let $\mathbb{F}_n$ denote the empirical distribution function of the X_i's: $\mathbb{F}_n(x) = n^{-1} \sum_{i=1}^n 1\{X_i \le x\}$.
- Let $\mathbb{F}_n^{-1}$ denote the empirical quantile function, and let F^{-1} denote the population quantile function,
  where $F^{-1}(t) \equiv \inf\{x : \ F(x) \ge t\}$, $0 < t < 1$.

- The standardized quantile process $\mathbb{Q}_n$ is defined by

$$\mathbb{Q}_n(t) \equiv g(t)\sqrt{n}(\mathbb{F}_n^{-1}(t) - F^{-1}(t)) \quad \text{for} \quad 0 < t < 1.$$

where

$$g(t) \equiv f(F^{-1}(t)).$$

is the *density quantile function* or *isoperimetric profile function*.

Now suppose that $X_i \equiv F^{-1}(\xi_i)$ for $1 \le i \le n$ where:

- $\xi_1, \ldots, \xi_n$ are i.i.d. Uniform$(0,1)$ random variables.
- $\mathbb{G}_n$ is the empirical d.f. of the ξ_i's.
- $\mathbb{G}_n^{-1}$ is the empirical quantile function of the ξ_i's.
- $\mathbb{V}_n(t) \equiv \sqrt{n}(\mathbb{G}_n^{-1}(t) - t)$ is the *uniform quantile process*.

Csörgő and Révész (1978) imposed the following mysterious condition in their study of the asymptotic equivalence of $\mathbb{V}_n$ and $\mathbb{Q}_n^0$, the version of $\mathbb{Q}_n$ with the X_i's constructed in terms of the ξ_i's as above.
Let $J(F) \equiv \{x \in \mathbb{R} : \ 0 < F(x) < 1\}$, and assume

$$\gamma(F) \equiv \sup_{x \in J(F)} F(x)(1 - F(x)) \frac{|f'(x)|}{f^2(x)} \le \text{some } M < \infty. \qquad (1)$$

Then, for any (small) $r > 0$

$$\|\mathbb{Q}_n^0 - \mathbb{V}_n\|_\infty = O\left(n^{-1/2}(\text{loglog}n)^M(\text{log}n)^{(1+r)(M-1)}\right) \quad \text{a.s.}$$

Define the $CR(x)$ and $CR_m(x)$ functions as follows:

$$CR(x) \equiv F(x)(1 - F(x))\frac{|f'(x)|}{f^2(x)},$$

$$CR_m(x) \equiv \min\{F(x), 1 - F(x)\}\frac{|f'(x)|}{f^2(x)}.$$

The condition (1) has also appeared in the study of transportation distances between the empirical measure and true measure $\mathbb{P}_n$ and P on $\mathbb{R}$; see e.g.

- del Barrio, E., Giné, E., and Utzet, F. (2005). Asymptotics for L2 functionals of the empirical quantile process, with applications to tests of fit based on weighted Wasserstein distances. *Bernoulli* **11**, 131 - 189.

- Bobkov, S. and Ledoux, M. (2014). One - Dimensional empirical measures, order statistics, and Kantorovich transport distances. *Memoirs of the American Mathematical Society*, to appear. (especially see p. 45)

2. Log-concavity and $s-$concavity

If a density f on $\mathbb{R}^d$ is of the form

$$f(x) \equiv f_\varphi(x) = \begin{cases} (\varphi(x))^{1/s}, & \varphi \ \textit{convex}, \ \text{if } s < 0 \\ \exp(-\varphi(x)), & \varphi \ \textit{convex}, \ \text{if } s = 0 \\ (\varphi(x))^{1/s}, & \varphi \ \textit{concave}, \ \text{if } s > 0, \end{cases}$$

then f is **s-concave**.

The classes of all densities f on $\mathbb{R}^d$ of these forms are called the classes of $s-$*concave* densities, $\mathcal{D}_s$. The following inclusions hold: if $-\infty < s < 0 < r < \infty$, then

$$\mathcal{D}_{-\infty} \supset \mathcal{D}_s \supset \mathcal{D}_0 \supset \mathcal{D}_r \supset \mathcal{D}_\infty$$

Properties of s-concave densities:

- Every $s-$concave density f is quasi-concave.
- The Student t_ν density, $t_\nu \in \mathcal{P}_s$ for $s \le -1/(1+\nu)$. Thus the Cauchy density ($= t_1$) is in $\mathcal{P}_{-1/2} \subset \mathcal{P}_s$ for $s \le -1/2$.
- The classes $\mathcal{D}_s$ have interesting closure properties under convolution and marginalization which follow from the Borell-Brascamp-Lieb inequality: let $0 < \lambda < 1$, $-1/d \le s \le \infty$, and let $f, g, h : \mathbb{R}^d \to [0, \infty)$ be integrable functions such that

$$h((1-\lambda)x + \lambda y) \ge M_s(f(x), g(x), \lambda) \quad \text{for all} \quad x, y \in \mathbb{R}^d$$

where

$$M_s(a, b, \lambda) = ((1-\lambda)a^s + \lambda b^s)^{1/s}, \qquad M_0(a, b, \lambda) = a^{1-\lambda} b^\lambda.$$

Then

$$\int_{\mathbb{R}^d} h(x)dx \ge M_{s/(sd+1)} \left(\int_{\mathbb{R}^d} f(x)dx, \int_{\mathbb{R}^d} g(x)dx, \lambda \right).$$

If p is an $s-$concave density with corresponding probability measure P, then for Borel sets A, B

$$\begin{aligned} P(\lambda A + (1-\lambda)B) &= \int_{\lambda A + (1-\lambda)B} p(x)dx \\ &\geq M_{s/(sd+1)}\left(\int_A p(x)dx, \int_B p(x)dx, \lambda\right) \\ &= M_{s/(sd+1)}(P(A), P(B), \lambda) \end{aligned}$$

When $d = 1$, let $s^* \equiv s/(1+s)$ for $s \in (-1, \infty)$. Thus $0^* = 0$.

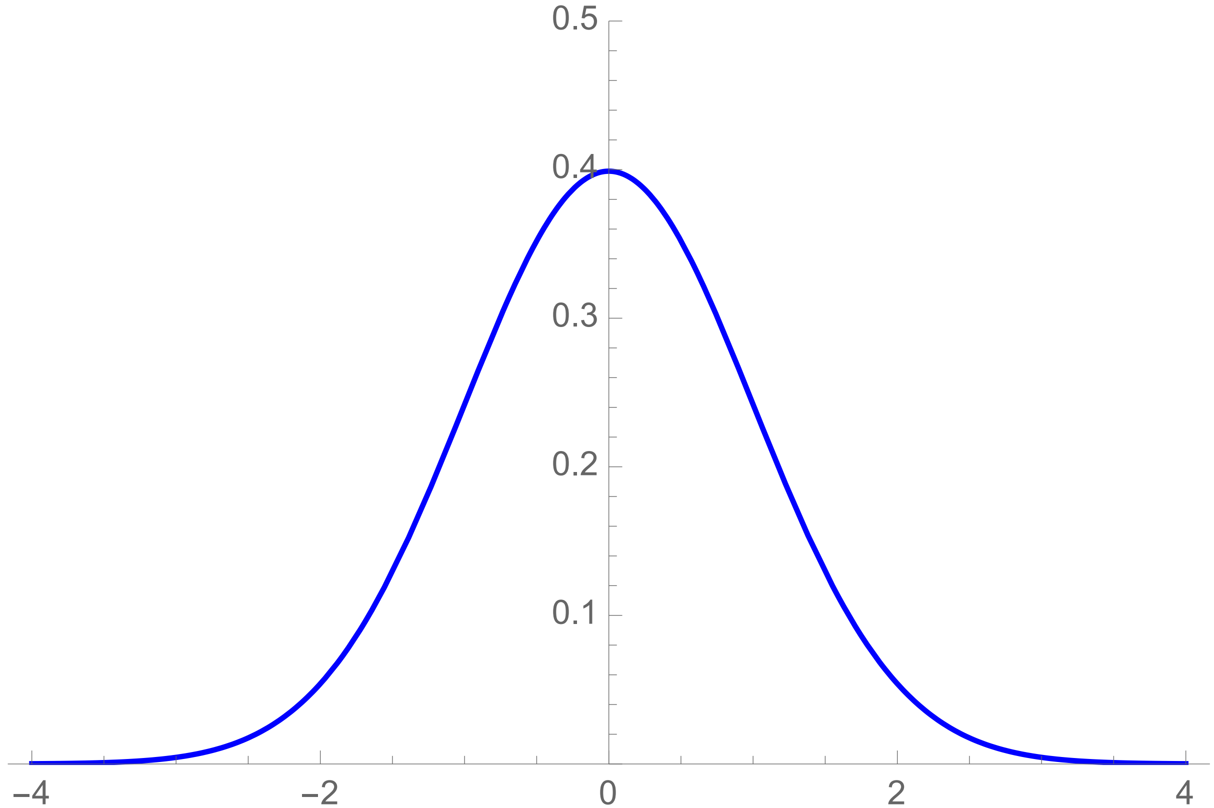
Definition: Dümbgen, Kolesnyk, and Wilke (2017)
A distribution function F on $\mathbb{R}$ is bi-log-concave if both log$F$ and log$(1-F)$ are concave functions from $\mathbb{R}$ to $[-\infty, 0]$.

- DKW (2017) noted that if F has log-concave density $f = F'$, then F is bi-log-concave. This follows from the Borell-Brascamp-Lieb inequality with $d = 1$, $s = 0$, and considering sets of the form $(-\infty, x]$ or (x, ∞).

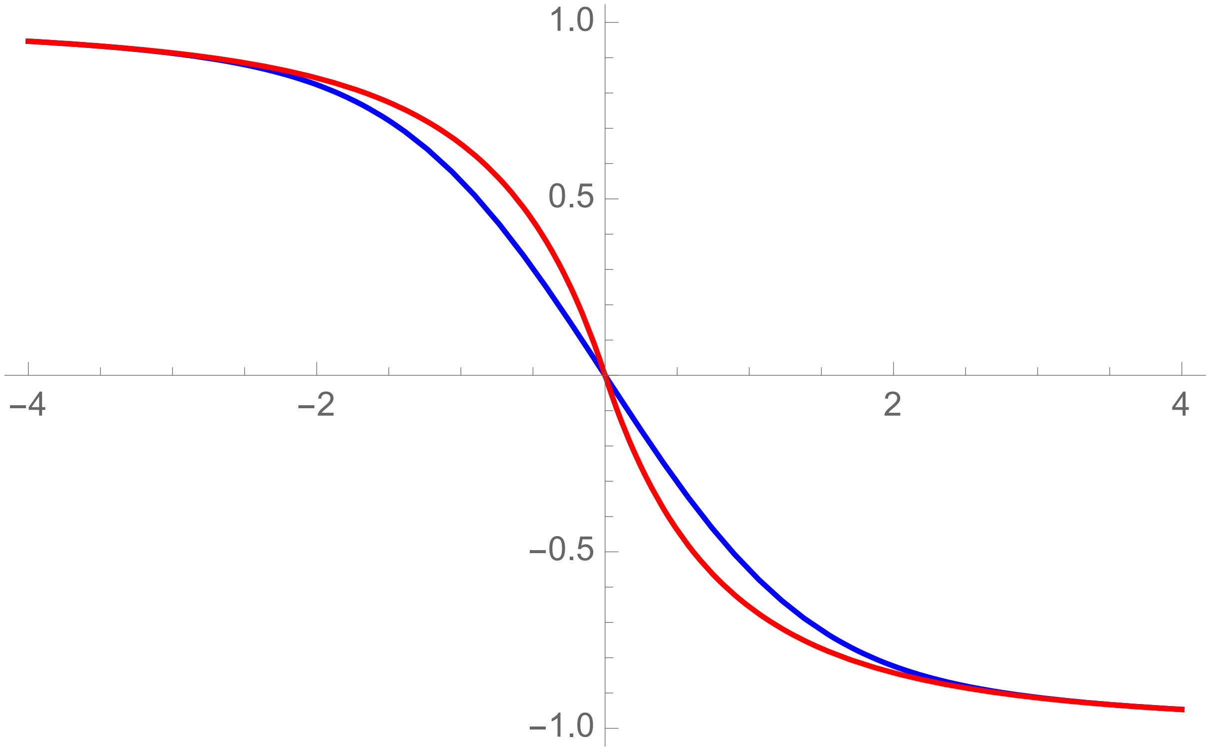
- But ... bi-log-concavity of F is a much weaker constraint:
  - ▷ While any log-concave density is unimodal,
  - ▷ a bi-log-concave distribution function F may have a density with an arbitrary number of modes; e.g.

