Liftings of BV-maps and lower semicontinuity

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(joint with G. Shaw)

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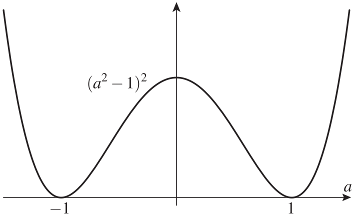
Rate-independent systems (Mielke–Theil, Mielke–Rossi–Savaré)

Prototypical equation:

$$\frac{\dot{u}(t)}{|\dot{u}(t)|} - \Delta u(t) + DW_0(u(t)) = f(t) \qquad \text{in } \Omega \times [0, T],$$

where W_0 = double well potential:



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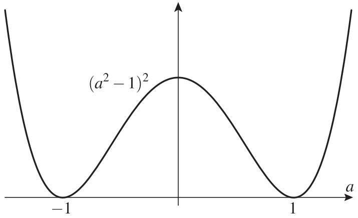

Interpretation:

$$\frac{\dot u(t)}{|\dot u(t)|} := \mathrm{Sgn}(\dot u(t)), \qquad \text{where} \qquad \mathrm{Sgn}(s) := \begin{cases} \{-1\} & \text{if } s < 0, \\ [-1, 1] & \text{if } s = 0, \\ \{1\} & \text{if } s > 0. \end{cases}$$

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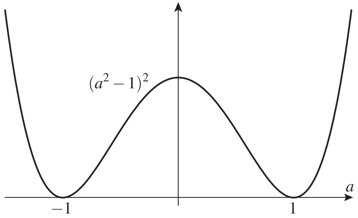

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- **Rate-independence:** Dissipation does not depend on rate (speed) of movement $\rightsquigarrow$ idealization!

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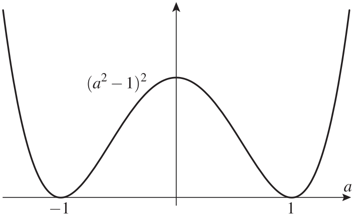

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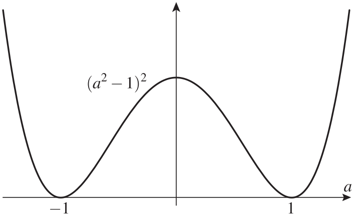

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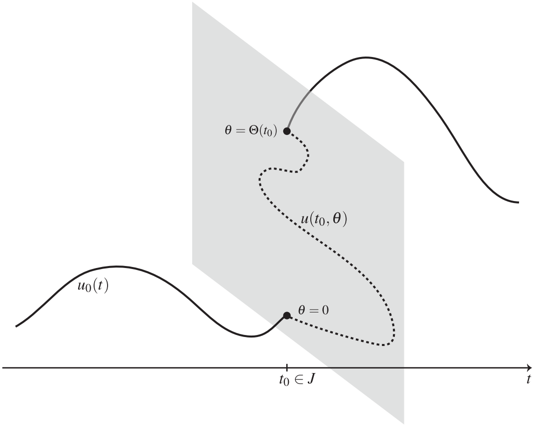
Features:

- **Rate-independence:** Dissipation does not depend on rate (speed) of movement $\rightsquigarrow$ idealization!
- Can only expect BV-regularity $\rightsquigarrow$ jumps
- The above equation says nothing about the behavior on jump transients!

Jump parametrization?



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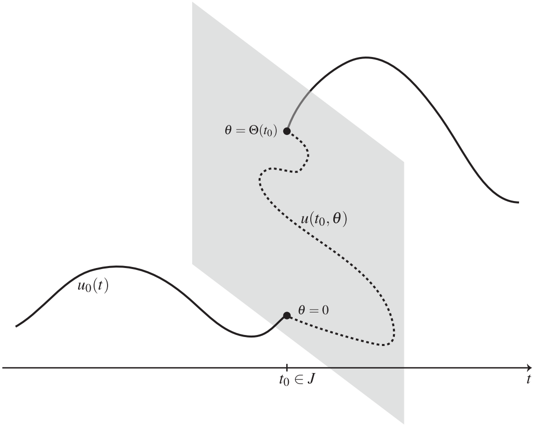

Two-speed solutions (R., Schwarzacher, Süli, Velázquez, 2017–):

