Dissipative boundary conditions and entropic solutions in dynamical perfect plasticity

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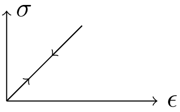
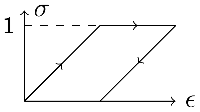
Modelling

Elasticity



Perfect plasticity



Constitutive law : $\sigma = \epsilon$

Loading threshold : $|\sigma| \leq 1$

Plastic variable : $p = \epsilon - \sigma$

Flow law : $\begin{cases} \dot{p} = 0 & \text{if } |\sigma| < 1, \\ \pm\dot{p} \geq 0 & \text{if } \sigma = \pm 1. \end{cases}$

Displacement : $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$

Stress : $\sigma : \Omega \to \mathbb{R}^2$

Equation of motion : $\ddot{u} - \mathrm{div}\sigma = 0$

Constitutive law : $\sigma = \nabla u$

Wave equation : $\ddot{u} - \Delta u = 0$

$$\ddot{u} - \mathrm{div}\sigma = 0,$$
$$|\sigma| \leq 1,$$
$$\nabla u = \sigma + p,$$
$$\left\{ \begin{array}{c} \dot{p} = 0 \text{ if } |\sigma| < 1 \\ \frac{\dot{p}}{|\dot{p}|} = \sigma \text{ if } |\sigma| = 1 \end{array} \right\} \Leftrightarrow \sigma \cdot \dot{p} = |\dot{p}|$$

Hyperbolic formulation : Friedrichs' systems

Elasticity : $\ddot{u} - \Delta u = 0$

Hyperbolic variable : $U := (\dot{u}, \sigma) = (\dot{u}, \nabla u)$

The wave equation is equivalent to

$$\partial_t U + \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{=:A_1} \partial_1 U + \underbrace{\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{=:A_2} \partial_2 U = 0,$$

or still, for all constant vectors $\kappa \in \mathbb{R}^3$ and all test functions φ with $\mathrm{Supp}(\varphi) \subset \Omega \times [0, +\infty)$,

$$\int_0^\infty \int_\Omega |U - \kappa|^2 \partial_t \varphi + \sum_{i=1}^2 A_i (U - \kappa) \cdot (U - \kappa) \partial_i \varphi + \int_\Omega |U_0 - \kappa|^2 \varphi(0) = 0.$$

Hyperbolic formulation : Friedrichs' systems

Plasticity : $\ddot{u} - \mathrm{div}\sigma = 0$, $\nabla \dot{u} = \dot{\sigma} + \dot{p}$, $\sigma \in B$, $\sigma \cdot \dot{p} = |\dot{p}| \geq 0$,

Hyperbolic variable : $U := (\dot{u}, \sigma) \in \mathbb{R} \times B =: K$

The system of perfect plasticity writes

$$\partial_t U + A_1 \partial_1 U + A_2 \partial_2 U + \begin{pmatrix} 0 \\ \dot{p} \end{pmatrix} = 0,$$

or still, for all constant vectors $\kappa = (k, \tau) \in K$ and all test functions $\varphi \geq 0$ with $\mathrm{Supp}(\varphi) \subset \Omega \times [0, +\infty)$,

$$\int_0^\infty \int_\Omega |U - \kappa|^2 \partial_t \varphi + \sum_{i=1}^{2} A_i (U - \kappa) \cdot (U - \kappa) \partial_i \varphi + \int_\Omega |U_0 - \kappa|^2 \varphi(0) \geq 0.$$

✓ Analogy with the entropic formulation of scalar conservation laws (Kruzkov) ; here an L^2 theory is natural because of quadratic entropies $U \mapsto |U - \kappa|^2$.

✓ For $\Omega = \mathbb{R}^n$ and general Friedrichs' systems WITH contraints :

- Després-Lagoutière-Seguin : $\exists!$ by means of a finite volume scheme and the doubling variable method ;
- B.-Mifsud-Seguin : $\exists$ by means of a constraint penalization and a vanishing diffusion.

✓ For $\Omega \subsetneq \mathbb{R}^n$ and general Friedrichs' systems WITHOUT contraints : entropic "dissipative" formulation by Després-Mifsud-Seguin for boundary conditions of the type

$$(A_\nu - M)U = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+,$$

where $A_\nu = \sum_{i=1}^{n} A_i \nu_i$ (ν unit outer normal to $\partial\Omega$) and M is matrix s.t.

