Keeping it together: A connectedness constraint in phase-field simulations

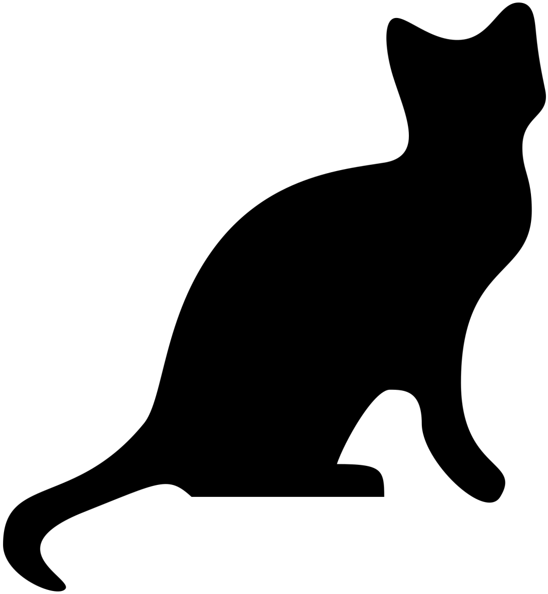
Stephan Wojtowytsch
joint work with Patrick Dondl, Matteo Novaga and Benedikt Wirth

Banff, May 24, 2018

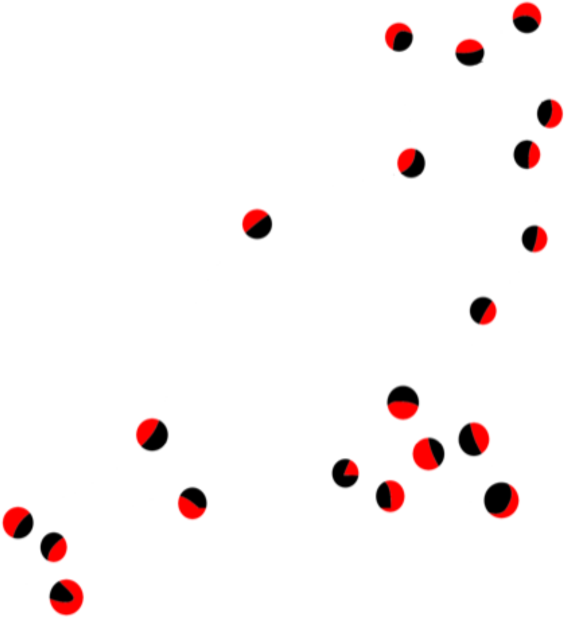
A cat

(shamelessly stolen from the internet)

A more relevant cat



An obscure(d) cat



Reconstructing a Cat

- $\Omega \Subset \mathbb{R}^2$ domain of the image
- On $U \subseteq \Omega$ image data exists
- $f : U \to [0, 1]$ image data
- Per = perimeter functional

Ansatz energy functional for $E \subset \Omega$ (binary approximation of f):

$$\mathcal{F}(E) = \mathrm{Per}(E) + \int_U |f - \chi_E|^p \,\mathrm{d}x.$$

Reconstructing a Cat

- $\Omega \Subset \mathbb{R}^2$ domain of the image
- On $U \subseteq \Omega$ image data exists
- $f : U \to [0, 1]$ image data
- Per = perimeter functional

Ansatz energy functional for $E \subset \Omega$ (binary approximation of f):

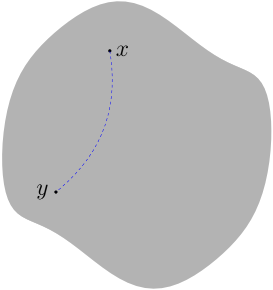
$$\mathcal{F}(E) = \mathrm{Per}(E) + \int_U |f - \chi_E|^p \, \mathrm{d}x.$$

Cats are connected $\Rightarrow$ E should have to be connected. (Cats appear even simply connected from most angles, but we may want to leave that part of the topology free. We'll come back to that.)

Quantitative (Dis-)connectedness

Let $F \subset \mathbb{R}^2$, $x, y \in F$. Consider

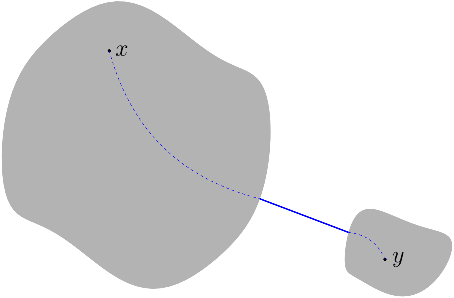
$$d^F(x,y) = \inf \left\{ \mathcal{H}^1(\gamma \setminus F) \mid \gamma \text{ Lipschitz curve from } x \text{ to } y \right\}.$$