- Strong solutions as long as possible
- Late jumps (similar to Mielke–Rossi–Savaré "Balanced Viscosity" theory)
- Jump resolution (viscous PDE on jump transients)

BV-maps with jumps: Relaxation

Let $\Omega \subset \mathbb{R}^d$ bounded Lipschitz domain, $d, m > 1$, and

$$\mathscr{F}[u] := \int_\Omega f(x, u(x), \nabla u(x)) \, \mathrm{d}x, \qquad u \in \mathrm{W}^{1,1}(\Omega; \mathbb{R}^m),$$

where $f \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to [0, \infty)$ with

$$0 \le f(x, y, A) \le C(1 + |y|^{d/(d-1)} + |A|).$$

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Relaxation of $\mathscr{F}$ at $u \in \mathrm{BV}(\Omega; \mathbb{R}^m)$:

$$\mathscr{F}_{**}[u] := \inf \left\{ \liminf_{j\to\infty} \mathscr{F}[u_j] \colon\; (u_j)_j \subset \mathrm{W}^{1,1}(\Omega; \mathbb{R}^m),\; u_j \rightsquigarrow u \right\}$$

with "$u_j \rightsquigarrow u$" meaning BV-weak* or L^1-strong convergence.

Q: What is $\mathscr{F}_{**}$? Does it have an integral representation?

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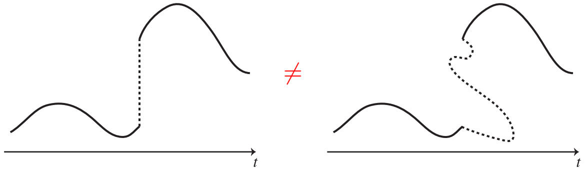
Q: What is $\mathscr{F}_{**}$? Does it have an integral representation? Jump paths matter!



Previous work: Fonseca–Müller '93, Ambrosio–Dal Maso '92 and many other works (Leoni, Bouchitté, Mascarenhas, . . .).

Relaxation theorem with respect to BV-weak* convergence

Theorem (R. & Shaw 2017)

Let $f\colon \overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to [0,\infty)$ where $d \geq 2$ and $m \geq 1$ be such that

(i) *f is a Carathéodory function whose recession function f^∞ exists as a limit,*

$$f^\infty(x,y,A) = \lim_{\substack{(x,y_k,A_k)\to(x,y,A)\\ t_k\to\infty}} \frac{f(x_k,y_k,t_kA_k)}{t_k};$$

(ii) $0 \leq f(x,y,A) \leq C(1 + |y|^{d/(d-1)} + |A|)$;

(iii) *$f(x,y,\bullet)$ is quasiconvex for every $(x,y) \in \overline{\Omega} \times \mathbb{R}^m$.*

Then the sequential **weak* relaxation** *$\mathscr{F}_{**}$ of $\mathscr{F}$ to $u \in \mathrm{BV}(\Omega;\mathbb{R}^m)$ is*

$$\mathscr{F}^{w*}_{**}[u] = \int_\Omega f(x,u,\nabla u)\,\mathrm{d}x + \int_\Omega f^\infty\left(x,u,\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}\right)\mathrm{d}|D^cu| + \int_J K_f[u]\,\mathrm{d}\mathscr{H}^{d-1}$$

where J is the jump set of u and

$$K_f[u](x) := \inf\left\{ \frac{1}{\omega_{d-1}}\int_{\mathbb{B}^d} f^\infty(x,\varphi(y),\nabla\varphi(y))\,\mathrm{d}y \;:\; \varphi \in \mathrm{C}^\infty(\mathbb{B}^d;\mathbb{R}^m),\ \varphi|_{\partial\mathbb{B}^d} = u^\pm(x) \text{ if } y\cdot n_u(x) \gtrless 0 \right\}$$

Toward L^1-relaxation

Task: Compute the Γ-limit of the sequence of functionals as $\varepsilon \downarrow 0$:

$$\mathscr{E}_\varepsilon[u] := \frac{1}{\varepsilon} \int_\Omega g(x, u)^2 \, \mathrm{d}x + \varepsilon \int_\Omega h(x, u, \nabla u)^2 \, \mathrm{d}x.$$

$\rightsquigarrow$ liquid crystals, fluid mixtures, phase transitions in solids, reaction–diffusion

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- **Main difficulty:** g may have zeroes $\rightsquigarrow$ need L^1-relaxation $\mathscr{F}^1_{**}$ of $\mathscr{F}$
- **Dal Maso '79 example:** there exists a continuous, convex (!), positively 1-homogeneous integrand $f \colon \Omega \times \mathbb{R}^d \to [0,\infty)$ for which $\mathscr{F}$ is not equal to $\mathscr{F}^1_{**}$ over $\mathrm{W}^{1,1}(\Omega;\mathbb{R})$.