$$f_{k,a}(x) = \left(1 + \frac{a \sin(2\pi k x)}{k\pi}\right) 1_{[0,1]}(x)/C(k,a).$$

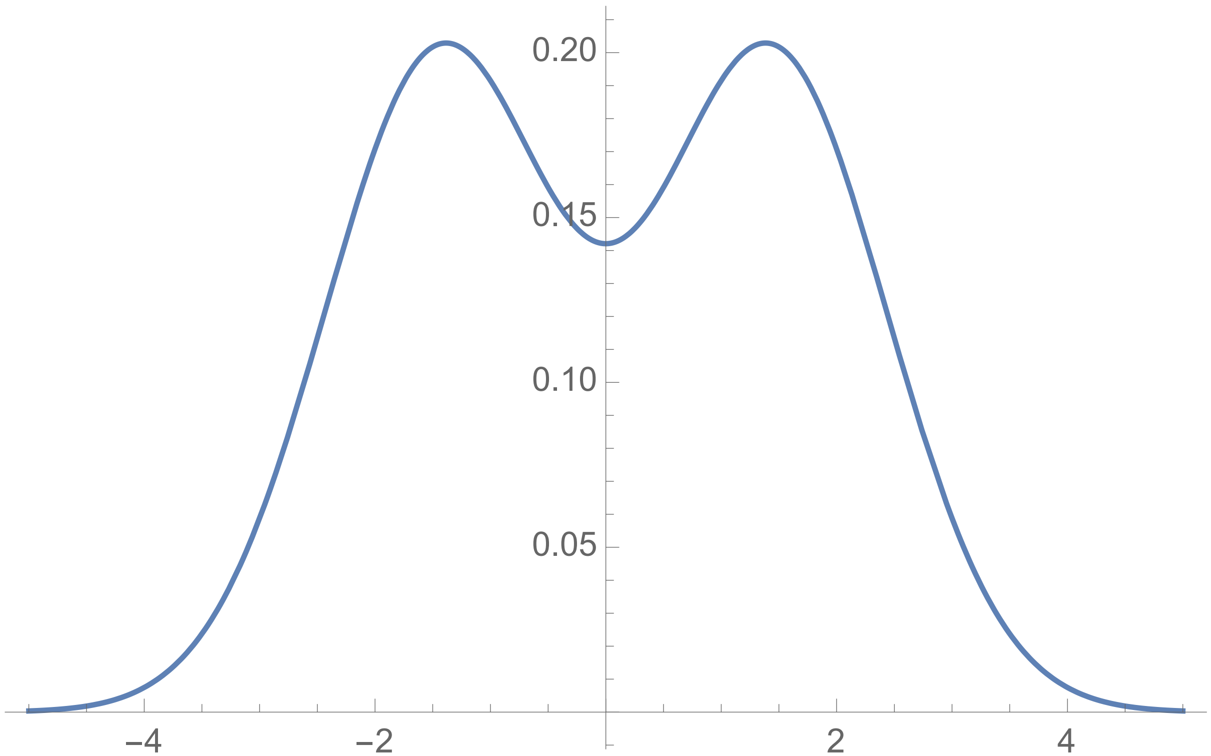
for a small, is bi-log-concave



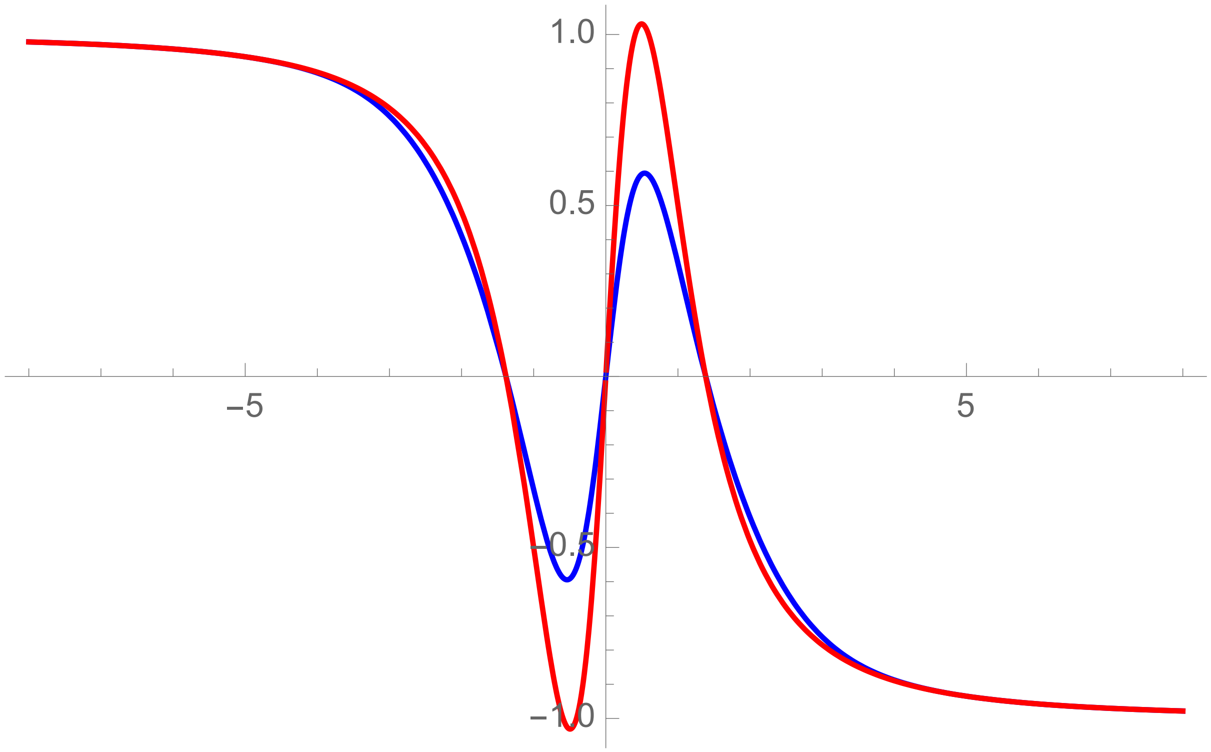
Gaussian density ϕ



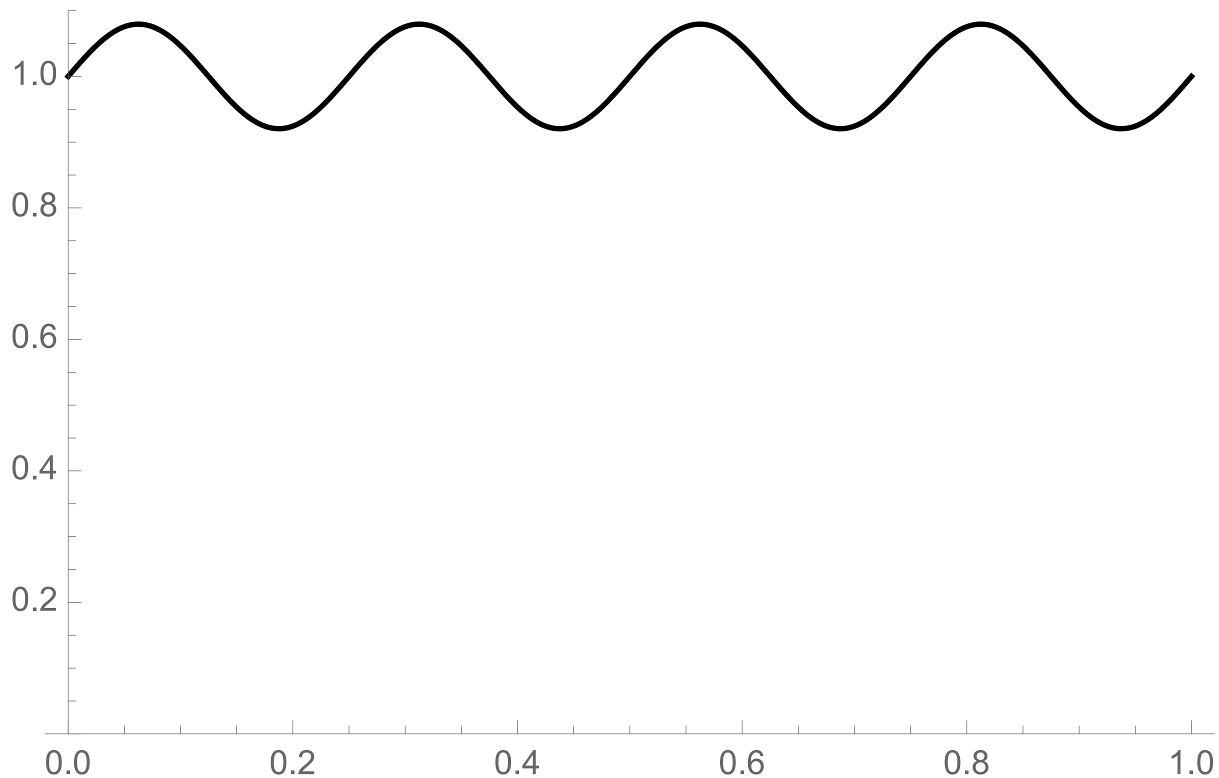
$CR(x)$ and $CR_m(x)$ for $f = \phi$



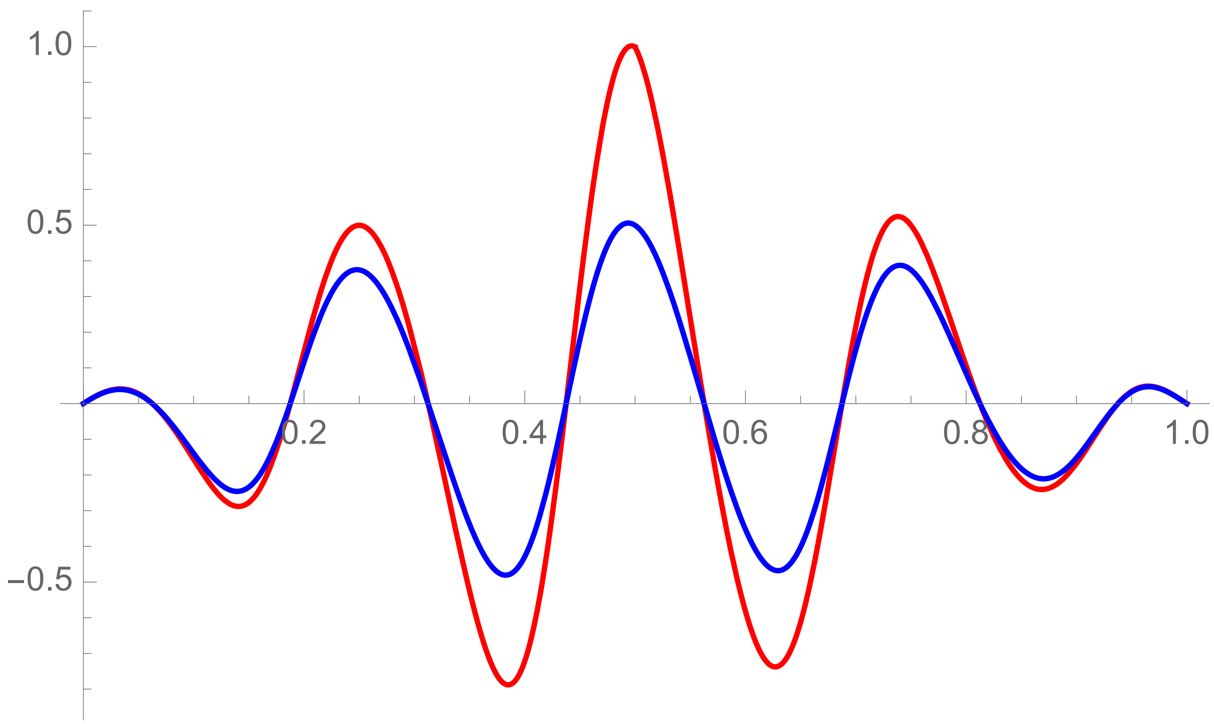
Mixed Gaussian density,
$2^{-1}\phi(x+\delta) + 2^{-1}\phi(x-\delta)$, $\delta = 1.37$



CR functions, mixed Gaussian density:
$2^{-1}\phi(x+\delta) + 2^{-1}\phi(x-\delta)$, $\delta = 1.37$



Multi-modal perturbed uniform density, $f_{k,a}$ with $k = 4$, $a = 0.795$



C-R functions, multi-modal perturbed uniform density:
$f_{k,a}$ with $k = 4$, $a = 0.0795$

Let

$$J(F) \equiv \{x \in \mathbb{R} : \ 0 < F(x) < 1\}.$$

Then a distribution function F is non-degenerate if $J(F) \neq \emptyset$.