Quantitative (Dis-)connectedness

Let $F \subset \mathbb{R}^2$, $x, y \in F$. Consider

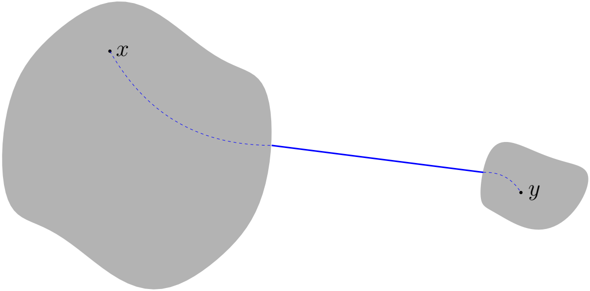
$$\mathcal{C}(F) = \int_E \int_E d^F(x,y)\,\mathrm{d}x\ \mathrm{d}y.$$



Quantitative (Dis-)connectedness

Let $F \subset \mathbb{R}^2$, $x, y \in F$. Consider

$$\mathcal{C}(F) = \int_E \int_E d^F(x,y)\,\mathrm{d}x\ \mathrm{d}y.$$

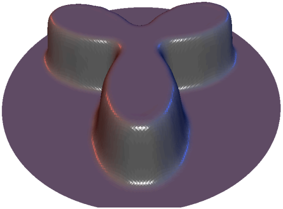


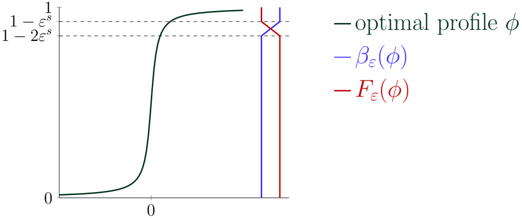
Phase-fields

$$\mathrm{Per}_\varepsilon(u) = \begin{cases} \frac{1}{c_0} \int_\Omega \frac{\varepsilon}{2} \, |\nabla u|^2 + \frac{W(u)}{\varepsilon} \, \mathrm{d}x & u \in W_0^{1,2}(\Omega) \\ +\infty & \text{else} \end{cases}$$

where $W(u) = u^2\,(1-u)^2$ and

$$c_0 = \int_0^1 \sqrt{2\,W(s)}\,\mathrm{d}s.$$





Phase-field Connectedness

Let $0 < s < 1/2$; $\beta_\varepsilon, F_\varepsilon$ monotone increasing/decreasing Lipschitz functions such that

$$\beta_\varepsilon(z) = \begin{cases} 0 & z \leq 1 - 2\varepsilon^s \\ 1 & z \geq 1 - \varepsilon \end{cases}, \qquad F_\varepsilon(z) = \begin{cases} 1 & z \leq 1 - 2\varepsilon^s \\ 0 & z \geq 1 - \varepsilon \end{cases}$$

and

$$\mathcal{C}_\varepsilon(u) = \int_\Omega \int_\Omega \beta_\varepsilon(u(x))\, \beta_\varepsilon(u(y))\, d^{F_\varepsilon(u)}(x, y) \,\mathrm{d}x \,\mathrm{d}y$$

where

$$d^{F_\varepsilon(u)}(x, y) = \inf \left\{ \int_\gamma F_\varepsilon(u) \,\mathrm{d}\mathcal{H}^1 \,\middle|\, \gamma \text{ Lipschitz curve from } x \text{ to } y \right\}.$$

Phase-field energy

$$\mathcal{F}_\varepsilon(u) = \mathrm{Per}_\varepsilon(u) + \varepsilon^{-\kappa}\, \mathcal{C}_\varepsilon(u)$$

Theorem (Dondl-Novaga-Wirth-W $\geq$ '18)

$$\left[\Gamma(L^1) - \lim_{\varepsilon\to 0} \mathcal{F}_\varepsilon\right](u) = \begin{cases} \overline{\mathrm{Per}}_{C,cc,r}(\{u=1\}) & u \in BV(\Omega,\{0,1\}) \\ +\infty & \text{else} \end{cases}$$

Relaxed perimeter

$$\overline{\mathrm{Per}}_{C,cc,r}(E) = \inf\left\{\liminf_{n\to\infty} \mathrm{Per}(E_n) \mid E_n \Subset \Omega \text{ smooth, connected, } E_n \xrightarrow{L^1} E\right\}$$