The story so far

Main works: Fonseca & Müller '92, Fonseca & Leoni '01.

(a) Need g bounded.

(b) Need fairly strong continuity assumptions in x.

(c) Need joint lower semicontinuity in (x, y).

Interesting integrands that are not covered:

- Models of chemical reactions (Rubinstein–Sternberg–Keller 1989, Lin–Pan–Wang 2012) or harmonic maps (Chen–Struwe 1989) lead to

$$g(x, y) := \mathrm{dist}(y, K)^p, \qquad h(x, u, A) := |A|$$

  with $K =$ compact Riemannian manifold.

- Inhomogeneity, e.g.

$$g(x, y) := |y|^{1-|x|}, \qquad h(x, u, A) := |A|.$$

Partial coercivity

Assume that $g\colon \overline{\Omega} \times \mathbb{R}^m \to [0, \infty)$ is continuous and:

(a) partial coercivity:

$$g(x, y)|A| \leq f(x, y, A) \leq Cg(x, y)(1 + |A|)$$

(b) there exists $R > 0$ and $M > 1$ for which

$$g(x, y) \leq Mg(x, ty) \quad \text{for all } x \in \Omega, \quad |y| \geq R \text{ and } t \geq 1,$$

(c) for every compact $K \subset \mathbb{R}^m$ and $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$|(f - f^\infty)(x, y, A)| \leq \varepsilon g(x, y)(1 + |A|)$$

for $(x, y, A) \in \overline{\Omega} \times K \times \mathbb{R}^{m\times d}$ with $|A| \geq R_\varepsilon$.

Relaxation theorem with respect to L^1-convergence

Theorem (R. & Shaw 2018)

Let $f\colon \overline{\Omega}\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\to[0,\infty)$ *where* $d\geq 2$ *and* $m\geq 1$ *be such that*

(i) f *is a Carathéodory function whose recession function* f^∞ *exists as a limit;*

(ii) f *is partially coercive via* g $(g(x,y)|A|\leq f(x,y,A)\leq Cg(x,y)(1+|A|))$*;*

(iii) $f(x,y,\bullet)$ *is quasiconvex for every* $(x,y)\in\overline{\Omega}\times\mathbb{R}^m$*.*

Define

$$\mathscr{G}:=\left\{u\in\mathrm{L}^1(\Omega;\mathbb{R}^m)\ :\ \int_\Omega g(x,u(x))\,\mathrm{d}x<\infty\right\}.$$

Then, the L^1-**relaxation** *of* $\mathscr{F}$ *from* $\mathrm{W}^{1,1}(\Omega;\mathbb{R}^m)\cap\mathscr{G}$ *to* $\mathrm{BV}(\Omega;\mathbb{R}^m)\cap\mathscr{G}$ *is*

$$\mathscr{F}^1_{**}[u]=\int_\Omega f(x,u,\nabla u)\,\mathrm{d}x+\int_\Omega f^\infty\left(x,u,\frac{\mathrm{d}D^cu}{\mathrm{d}|D^cu|}\right)\mathrm{d}|D^cu|+\int_J H_f[u]\,\mathrm{d}\mathscr{H}^{d-1}$$

where $H_f[u]$ *is given on the next slide.*

Surface densities

Given $u \in \mathrm{BV}(\Omega;\mathbb{R}^m)$ and $x \in J = J_u$, let $\mathscr{A}_u(x)$ by

$$\mathscr{A}_u(x) := \left\{\varphi \in (\mathrm{C}^\infty \cap \mathrm{L}^\infty)\,(\mathbb{B}^d;\mathbb{R}^m) \colon\ \varphi = u_x^\pm \text{ on } \partial\mathbb{B}^d\right\},$$