Theorem 1. (DKW-2017).
If F is non-degenerate, the following four statements are equivalent:
(i) F is bi-log-concave, and we write
$F \in \mathcal{F}_{blc} = \mathcal{F}_{bi-s^*}$.
(ii) F is continuous on $\mathbb{R}$ and differentiable on $J(F)$ with derivative $f = F'$ such that

$$F(x+t) \begin{cases} \leq F(x)\text{exp}\left(\frac{f(x)}{F(x)}t\right), \\ \geq 1 - (1 - F(x))\text{exp}\left(-\frac{f(x)}{1-F(x)}t\right) \end{cases}$$

for all $x \in J(F)$ and $t \in \mathbb{R}$.

(iii) F is continuous on $\mathbb{R}$ and differentiable on $J(F)$ with derivative $f = F'$ such that the hazard function $f/(1-F)$ is non-decreasing and the reverse hazard function f/F in non-increasing on $J(F)$.

(iv) F is continuous on $\mathbb{R}$ and differentiable on $J(F)$ with bounded and strictly positive derivative $f = F'$. Furthermore, f is locally Lipschitz continuous on $J(F)$ with L^1−derivative $f' = F''$ satisfying

$$\frac{-f^2}{1-F} \leq f' \leq \frac{f^2}{F},$$

or, equivalently,

$$\frac{-f}{1-F} \leq \frac{f'}{f} \leq \frac{f}{F}.$$

Corollary. If F is bi-log-concave

$$\tilde{\gamma}(F) \equiv \sup_{x \in \mathbb{R}} \{F(x) \wedge (1 - F(x))\} \frac{|f'(x)|}{f^2(x)} \le 1,$$

$$\gamma(F) \equiv \sup_{x \in \mathbb{R}} F(x)(1 - F(x)) \frac{|f'(x)|}{f^2(x)} \le 1.$$

3. Questions and some examples

If a density f on $\mathbb{R}^d$ is of the form

$$f(x) \equiv f_\varphi(x) = \begin{cases} (\varphi(x))^{1/s}, & \varphi \ \textit{convex}, \text{ if } s < 0 \\ \exp(-\varphi(x)), & \varphi \ \textit{convex}, \text{ if } s = 0 \\ (\varphi(x))^{1/s}, & \varphi \ \textit{concave}, \text{ if } s > 0, \end{cases}$$

then f is **s-concave**.

The classes of all densities f on $\mathbb{R}^d$ of these forms are called the classes of $s-$*concave* densities, $\mathcal{P}_s$. The following inclusions hold: if $-\infty < s < 0 < r < \infty$, then

$$\mathcal{P}_\infty \subset \mathcal{P}_r \subset \mathcal{P}_0 \subset \mathcal{P}_s \subset \mathcal{P}_{-\infty}$$

Questions:

- **Q1:** What if the density f is $s-$concave with $s \neq 0$. In particular, what if $f \in \mathcal{D}_s$ with $s < 0$ where we know (Borell, Brascamp & Lieb, Rinott, ...)

$$\mathcal{D}_{-\infty} \supset \mathcal{D}_s \supset \mathcal{D}_0 \supset \mathcal{D}_r \supset \mathcal{D}_\infty$$

  for $-\infty < s < 0 < r < \infty$?

- **Q2:** If $f \in \mathcal{D}_s$, is there a class of bi-s^*-concave distribution functions F with the property that F and $1 - F$ are s^*-concave?

- **Q3:** Is there an analogue of Theorem 1 including an analogue of Theorem 1(iv) with the corollary that $\gamma(F)$ is bounded by some function of s?

From Borell, Brascamp, & Lieb, Rinott, we know that if $f \in \mathcal{D}_s$ for $s > -1$, then the measure $P_f(A) = \int_A f d\lambda$ for Borel sets A is $t-$concave with $t = s/(1+s) \equiv s^*$ for $s > -1$. Thus by taking $A = (-\infty, x]$, it follows that $x \mapsto F(x)$ is s^*-concave; similarly, taking $A = [x, \infty)$ it follows that $x \mapsto 1 - F(x)$ is s^*-concave.

Example 1. t_r densities: $s = -1/(1+r)$; $s^* = s/(1+s) = -1/r$. Suppose that

$$f_r(x) = \frac{C_r}{\left(1 + \frac{x^2}{r}\right)^{(r+1)/2}}$$

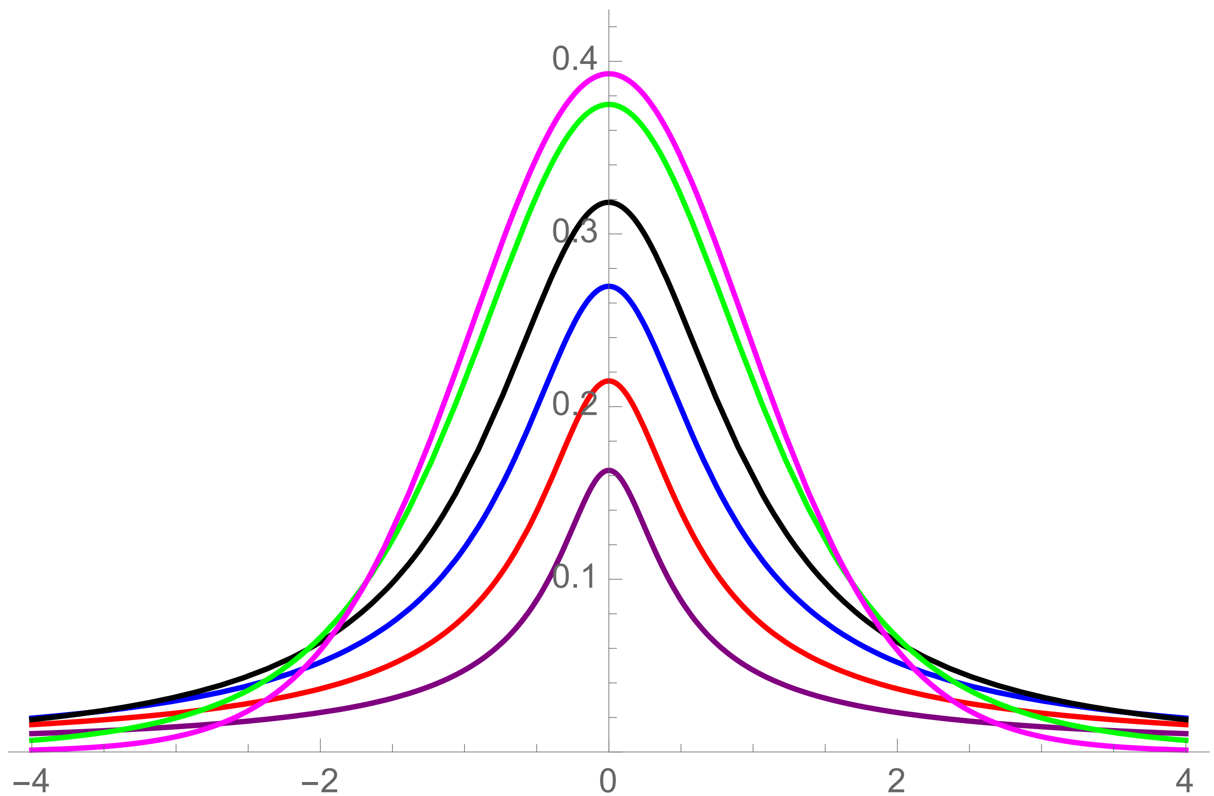
where $C_r = \Gamma((r+1)/2)/(\sqrt{\pi}\Gamma(r/2))$. Then f_r is $s-$concave for all $s \le -(1+r)^{-1}$. By the Borell-Brascamp-Lieb-Rinott correspondence between $s-$concave densities we know that

$$x \mapsto F_r(x)^{s^*} \quad \text{and} \quad x \mapsto (1 - F_r(x))^{s^*}$$

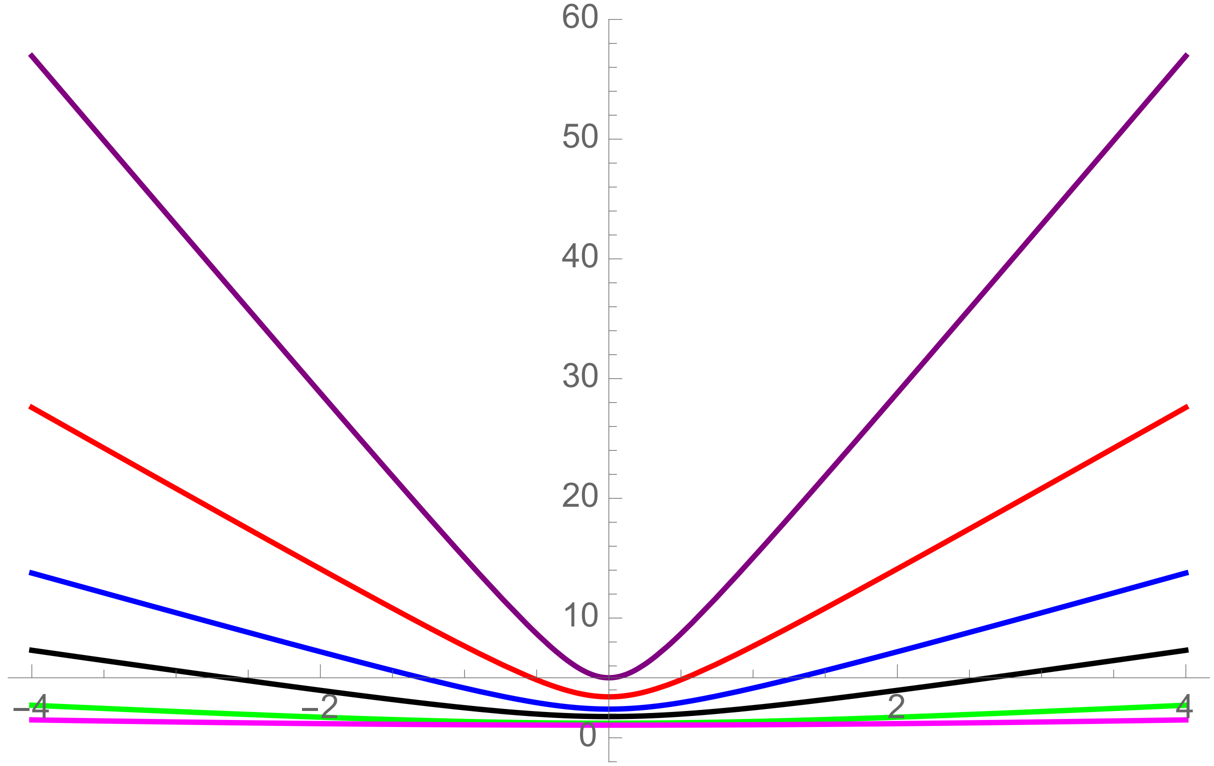
are convex.