Proof: $\Gamma - \limsup$

Let $E_n \Subset \Omega$ connected and smooth such that

$$\overline{\mathrm{Per}}_{C,cc,r}(E) = \lim_{n\to\infty} \mathrm{Per}(E_n).$$

Choose $\varepsilon_n < \min\{\varepsilon_{n-1}, 2^{-n}\}$ such that $B_{\sqrt{\varepsilon_n}}(\partial E_n)$ is diffeomorphic to a finite union of annuli in the usual way and set

$$u_\varepsilon(x) = q\left(\frac{\mathrm{sdist}(x, \partial E_n)}{\varepsilon}\right)$$

(+ boundary cutoff) for $\varepsilon_{n+1} < \varepsilon \leq \varepsilon_n$ where

$$q'' - W'(q) = 0 \quad \text{and } q(-\infty) = 0,\ q(0) = \frac{1}{2},\ q(+\infty) = 1.$$

Then $\mathcal{C}_\varepsilon(u_\varepsilon) \equiv 0$ and

$$\lim_{\varepsilon\to 0} \mathrm{Per}_\varepsilon(u_\varepsilon) = \lim_{n\to\infty} \mathrm{Per}(E_n) = \overline{\mathrm{Per}}_{C,cc,r}(E).$$

Proof: $\Gamma - \liminf$. Part I

Let $u_\varepsilon \to u$. Wlog: $\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty$, $u_\varepsilon \in C_c^\infty(\Omega, [0,1])$.

Proof: $\Gamma - \liminf$. Part I

Let $u_\varepsilon \to u$. Wlog: $\liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty$, $u_\varepsilon \in C_c^\infty(\Omega, [0,1])$. We know that

$$\begin{aligned}
\mathrm{Per}_\varepsilon(u_\varepsilon) &= \frac{1}{c_0}\int_\Omega \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}\,\mathrm{d}x \\
&\geq \frac{1}{c_0}\int_\Omega |\nabla u_\varepsilon|\sqrt{2\,W(u_\varepsilon)}\,\mathrm{d}x \\
&= \frac{1}{c_0}\int_\Omega |\nabla G(u_\varepsilon)|\,\mathrm{d}x \\
&= \frac{1}{c_0}\int_0^{c_0}\left[\int_\Omega \mathrm{d}|\nabla \chi_{\{G(u_\varepsilon)>t\}}|\right]\mathrm{d}t
\end{aligned}$$

Where $G(z) = \int_0^z \sqrt{2\,W(s)}\,\mathrm{d}s$ (Young's inequality and co-area formula).

Proof: $\Gamma - \liminf$. Part II

By Sard's theorem, there exists $t_\varepsilon \in (\delta, 1-\delta)$ such that

$$E_\varepsilon = \{G(u_\varepsilon) > t_\varepsilon\} \in C^\infty \quad \text{and} \quad \mathrm{Per}(E_\varepsilon) \leq \frac{1}{c_0 - 2\delta}\, \mathcal{F}_\varepsilon(u_\varepsilon).$$

Proof: $\Gamma - \liminf$. Part II

By Sard's theorem, there exists $t_\varepsilon \in (\delta, 1-\delta)$ such that

$$E_\varepsilon = \{G(u_\varepsilon) > t_\varepsilon\} \in C^\infty \quad \text{and} \quad \mathrm{Per}(E_\varepsilon) \le \frac{1}{c_0 - 2\delta}\,\mathcal{F}_\varepsilon(u_\varepsilon).$$

Note that $E_\varepsilon \approx \{u_\varepsilon > 1 - \varepsilon^s\} \neq \emptyset$ in L^1 since

$$\left|\{\varepsilon^s < u_\varepsilon < 1 - \varepsilon^s\}\right| \le \frac{1}{W(\varepsilon^s)} \int_\Omega W(u_\varepsilon)\,\mathrm{d}x \le C\varepsilon^{1-2s}$$

so $E_\varepsilon \xrightarrow{L^1} \{u = 1\}$.

$\Rightarrow$ if we knew that E_ε is connected for all ε, we would be done (letting $\delta \to 0$ in a second step).

Proof: $\Gamma - \liminf$. Part III

Order the (finitely many) connected components $U_{1,\varepsilon}, U_{2,\varepsilon}, \ldots$ of E_ε by size and choose N_ε such that

$$|U_{i,\varepsilon}| \begin{cases} \geq \frac{1}{|\log \varepsilon|} & 1 \leq i \leq N_\varepsilon \\ < \frac{1}{|\log \varepsilon|} & N_\varepsilon < i \end{cases}.$$