Define

$$K_f[u](x) := \inf\left\{\, \omega_{d-1}^{-1} \int_{\mathbb{B}^d} f^\infty(x,\varphi(z),\nabla\varphi(z))\,\mathrm{d}z \ \ \textbf{:}\ \ \varphi \in \mathscr{A}_u(x) \,\right\},$$

$$H_f^r[u](x) := \inf\left\{\, \omega_{d-1}^{-1} \int_{\mathbb{B}^d} f^\infty\,(x + rz,\varphi(z),\nabla\varphi(z))\,\mathrm{d}z \ \ \textbf{:}\ \ \varphi \in \mathscr{A}_u(x),\ \|\varphi\|_{\mathrm{L}^1} \le 2\|u_x^\pm\|_{\mathrm{L}^1} \right\},$$

$$H_f[u](x) := \liminf_{r\to 0} H_f^r[u](x).$$

Example in paper: In general, $K_f \neq H_f$, hence $\mathscr{F}_{**}^{w*}$ and $\mathscr{F}_{**}^{1}$ differ

In previous works (Fonseca, Müller, Leoni, Bouchitté, Mascarenhas . . .): technical assumptions are strong enough to force $K_f = H_f$.

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- Liftings.

Liftings (R. & Shaw 2017 based on special case by Jung & Jerrard '04)

$$\mathrm{BV}_{\#}(\Omega;\mathbb{R}^m) := \left\{ u \in \mathrm{BV}(\Omega;\mathbb{R}^m) \;\; : \;\; ⨍_\Omega u(x)\,\mathrm{d}x = 0 \right\}.$$

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Definition

A **lifting** $\gamma \in \mathbf{Lift}(\Omega \times \mathbb{R}^m)$ is a measure $\gamma \in \mathbf{M}(\Omega \times \mathbb{R}^m; \mathbb{R}^{m\times d})$ for which there exists a (unique) $u \in \mathrm{BV}_{\#}(\Omega;\mathbb{R}^m)$ such that the **chain rule** holds:

$$\int_\Omega \nabla_x \varphi(x, u(x))\,\mathrm{d}x + \int_{\Omega\times\mathbb{R}^m} \nabla_y \varphi(x,y)\,\mathrm{d}\gamma(x,y) = 0 \quad \text{for all } \varphi \in \mathrm{C}^1_0(\Omega \times \mathbb{R}^m).$$

This u is called the **barycenter** $[\gamma]$ of γ.
Weak* convergence of liftings means weak* convergence in $\mathbf{M}(\Omega \times \mathbb{R}^m; \mathbb{R}^{m\times d})$.

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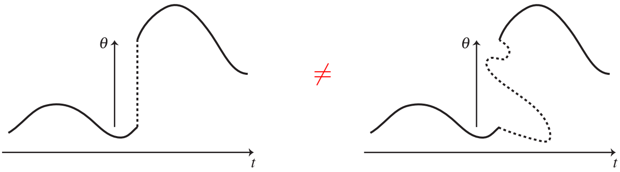

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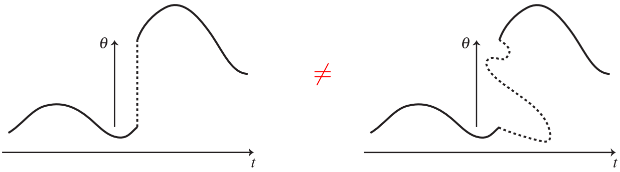


Lemma

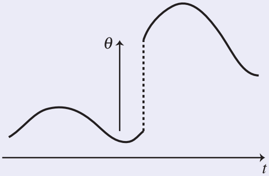
$\pi_\#\gamma = Du$ in $\mathbf{M}(\Omega;\mathbb{R}^{m\times d})$ and $\pi_\#|\gamma| \geq |Du|$ in $\mathbf{M}^+(\Omega)$.

Elementary liftings

Definition (Elementary/Minimal Liftings)

Given $u \in \mathrm{BV}_{\#}(\Omega;\mathbb{R}^m)$, the associated **elementary lifting** $\gamma[u] \in \mathbf{Lift}(\Omega \times \mathbb{R}^m)$ is

$$\gamma[u] := |Du| \otimes \left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|} \int_0^1 \delta_{u^\theta} \, \mathrm{d}\theta \right),$$



where u^θ is the jump interpolant,

$$u^\theta(x) := \begin{cases} \theta u^-(x) + (1-\theta)u^+(x) & \text{if } x \in J_u, \\ \widetilde{u}(x) & \text{otherwise.} \end{cases}$$

that is,

$$\langle \varphi, \gamma[u] \rangle = \int_\Omega \int_0^1 \varphi(x, u^\theta(x)) \, \mathrm{d}\theta \, \mathrm{d}Du(x) \quad \text{for all } \varphi \in \mathrm{C}_0(\Omega \times \mathbb{R}^m).$$