Here are some plots, for $r \in \{1/8, 1/4, 1/2, 1, 4, 16\}$,
and hence $s \in \{-8/9, -4/5, -2/3, -1/2, -1/3, -1/5\}$:

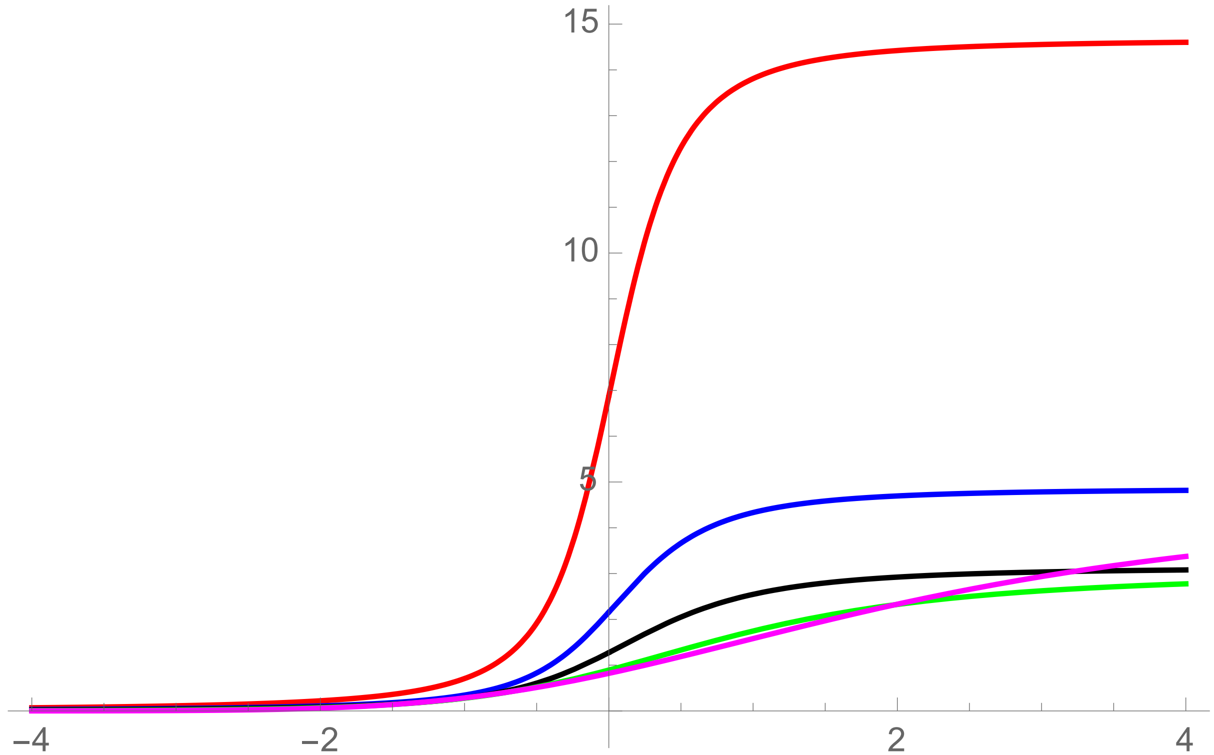
- f_r,
- f_r^s, $s = -1/(1+r)$.
- $f_r/(1-F_r)^{1-s^*}$, $s^* = s/(1+s) = -1/r$.
- $CRm(x, f) \equiv \min\{F(x), 1-F(x)\} f'(x)/f(x)^2$ for $f = f_r$.
  $CR(x, f) \equiv F(x)(1-F(x)) f'(x)/f(x)^2$.



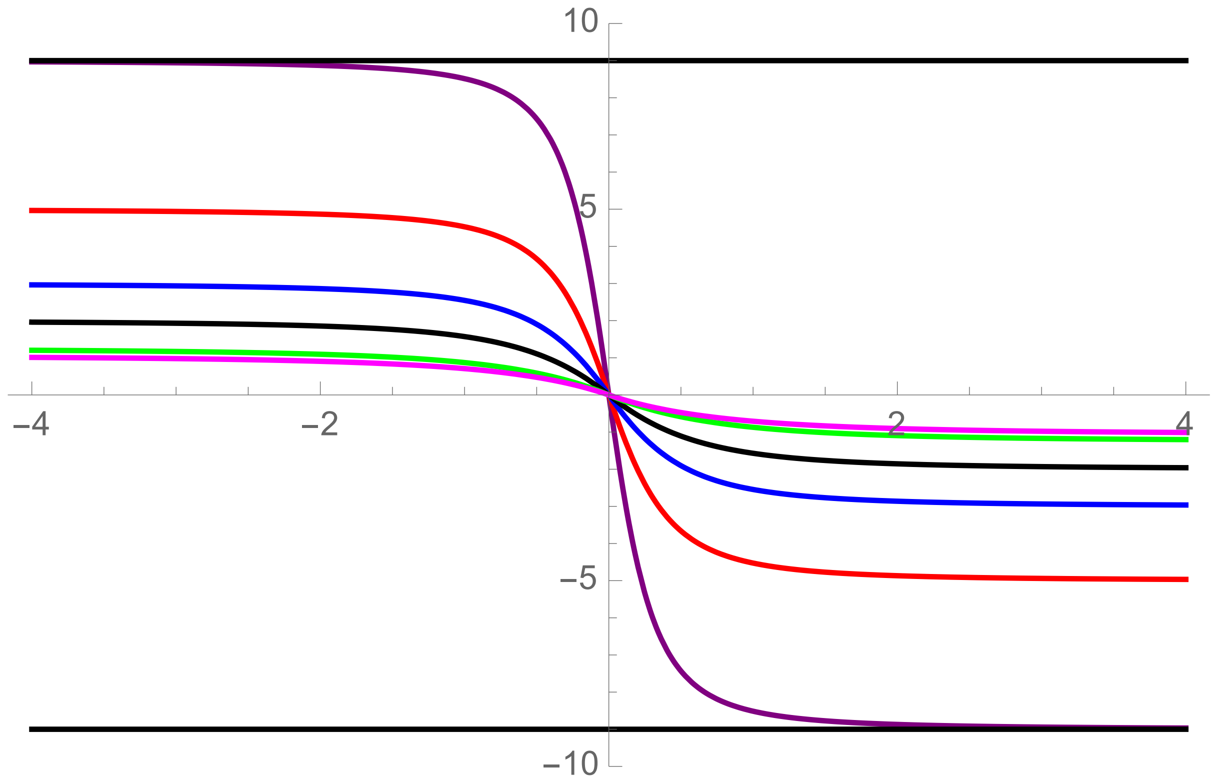
f_r, $r \in \{1/8, 1/4, 1/2, 1, 4, 16\}$ or $s \in \{-8/9, -4/5, -2/3, -1/2, -1/3, -1/5\}$



f_r^s, $r \in \{1/8, 1/4, 1/2, 1, 4, 16\}$ or $s \in \{-8/9, -4/5, -2/3, -1/2, -1/3, -1/5\}$



$f_r/(1-F_r)^{1-s^*}$, $r \in \{1/8, 1/4, 1/2, 1, 4, 16\}$



$$CR_m(x, f) \equiv \min\{F(x), 1 - F(x)\} f'(x)/f(x)^2 \text{ for } f = f_r.$$

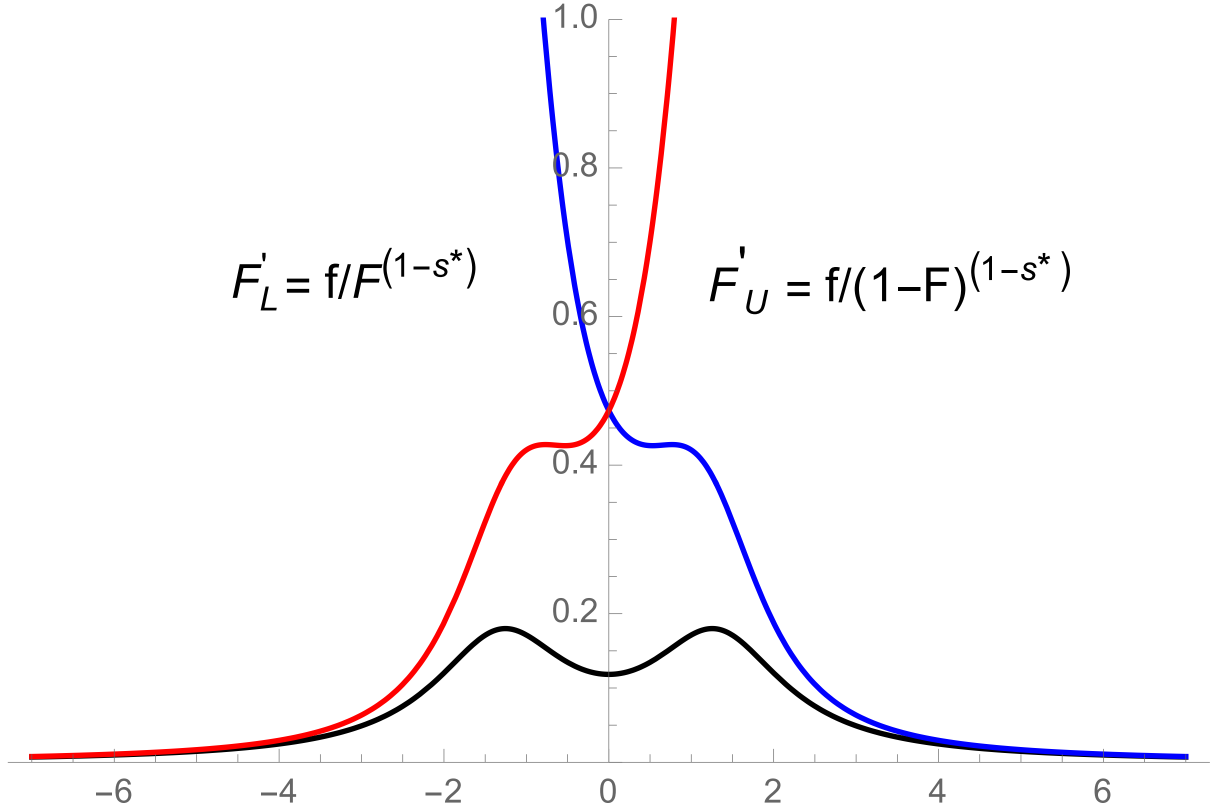
Here the black bounding lines at the top and bottom are given by

$$1 - s^* = \frac{1}{1+s} = \frac{1}{1-8/9} = 9 \quad \text{since} \quad s = -\frac{1}{1+1/8} = -\frac{8}{9}.$$

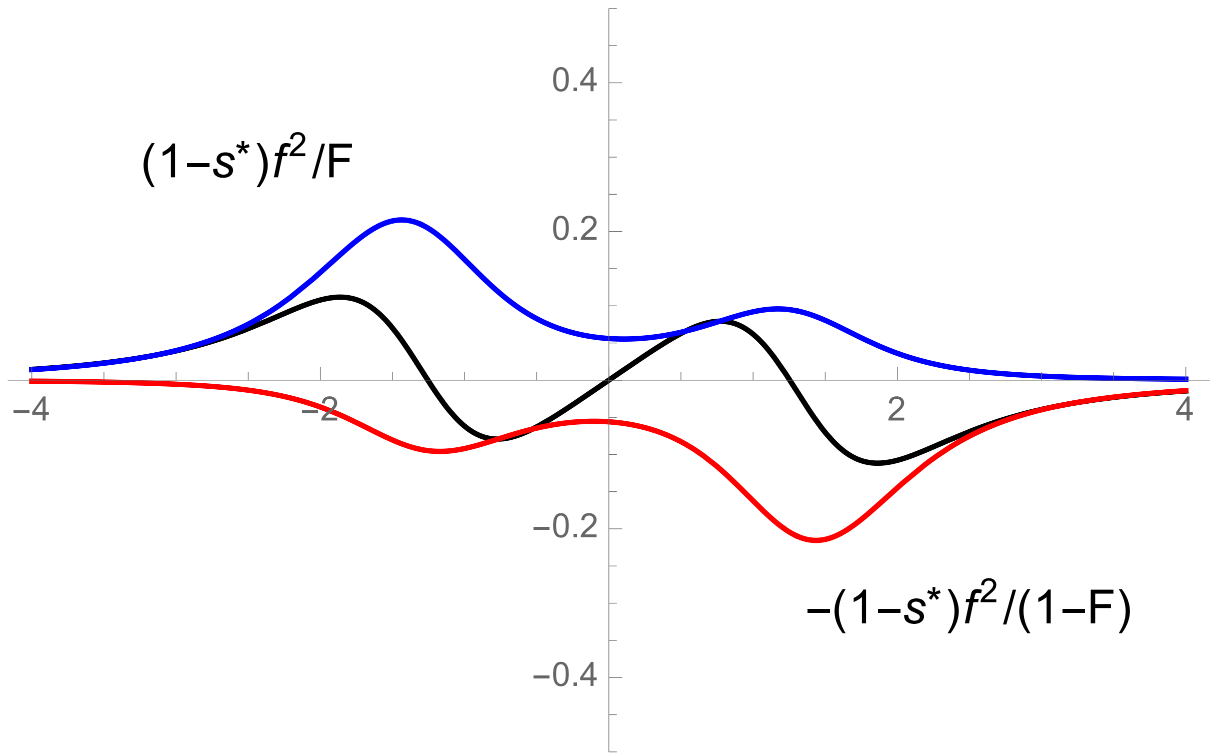
Example 2. (Mixtures of t_r) Suppose that

$$f(x) = f(x; r, \delta) \equiv \frac{1}{2} g_r(x - \delta) + \frac{1}{2} g_r(x + \delta)$$