Note that

$$|E_\varepsilon| \geq N_\varepsilon \cdot |U_{N_\varepsilon,\varepsilon}| \quad \Rightarrow \quad N_\varepsilon \leq \frac{|\Omega|}{|\log \varepsilon|}$$

and

$$\begin{aligned} \left| \bigcup_{i>N_\varepsilon} U_{i,\varepsilon} \right| &\leq \sqrt{|U_{N_\varepsilon,\varepsilon}|} \sum_{i>N_\varepsilon} \sqrt{|U_{i,\varepsilon}|} \\ &\leq \frac{1}{\sqrt{|\log \varepsilon|}} \sum_{i>N_\varepsilon} \frac{1}{\sqrt{4\pi}} \mathrm{Per}(U_{i,\varepsilon}) \\ &\leq \frac{\mathrm{Per}(E_\varepsilon)}{\sqrt{4\pi}} \frac{1}{\sqrt{|\log \varepsilon|}}. \end{aligned}$$

Proof: $\Gamma - \liminf$. Part IV

Note that for $1 \le i \neq j \le N_\varepsilon$

$$
\begin{aligned}
C_\varepsilon(u_\varepsilon) &\ge \int_{C_{i,\varepsilon}\cap\{u_\varepsilon>1-\varepsilon^s\}} \int_{C_{j,\varepsilon}\cap\{u_\varepsilon>1-\varepsilon^s\}} d^{F_\varepsilon(u_\varepsilon)}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&\ge \frac{1}{|\log\varepsilon|^2}\mathrm{dist}^{F_\varepsilon(u_\varepsilon)}(C_{i,\varepsilon}, C_{j,\varepsilon})
\end{aligned}
$$

so there exists a C^∞-curve $\gamma_{i,j,\varepsilon}$ from $C_{i,\varepsilon}$ to $C_{j,\varepsilon}$ such that

$$
\mathcal{H}^1\big(\gamma_{i,j,\varepsilon} \setminus \{u_\varepsilon > 1-\varepsilon^s\}\big) \le C\varepsilon^\kappa\,|\log\varepsilon|^2.
$$

Proof: $\Gamma - \liminf$. Part IV

Note that for $1 \leq i \neq j \leq N_\varepsilon$

$$C_\varepsilon(u_\varepsilon) \geq \int_{C_{i,\varepsilon}\cap\{u_\varepsilon > 1-\varepsilon^s\}} \int_{C_{j,\varepsilon}\cap\{u_\varepsilon > 1-\varepsilon^s\}} d^{F_\varepsilon(u_\varepsilon)}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$
$$\geq \frac{1}{|\log\varepsilon|^2}\mathrm{dist}^{F_\varepsilon(u_\varepsilon)}(C_{i,\varepsilon}, C_{j,\varepsilon})$$

so there exists a C^∞-curve $\gamma_{i,j,\varepsilon}$ from $C_{i,\varepsilon}$ to $C_{j,\varepsilon}$ such that

$$\mathcal{H}^1\big(\gamma_{i,j,\varepsilon} \setminus \{u_\varepsilon > 1-\varepsilon^s\}\big) \leq C\varepsilon^\kappa\,|\log\varepsilon|^2.$$

Since the curve is smooth, we can fatten it a little bit and get roughly twice the length as perimeter and almost no area.

Proof: $\Gamma - \liminf$. Part V

- Connect $U_{1,\varepsilon}, \ldots, U_{N_\varepsilon,\varepsilon}$ by curves $\gamma_{i,j,\varepsilon}$ and fatten them
- add missing components of E_ε back to the set if a relevant curve passes through them
- smooth out the corners
- Note that the resulting set is connected, smooth, has at most slightly larger perimeter, and still converges to $\{u = 1\}$.

$\square$

Diffuse Cat

Unfortunately work in progress.

Cat-free numerics

But we have some other simulations! For $U = \Omega$, so

$$\mathcal{F}_\varepsilon(u) = \mathrm{Per}_\varepsilon(u) + \varepsilon^{-1}\,\mathcal{C}_\varepsilon(u) + \int_\Omega |u - f|^2 \,\mathrm{d}x.$$

Note: The connectedness constraint simplifies significantly since $d^{F(u)}(x, y) = d^{F(u)}(x', y')$ whenever x, x' and y, y' are in the same connected component of $\{u > 1 - \varepsilon^s\}$, so we only need to compute a hand full of distances (Dijkstra's algorithm).

Extensions and Future Work

1. We can additionally keep $\{u_\varepsilon < \varepsilon^s\}$ (almost) connected. Simple connectedness?
2. We could take $W^{1,2}$ as the domain of the phase fields instead of $W^{1,2}_0$. Relative perimeters?
3. In three dimensions, this problem becomes meaningless. But we can even keep interfaces connected if we use Willmore's energy rather than the perimeter (analysis becomes harder since curves for distance function become co-dimension two objects).

Disclaimer

- All cat pictures in this presentation are available under a creative commons license. (wikipedia.org, pxhere.com)
- No cats were harmed in the making of this presentation.

Thank you for your attention!