Chain rule

The liftings chain rule for the elementary lifting

$$\gamma[u](\mathrm{d}x, \mathrm{d}y) := Du(\mathrm{d}x) \otimes \int_0^1 \delta_{u^\theta(x)}(\mathrm{d}y)\ \mathrm{d}\theta,$$

follows from usual BV-chain rule:

Chain rule

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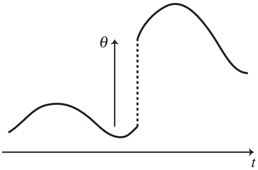
follows from usual BV-chain rule:

For $\varphi \in \mathrm{C}_0^1(\Omega \times \mathbb{R}^m)$:

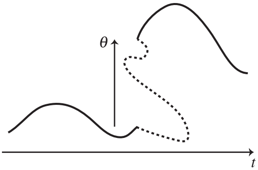
$$\begin{aligned}
&\int_\Omega \nabla_x \varphi(x, u(x))\, \mathrm{d}x + \int_{\Omega\times\mathbb{R}^m} \nabla_y \varphi(x, y)\, \mathrm{d}\gamma(x, y) \\
&\quad = \int_\Omega \nabla_x \varphi(x, u(x))\, \mathrm{d}x + \int_\Omega \int_0^1 \nabla_y \varphi(x, u^\theta(x))\, \mathrm{d}\theta\, \mathrm{d}Du(x) \\
&\quad = \int_\Omega \nabla_x \big[\varphi(x, u(x))\big]\, \mathrm{d}x \\
&\quad = 0.
\end{aligned}$$

Non-elementary liftings



$\gamma_1 := |Du| \otimes \left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|} \int_0^1 \delta_{u^\theta_{\mathrm{affine}}} \, \mathrm{d}\theta \right)$

$\neq$



$\gamma_2 := |Du| \otimes \left(\frac{\mathrm{d}Du}{\mathrm{d}|Du|} \int_0^1 \delta_{u^\theta_{\mathrm{squiggle}}} \, \mathrm{d}\theta \right)$

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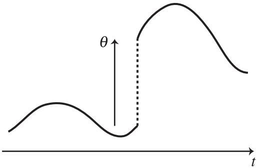

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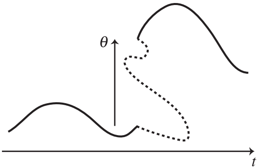

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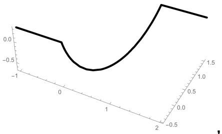
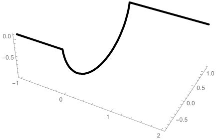
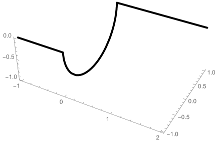
Example:

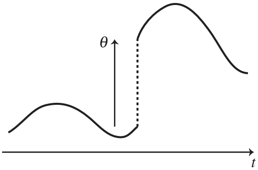
$u_j =$ 

, 

, 

$\gamma[u_j] \xrightarrow{*} \gamma \neq \gamma[u]$ for some $\gamma \in \mathbf{Lift}((-1,1) \times \mathbb{R}^2)$.

Non-elementary liftings



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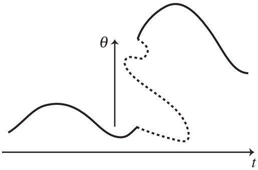

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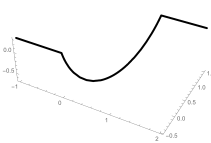
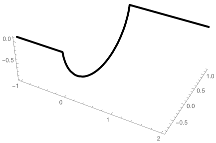
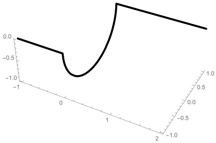
$u_j =$  ,  , 

$\gamma[u_j] \overset{*}{\rightharpoonup} \gamma \neq \gamma[u]$ for some $\gamma \in \mathbf{Lift}((-1,1) \times \mathbb{R}^2)$.