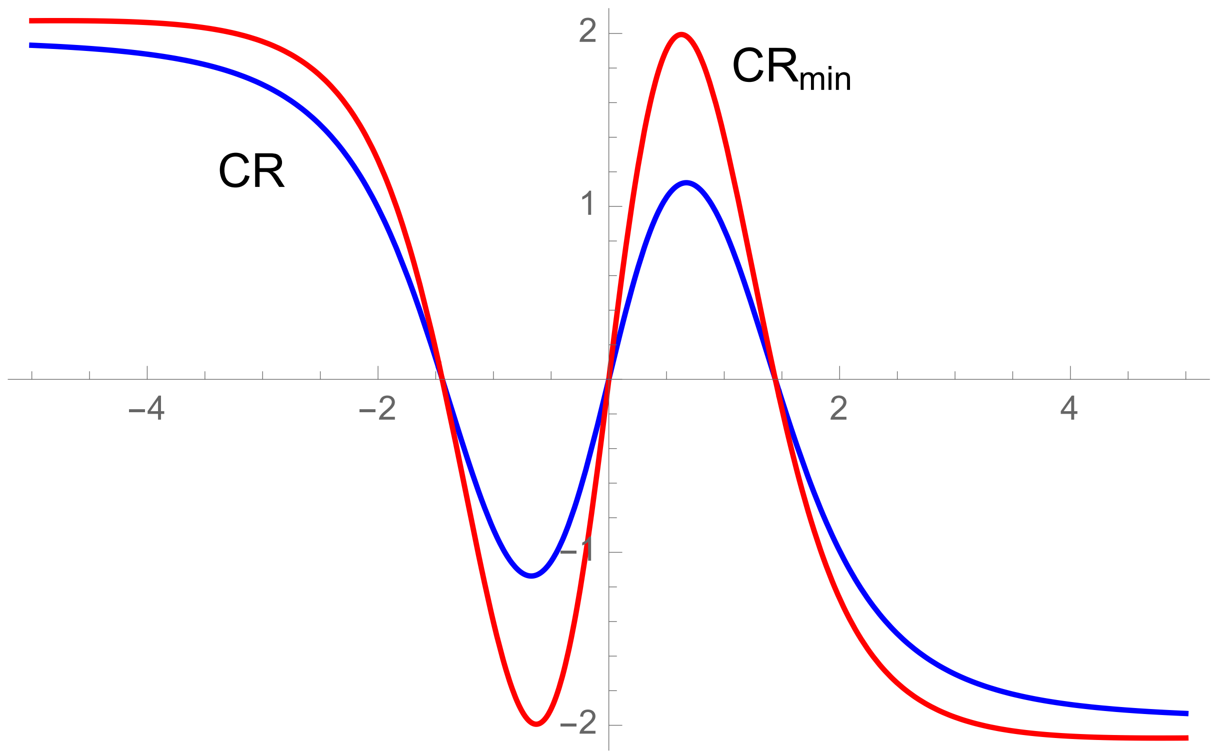
where g_r is the t_r−density in Example 1 and where $\delta > 0$ is not too large. For example here are Figures 2 - 5 of Laha & W (2017). Showing g_r with $r = 1$ and $\delta = 1.3$



$F_L' = f/F^{(1-s^*)}$
$F_U' = f/(1-F)^{(1-s^*)}$
1.0
0.8
0.6
0.4
0.2
−6
−4
−2
0
2
4
6



f', black; bounds f^2/F and $f^2/(1-F)$ from bi$-s^*-$concave characterization

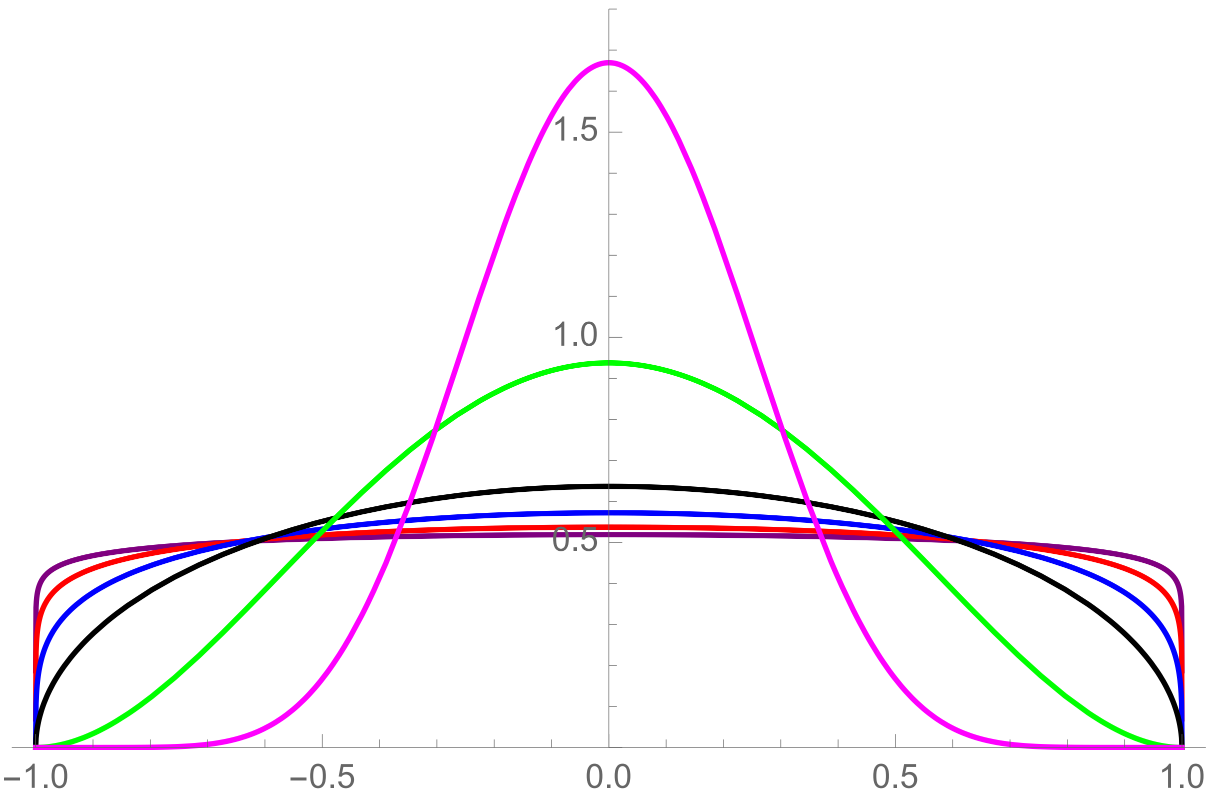

2
1
−1
−2
−4
−2
2
4
CR
CRmin

Example 3. (symmetric beta densities). Now consider the family of $s-$concave densities with $s > 0$ given for any $r \in (0, \infty)$ by

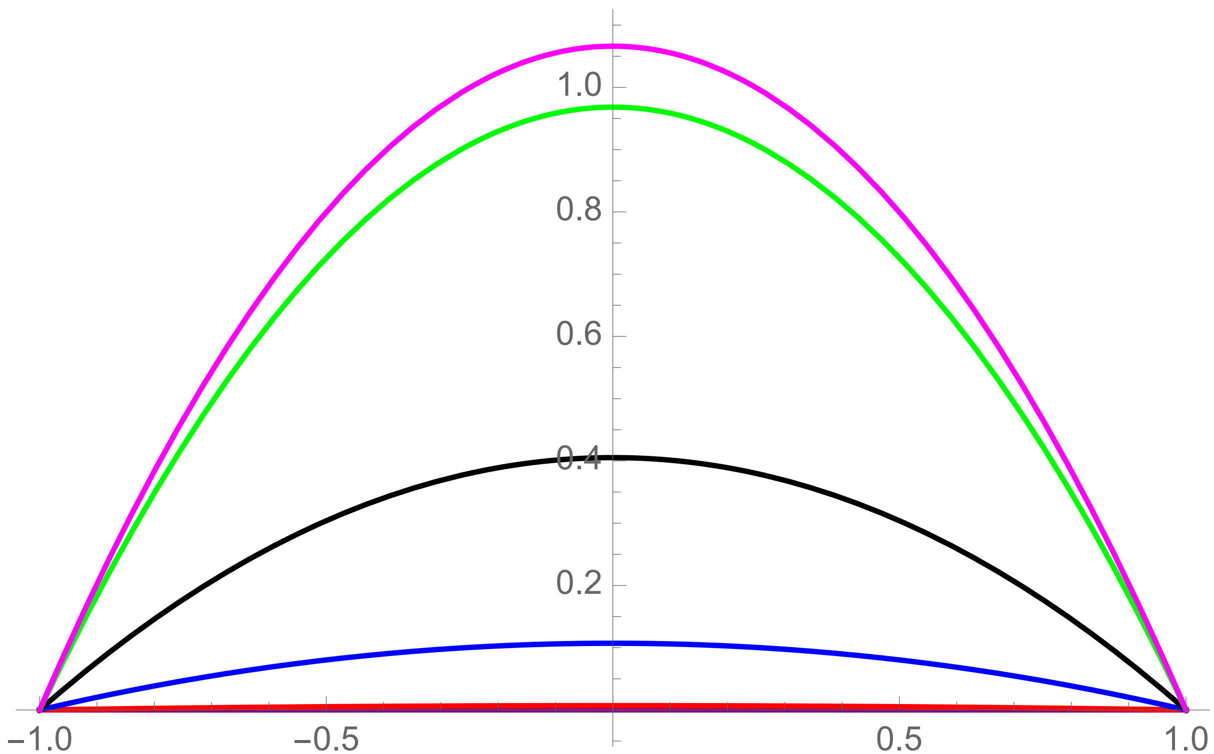
$$f_r(x) = \sqrt{r} C_r (1 - x^2)^{r/2} 1_{[-1,1]}(x)$$

where $C_r \equiv \Gamma((3 + r)/2)/(\sqrt{\pi r}\Gamma(1 + r/2))$. Then $f_r \in \mathcal{P}_s$. with $s = 2/r$. Here are some plots, for $r \in \{1/8, 1/2, 2, 4, 8, 16\}$, and hence $s \in \{16, 4, 1, 1/2, 1/4, 1/8\}$:

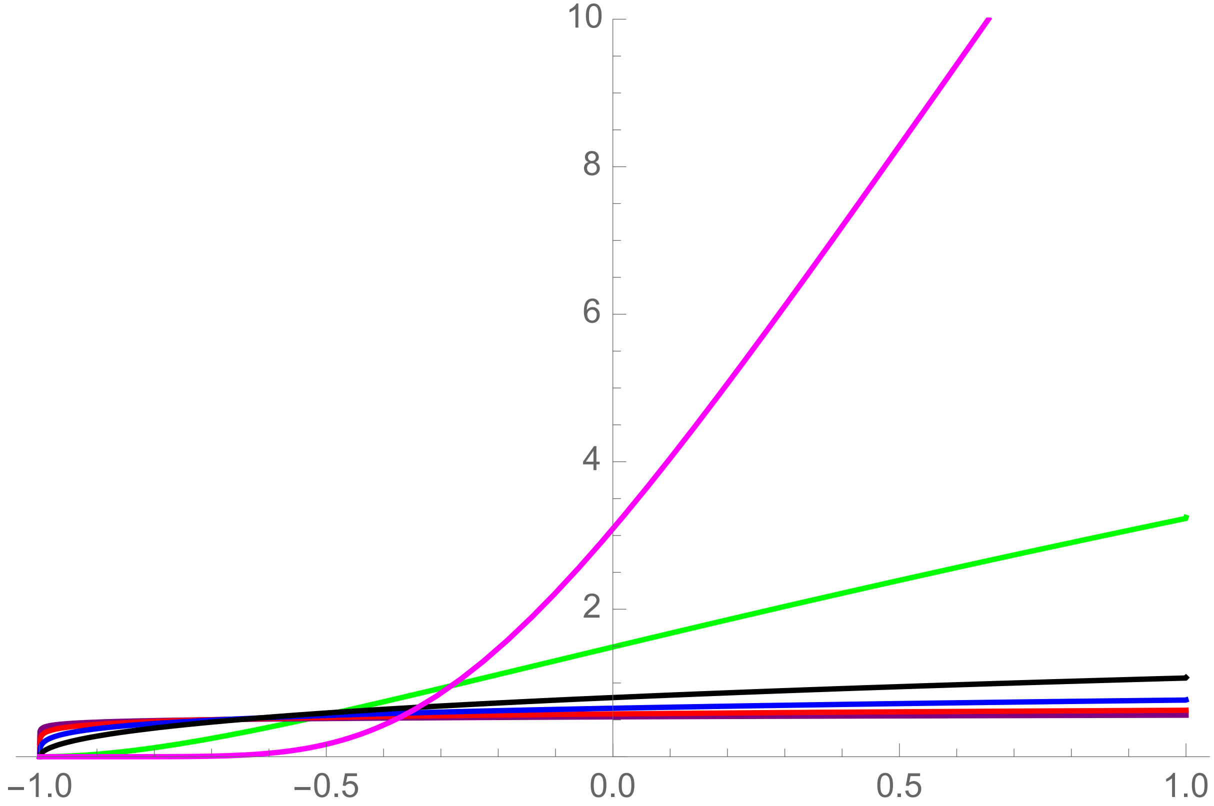
- f_r,
- f_r^s, $s = 2/r$.
- $f_r/(1 - F_r)^{1-s^*}$, $s^* = s/(1 + s) =$.
- $CRm(x, f) \equiv \min\{F(x), 1 - F(x)\} f'(x)/f(x)^2$ for $f = f_r$. $CR(x, f) \equiv F(x)(1 - F(x))f'(x)/f(x)^2$.

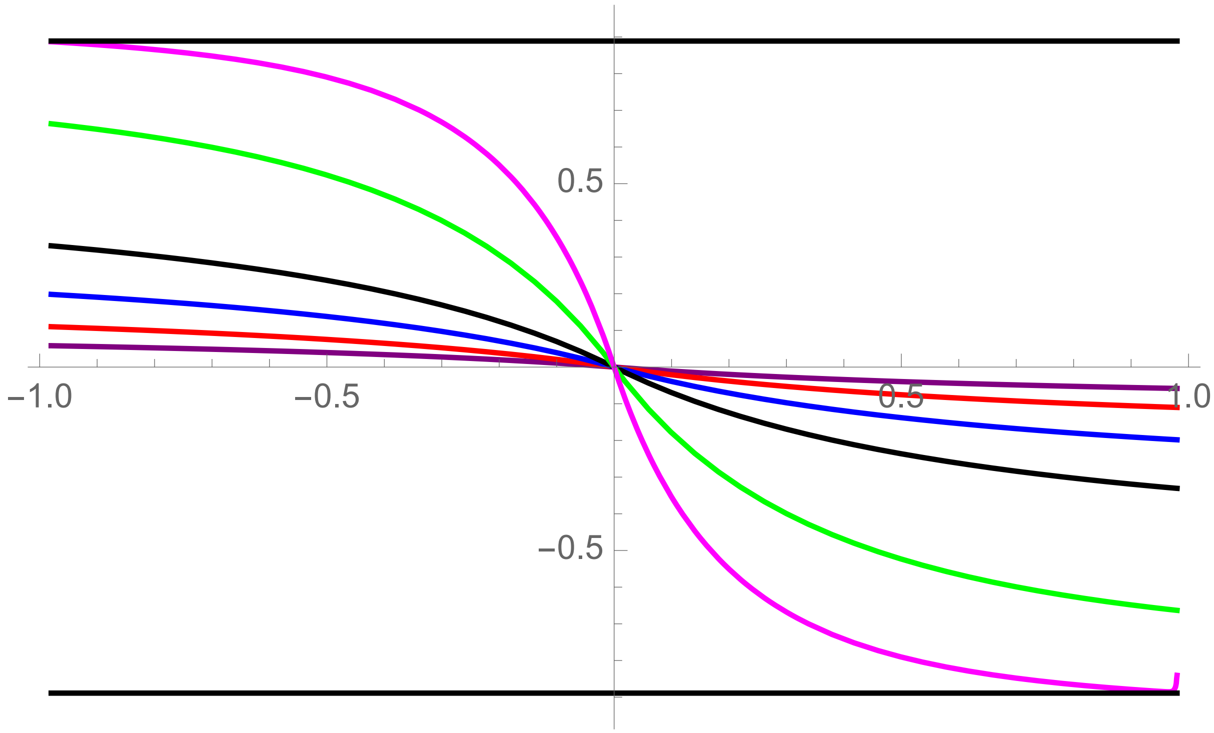


Symmetrized beta densities f_r with $r \in \{1/8, 1/4, 1/2, 2, 4, 16\}$



Powers of symmetrized beta densities $f_r^s = f_r^{2/r}$
with $r \in \{1/8, 1/4, 1/2, 2, 4, 16\}$


10
8
6
4
2
–1.0
–0.5
0.0
0.5
1.0



$CRm(x)$ for f_r symmetrized Beta, $r \in \{1/8, 1/4, 1/2, 2, 4, 16\}$
Here the black bounding lines at the top and bottom are given by the bound for the biggest class, namely for $r = 16$, so $s = 1/8$ and

$$1 - s^* = \frac{1}{1+s} = \frac{1}{1+1/8} = \frac{8}{9} \quad \text{since} \quad s = \frac{2}{16} = \frac{1}{8}.$$

3. Bi-s^*-concave distributions

Definition.

- For $s \in (-1, \infty)$, let $s^* \equiv s/(1+s) \in (-\infty, 1]$.
- For $s \in (-1, 0)$, a distribution function F on $\mathbb{R}$ is bi-s^*-concave if both $x \mapsto F^{s^*}(x)$ and $x \mapsto (1-F)^{s^*}(x)$ are convex functions of $x \in J(F)$.
- For $s \in (0, \infty)$, F on $\mathbb{R}$ is bi-s^*-concave if $x \mapsto F^{s^*}(x)$ is concave for $x \in (\inf J(F), \infty)$ and and $x \mapsto (1-F)^{s^*}(x)$ is concave for $x \in (-\infty, \sup J(F))$.
- For $s = 0$, F on $\mathbb{R}$ is bi-0-concave or bi-log-concave if both $x \mapsto \log F(x)$ and $x \mapsto \log(1 - F(x))$ are concave functions of $x \in J(F)$.

Theorem 2. (Bi-s^*-characterization theorem) Let $s \in (-1, \infty]$. For a non-degenerate distribution function F the following four statements are equivalent:

(i) F is bi-s^*-concave.

(ii) F is continuous on $\mathbb{R}$ and differentiable on $J(F)$ with derivative $f = F'$. Moreover when $s \le 0$,

$$F(x+t) \begin{cases} \le F(x) \cdot \left(1 + s^* \frac{f(x)}{F(x)} t\right)_+^{1/s^*} \\ \ge 1 - (1 - F(x)) \cdot \left(1 - s^* \frac{f(x)}{1-F(x)} t\right)_+^{1/s^*} \end{cases} \quad (2)$$

for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$. When $s > 0$,

$$F(x+t) \begin{cases} \le F(x) \cdot \left(1 + s^* \frac{f(x)}{F(x)} t\right)_+^{1/s^*}, & \text{for } t \in (a - x, \infty) \\ \ge 1 - (1 - F(x)) \cdot \left(1 - s^* \frac{f(x)}{1-F(x)} t\right)_+^{1/s^*}, & \text{for } t \in (-\infty, b - x) \end{cases} \quad (3)$$

for all $x \in J(F)$.

(iii) F is continuous on $\mathbb{R}$ and differentiable on $J(F)$ with derivative $f = F'$ such that the s^*-hazard function $f/(1-F)^{1-s^*}$ is non-decreasing, and the reverse s^*-hazard function f/F^{1-s^*} is non-increasing on $J(F)$.

(iv) F is continuous on $\mathbb{R}$ and differentiable on $J(F)$ with bounded and strictly positive derivative $f = F'$. Furthermore, f is locally Lipschitz-continuous on $J(F)$ with L^1-derivative $f' = F''$ satisfying

$$-(1-s^*)\frac{f^2}{1-F} \leq f' \leq (1-s^*)\frac{f^2}{F}. \qquad (4)$$

Corollary.
Suppose that F is bi-s^*-concave for $s \in (-1, \infty]$. Then

$$\gamma(F) = \sup_{x \in J(F)} F(x)(1 - F(x)) \frac{|f'(x)|}{f^2(x)} = 1 - s^* = \frac{1}{1+s},$$

and

$$\tilde{\gamma}(F) = \sup_{x \in J(F)} \min\{F(x), 1 - F(x)\} \frac{|f'(x)|}{f^2(x)} \leq 1 - s^* = \frac{1}{1+s}.$$

4. Improved Confidence bands

Suppose F is bi-logconcave. Given valid nonparametric confidence bands for F, can we improve them under the assumption of bi-log-concavity?

YES! DKW (2017)

Suppose that (L_n, U_n) is a $(1-\alpha)$-confidence band for F with $0 < \alpha \le 1/2$. Thus $L_n = L_{n,\alpha}(\cdot|X_1,\ldots,X_n) < 1$ and $U_n = U_{n,\alpha}(\cdot|X_1,\ldots,X_n)$ are non-decreasing functions on $\mathbb{R}$ with $L_n \le U_n$ pointwise and

$$P_F(L_n(x) \le F(x) \le U_n(x) \text{ for all } x \in \mathbb{R}) = 1-\alpha.$$

Example: (Kolmogorov-Smirnov band). Let $\mathbb{F}_n$ be the empirical distribution function as in Section 1. Then

$$[L_n(x), U_n(x)] \equiv \left[\mathbb{F}_n(x) - \frac{\kappa_{\alpha,n}^{KS}}{n^{1/2}}, \mathbb{F}_n(x) + \frac{\kappa_{\alpha,n}^{KS}}{n^{1/2}}\right] \cap [0,1]$$

where $\kappa_{\alpha,n}^{KS}$ denotes the $(1-\alpha)$-quantile of

$$\sup_{x \in \mathbb{R}} n^{1/2} |\mathbb{F}_n(x) - F(x)|;$$

cf. Shorack and W (1986) (and note that $\kappa_{\alpha,n}^{KS} \le \sqrt{\log(2/\alpha)/2}$ by Massart's inequality).