Lemma

Every lifting $\gamma \in \mathbf{Lift}(\Omega \times \mathbb{R})$ *is elementary:* $\gamma = \gamma[u]$ *for some* $u \in \mathrm{BV}_{\#}(\Omega; \mathbb{R})$.

Compactness for liftings

Lemma (Compactness)

Let $(\gamma_j)_j \subset \mathbf{Lift}(\Omega \times \mathbb{R}^m)$ *be such that* $\sup_j |\gamma_j|(\Omega \times \mathbb{R}^m) < \infty$*. Then there exists a subsequence* $(\gamma_{j_k})_k \subset (\gamma_j)_j$ *and a limit* $\gamma \in \mathbf{Lift}(\Omega \times \mathbb{R}^m)$ *such that*

$$\gamma_{j_k} \xrightarrow{*} \gamma \text{ in } \mathbf{M}(\Omega \times \mathbb{R}^m; \mathbb{R}^{m \times d}) \text{ and } [\gamma_{j_k}] \xrightarrow{*} [\gamma] \text{ in } \mathrm{BV}_\#(\Omega; \mathbb{R}^m).$$

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Corollary (Lifting generation from BV)

Let $(u_j)_j \subset \mathrm{BV}_\#(\Omega; \mathbb{R}^m)$ *be a bounded sequence with* $u_j \overset{*}{\rightharpoonup} u$ *in* $\mathrm{BV}_\#(\Omega; \mathbb{R}^m)$*. Then there exists a (non-relabelled) subsequence and a limit* $\gamma \in \textbf{Lift}(\Omega \times \mathbb{R}^m)$ *with* $[\gamma] = u$ *such that*

$$\gamma[u_j] \overset{*}{\rightharpoonup} \gamma \text{ in } \textbf{Lift}(\Omega \times \mathbb{R}^m).$$

Structure theorem

Graph map: $\mathrm{gr}^u : x \mapsto (x, u(x))$ for $u \in \mathrm{BV}(\Omega; \mathbb{R}^m)$

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Pushforward: If $\mu \in \mathbf{M}(\Omega)$ satisfying $|\mu| \ll \mathscr{H}^{d-1}$ and $|\mu|(J_u) = 0$, then the pushforward $\mathrm{gr}^u_{\#}\mu$ of μ under gr^u is well-defined as a measure on $\Omega \times \mathbb{R}^m$.
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Theorem (Structure Theorem for Liftings, R. & Shaw 2017)

If $\gamma \in \mathbf{Lift}(\Omega \times \mathbb{R}^m)$ with $u = [\gamma]$, then γ admits the following decomposition into mutually singular measures:

$$\gamma = \gamma[u] \llcorner ((\Omega \setminus \mathscr{J}_u) \times \mathbb{R}^m) + \gamma^{\mathrm{gs}}.$$

Moreover, $\gamma^{\mathrm{gs}} \in \mathbf{M}(\Omega \times \mathbb{R}^m; \mathbb{R}^{m \times d})$ satisfies

$$\mathrm{div}_y \, \gamma^{\mathrm{gs}} = -|D^j u| \otimes \frac{n_u}{|u^+ - u^-|}(\delta_{u^+} - \delta_{u^-}),$$

*and it is **graph-singular** with respect to u in the sense that γ^{gs} is singular with respect to all measures of the form $\mathrm{gr}^u_\# \lambda$ where $\lambda \in \mathbf{M}(\Omega)$ satisfies both $\lambda \ll \mathscr{H}^{d-1}$ and $\lambda(J_u) = 0$.*

Perspective functionals

Proposition

Let $\gamma \in \textbf{Lift}(\Omega \times \mathbb{R}^m)$ *with* $u = [\gamma]$ *be* **minimal** *in the sense that* $|\gamma|(\Omega \times \mathbb{R}^m) = |Du|(\Omega)$. *Then* γ *must be elementary,* $\gamma = \gamma[u]$. *In particular, if* $u_j \to u$ *in* $\mathrm{BV}_{\#}(\Omega; \mathbb{R}^m)$ *strictly, then* $\gamma[u_j] \to \gamma[u]$ *strictly in* $\textbf{Lift}(\Omega \times \mathbb{R}^m)$.