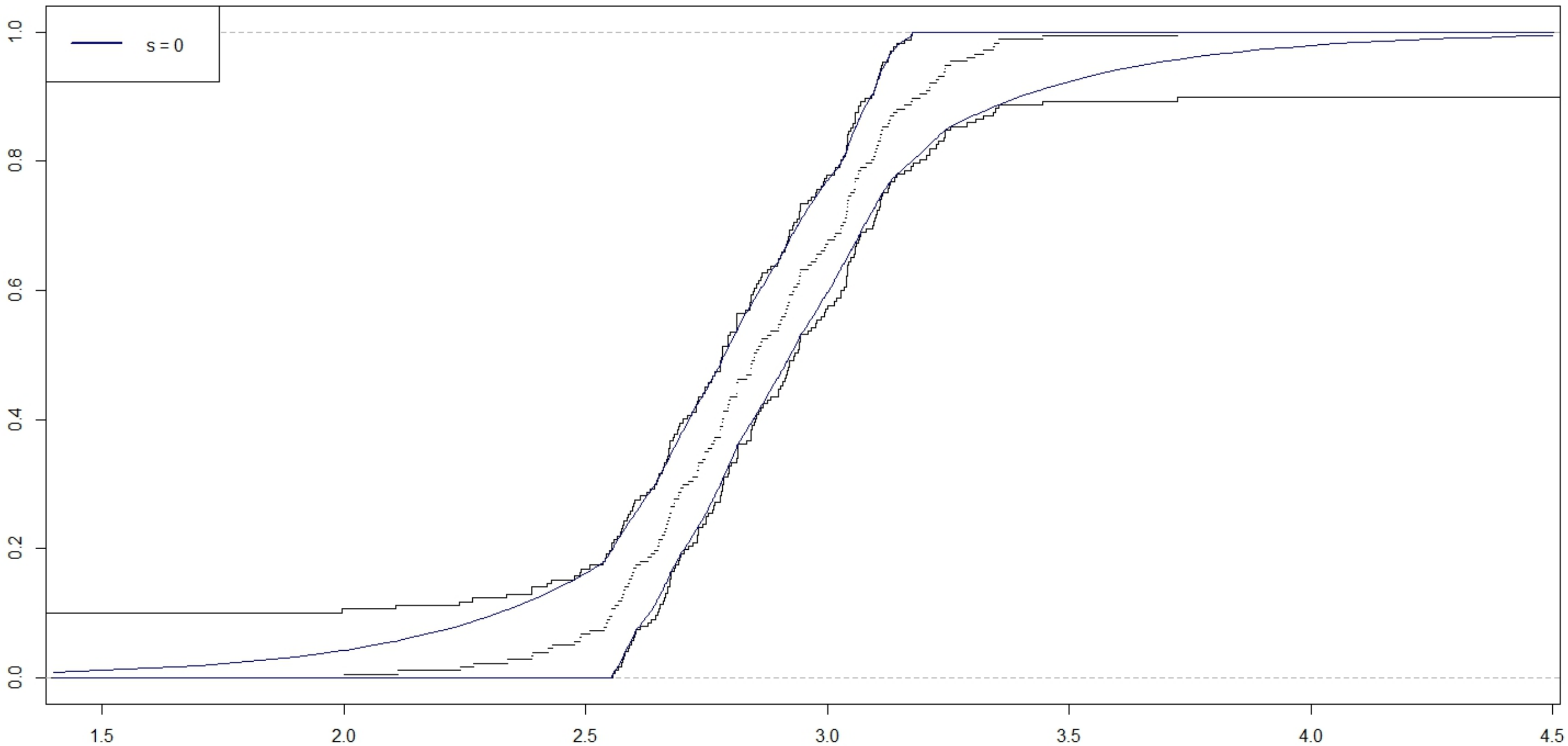
If $F \in$F$_{blc}$, find a new band for F as follows:

$$L_n^0(x) \equiv \inf\{G(x) : \; G \in \mathcal{F}_{blc}, \; L_n \le G \le U_n\},$$
$$U_n^0(x) \equiv \sup\{G(x) : \; G \in \mathcal{F}_{blc}, \; L_n \le G \le U_n\}.$$

If no bi-log-concave distribution function fits into the band (L_n, U_n), set $L_n^0 \equiv 1$ and $U_n^0 \equiv 0$ and conclude with confidence $1-\alpha$ that $F \notin \mathcal{F}_{blc}$. But if $F \in \mathcal{F}_{blc}$, this happens with probability at most α: by the construction of (L_n^0, U_n^0)

$$P_F(L_n^0) \le F \le U_n^0) = P(L_n \le F \le U_n) = 1 - \alpha \quad \text{if} \quad F \in \mathcal{F}_{blc}.$$

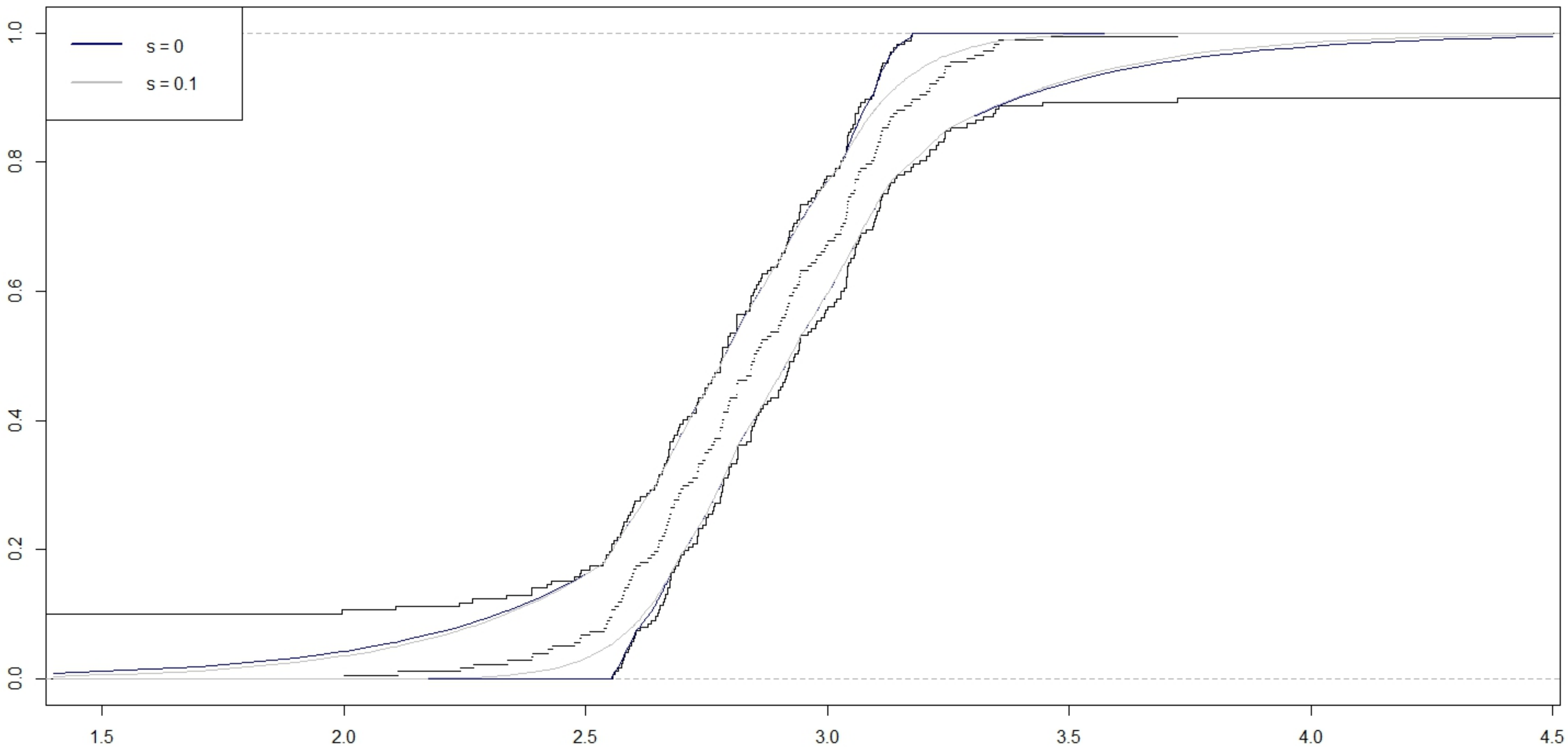
Similarly, if $F \in \mathcal{F}_{bi-s^*}$, find a new band for F by refining the given band.



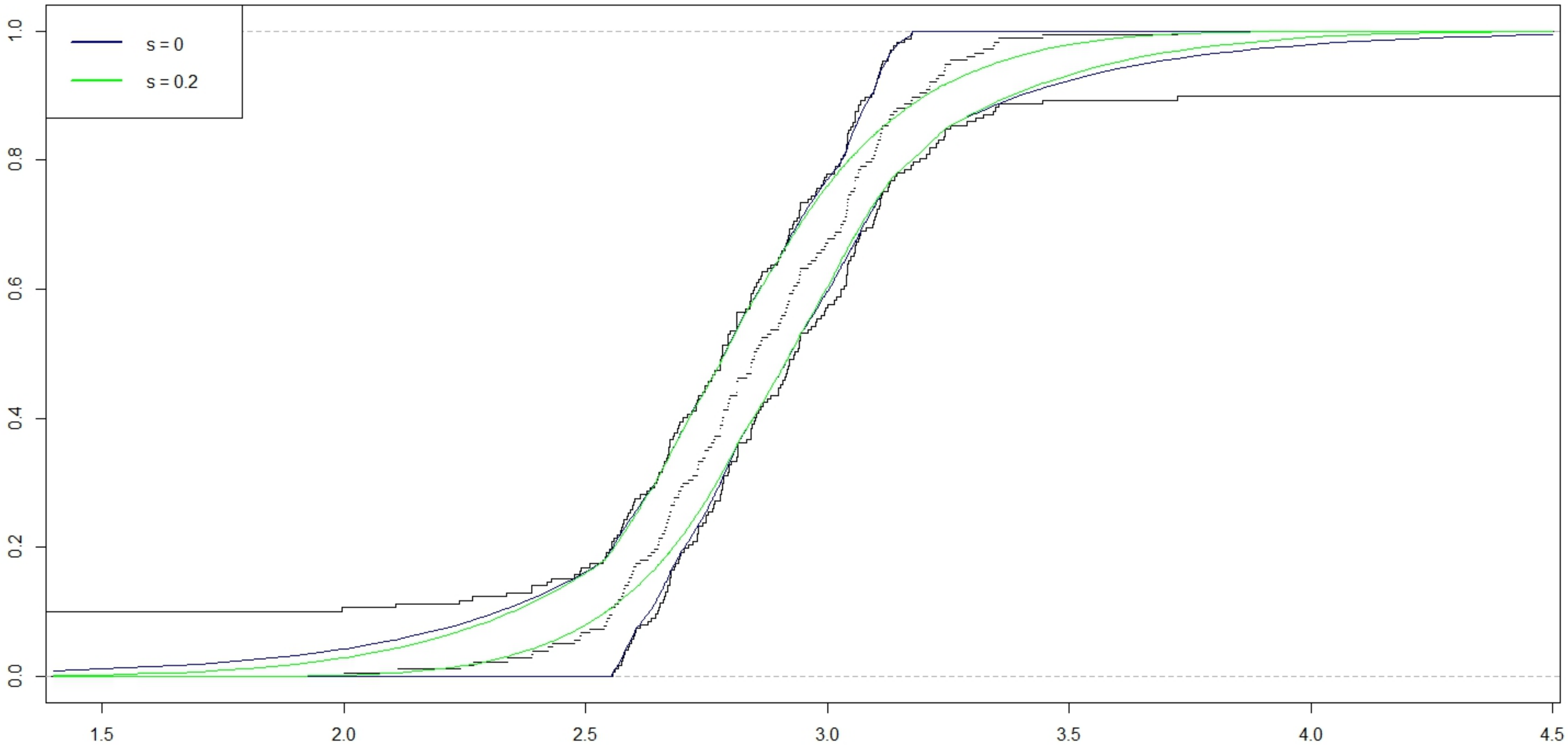
CEO salary data; Woolridge 2002: $n = 177$; $X_i = \log_{10} Y_i$

$Y_i =$ annual salaries of CEO's in multiples of USD

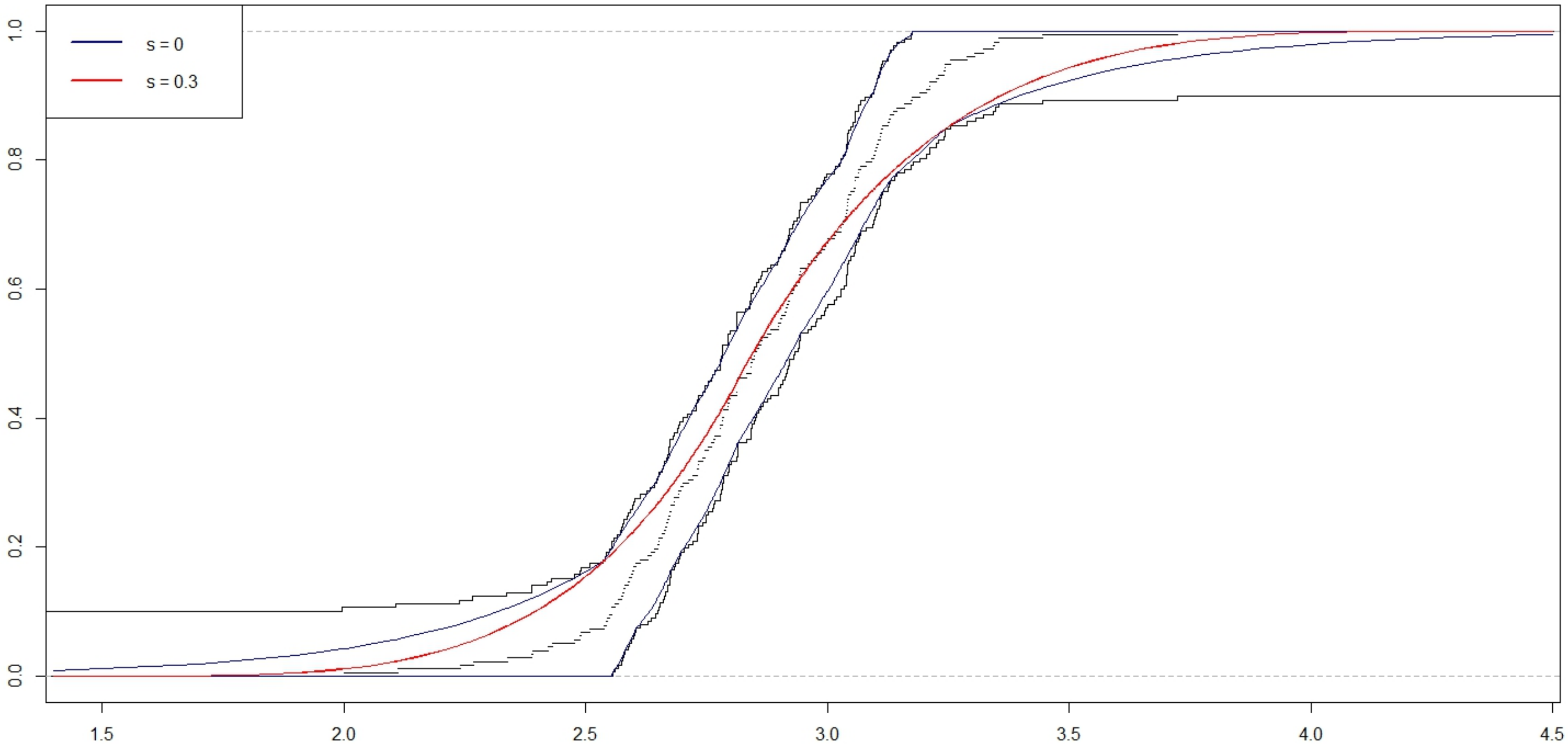
95% unconstrained band (black); log-concave-constrained band (blue)



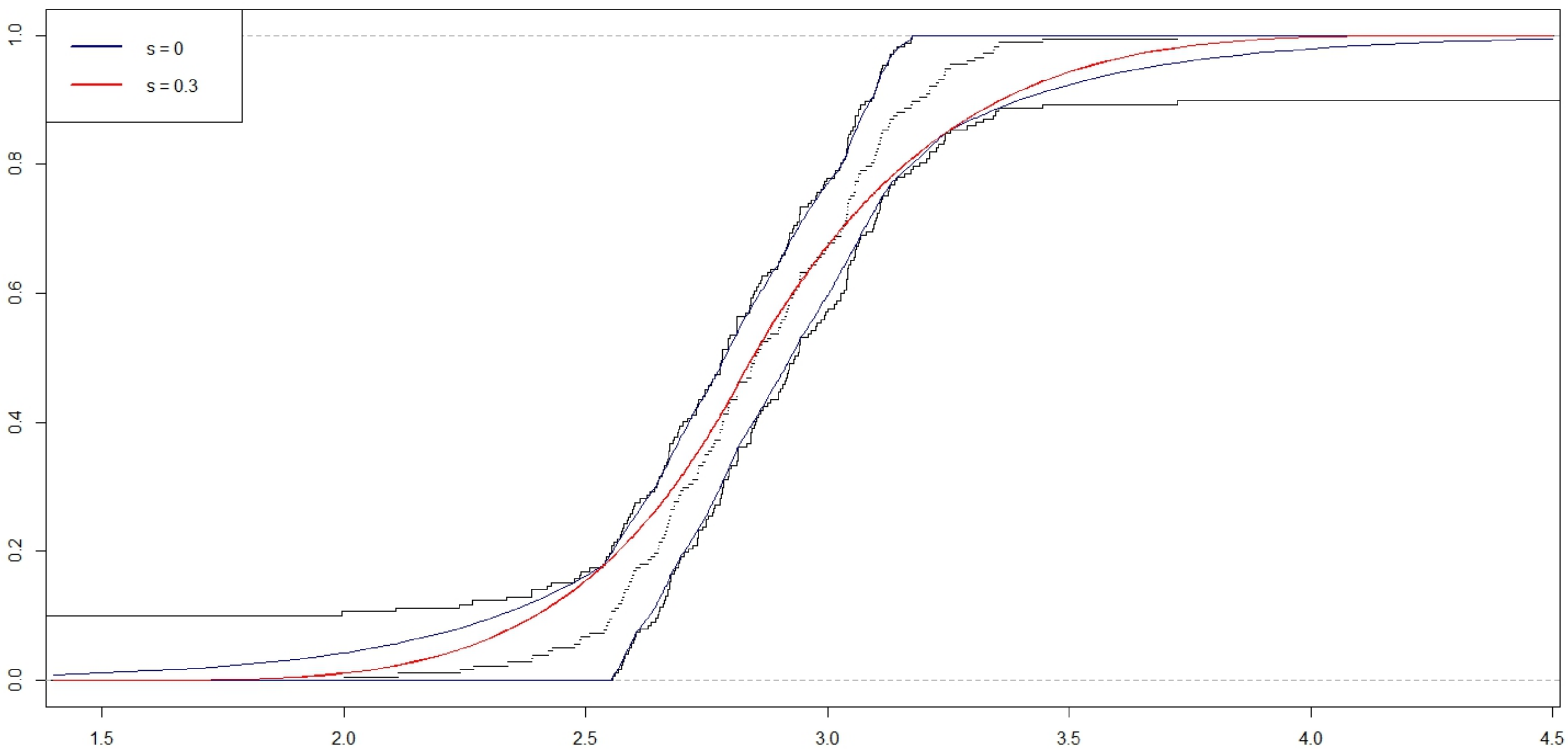
95% unconstrained band (black);
bi$-s^*-$constrained band, $s = 0.10$ (gray)



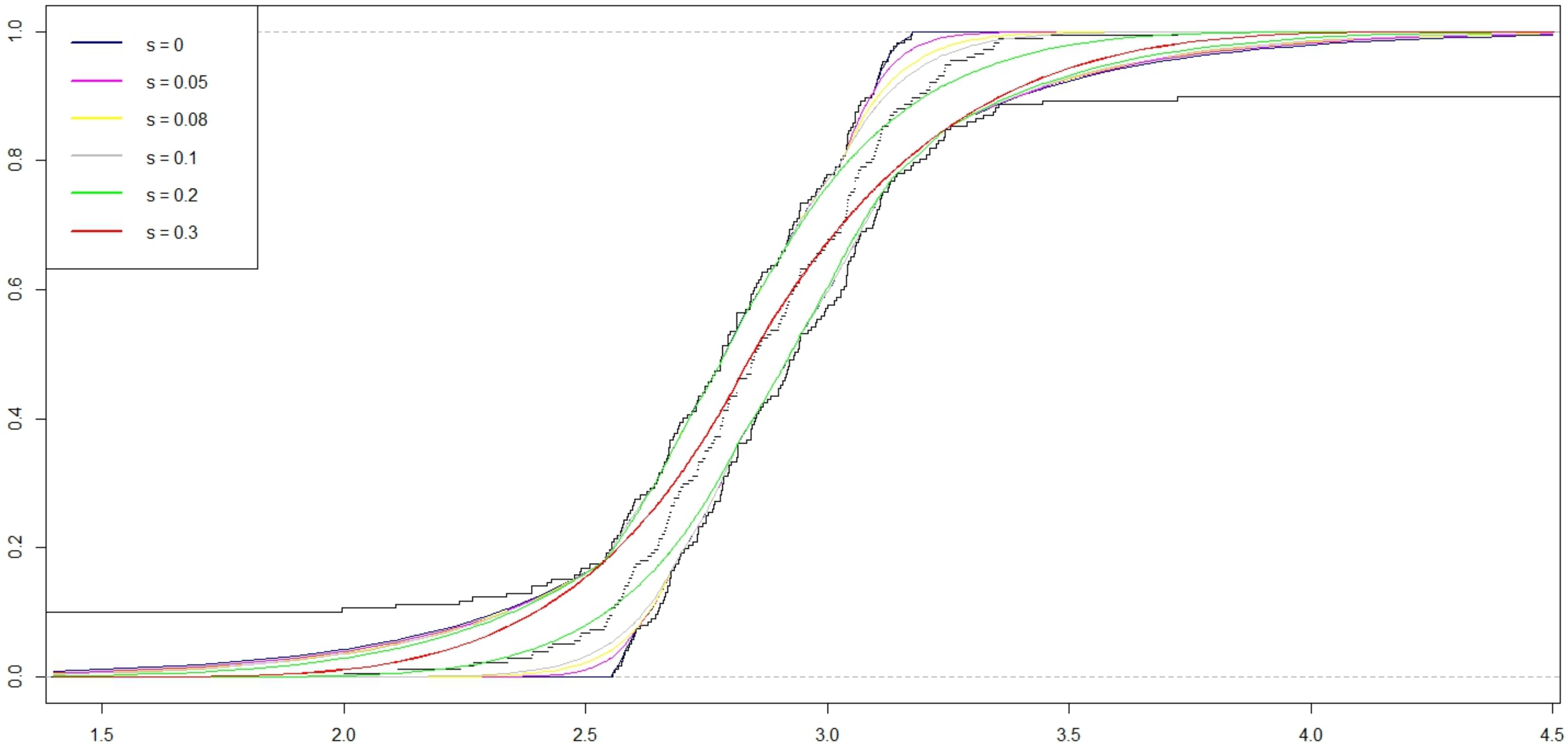
95% unconstrained band (black); log-concave-constrained band (blue); bi$-s^*-$constrained band, $s = 0.20$ (green)



95% unconstrained band (black); log-concave-constrained band (blue); bi$-s^*-$constrained band, $s = 0.30$ (red)



95% unconstrained band (black); log-concave-constrained band (blue); bi$-s^*-$constrained band, $s = 0.30$ (red)



95%unconstrained band (black); log-concave-constrained band (blue);

5. Questions and further problems

Questions and further problems: **Q1.** What can be said about estimation of s^* (and s)?

Q2. Can anything be said when f is $s-$concave with $s \le -1$?

Q3. Bi-log-concave or bi-s^*-concave in higher dimensions?

Q4. What are the "right" hypotheses for the study of transportation (Wasserstein) distances for empirical measures on $\mathbb{R}^d$ with $d \ge 2$?

Selected references:

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Thank You!