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Define $\mathscr{F}_{\mathrm{L}} \colon \textbf{Lift}(\Omega \times \mathbb{R}^m) \to \mathbb{R}$ by

$$\mathscr{F}_{\mathrm{L}}[\gamma] = \int_\Omega f(x, [\gamma](x), \nabla[\gamma](x)) \,\mathrm{d}x + \int_{\Omega \times \mathbb{R}^m} f^\infty(x, y, \gamma^s) \, .$$

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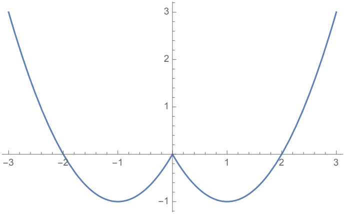
Strategy: Study $\mathscr{F}$ via $\mathscr{F}_{\mathrm{L}}$ (via blowups / Young measures for liftings ...).

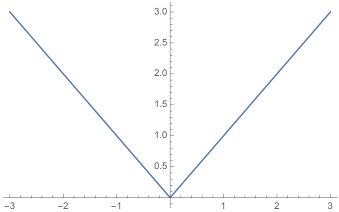
Thank you for your attention!

Solution concepts for a motivating example

Zero-dimensional (ODE) setup:

$$\mathscr{W}_0(z) = W_0(z) := \min\{z(z+2), z(z-2)\}, \qquad \mathscr{R}_1(z) = R_1(z) := |z|$$



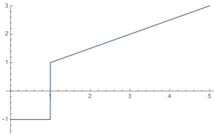


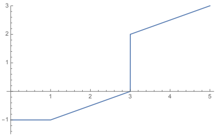
$$\begin{cases} \mathrm{Sgn}(\dot{u}(t)) + DW_0 \ni f(t) := t \\ u(0) = -1 \end{cases}$$

where

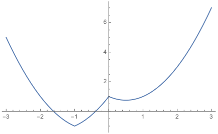
$$\mathrm{Sgn}(s) := \begin{cases} \{-1\} & \text{if } s < 0, \\ [-1, 1] & \text{if } s = 0, \\ \{1\} & \text{if } s > 0. \end{cases}$$

Weak & balanced viscosity / two-speed solution

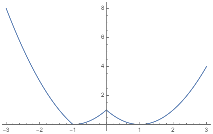
$u^{\mathrm{weak}}(t) :=$ 

$u^{\mathrm{twospeed}}(t) :=$ 

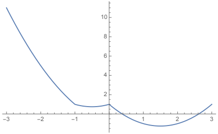
Effective energy: $W_0(z) + |z - (-1)| - t \cdot z$



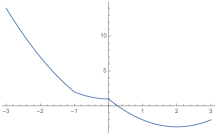
$t = 0$



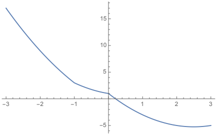
$t = 1$



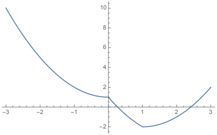
$t = 2$



$t = 3$



$t = 4$



$t = 1^+$ (after jump)

Perspective integrands / measures

- For an integrand $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}$, define the **perspective integrand**

$$(Pf)(x,y,(A,t)) := \begin{cases} |t| f(x,y,|t|^{-1}A) & \text{if } |t|>0, \\ f^\infty(x,y,A) & \text{if } t=0. \end{cases}$$

- Pf is positively one-homogeneous in the (A,t)-argument.
- The **perspective measure** $P\gamma \in \mathbf{M}(\Omega \times \mathbb{R}^m; \mathbb{R}^{m\times d} \times \mathbb{R})$ of a lifting $\gamma \in \mathbf{Lift}(\Omega \times \mathbb{R}^m)$ is

$$P\gamma := \left(\gamma, \mathrm{gr}^{[\gamma]}_\#(\mathscr{L}^d \llcorner \Omega)\right).$$

- By a (hard) **Structure Theorem**: $P\gamma$ admits the following decomposition with respect to $\mathrm{gr}^u_\#(\mathscr{L}^d \llcorner \Omega)$, where $u = [\gamma]$:

$$P\gamma = (\nabla u, 1)\,\mathrm{gr}^u_\#(\mathscr{L}^d \llcorner \Omega) + (\gamma^s, 0).$$

- If $u_j \to u$ area-strictly in $\mathrm{BV}_\#(\Omega;\mathbb{R}^m)$, then $P\gamma_j \to P\gamma$ strictly.