- $M^T = M$;
- $M \geq 0$;
- $\mathrm{Ker} A_\nu \subset \mathrm{Ker} M$;
- $\mathbb{R}^3 = \mathrm{Ker}(A_\nu - M) + \mathrm{Ker}(A_\nu + M)$.

Entropic formulation

For all constant vectors $\kappa \in K$ and all test functions $\varphi \geq 0$,

$$\int_0^\infty \int_\Omega |U-\kappa|^2 \partial_t \varphi + \sum_{i=1}^2 A_i(U-\kappa)\cdot(U-\kappa)\partial_i \varphi + \int_\Omega |U_0-\kappa|^2 \varphi(0) + \int_0^\infty \int_{\partial\Omega} M\kappa^+ \cdot \kappa^+ \varphi \geq 0,$$

where

$$\begin{array}{ccccccc} \mathbb{R}^3 & = & \mathrm{Ker} A_\nu & \oplus & \mathrm{Ker}(A_\nu - M) \cap \mathrm{Im} A_\nu & \oplus & \mathrm{Ker}(A_\nu + M) \cap \mathrm{Im} A_\nu \\ \kappa & = & \kappa_0 & + & \kappa^- & + & \kappa^+ . \end{array}$$

✓ L^2 formulation of a boundary value problem by analogy with the L^∞ theory of Otto for scalar conservation laws in bounded domains.
✓ In the case of elasticity and/or plasticity,

$$(A_\nu - M)U = 0 \iff \sigma\cdot\nu + \lambda \dot{u} = 0, \quad \lambda > 0.$$

Well posedness ($\lambda = 1$)

We look for (u, σ, p) such that

$$
\begin{cases}
\ddot{u} - \mathrm{div}\sigma = 0 \text{ in } \Omega \times (0, T), \\
\nabla u = \sigma + p \text{ in } \Omega \times (0, T), \\
|\sigma| \leq 1 \text{ in } \Omega \times (0, T), \\
\sigma \cdot \dot{p} = |\dot{p}| \text{ in } \Omega \times (0, T), \\
\sigma \cdot \nu + \dot{u} = 0 \text{ on } \partial\Omega \times (0, T), \\
+I.C.
\end{cases}
$$

Elasto-visco-plastic approximation : We look for $(u_\varepsilon, \sigma_\varepsilon, p_\varepsilon)$ such that

$$
\begin{cases}
\ddot{u}_\varepsilon - \mathrm{div}(\sigma_\varepsilon + \varepsilon \nabla \dot{u}_\varepsilon) = 0 \text{ in } \Omega \times (0, T), \\
\nabla u_\varepsilon = \sigma_\varepsilon + p_\varepsilon \text{ in } \Omega \times (0, T), \\
\dot{p}_\varepsilon = \frac{\sigma_\varepsilon - P(\sigma_\varepsilon)}{\varepsilon} \text{ in } \Omega \times (0, T), \\
(\sigma_\varepsilon + \varepsilon \nabla \dot{u}_\varepsilon) \cdot \nu + \dot{u}_\varepsilon = 0 \text{ on } \partial\Omega \times (0, T), \\
+I.C.
\end{cases}
$$

✓ Existence and uniqueness :

$$u_\varepsilon \in W^{2,\infty}(L^2) \cap H^2(H^1), \quad \sigma_\varepsilon \in W^{1,\infty}(L^2), \quad p_\varepsilon \in H^1(L^2).$$

✓ Energy balance : $\forall\ t \geq 0$,

$$\frac{1}{2}\int_\Omega |\dot u_\varepsilon(t)|^2 + \frac{1}{2}\int_\Omega |\sigma_\varepsilon(t)|^2 + \int_0^t \int_\Omega |\dot p_\varepsilon| + \int_0^t \int_{\partial\Omega} |\dot u_\varepsilon|^2 \\ + \varepsilon \int_0^t \int_\Omega |\nabla \dot u_\varepsilon|^2 + \varepsilon \int_0^t \int_\Omega |\dot p_\varepsilon|^2 = \frac{1}{2}\int_\Omega |v_0|^2 + \frac{1}{2}\int_\Omega |\sigma_0|^2$$

✓ Weak convergences : $\exists\ u \in W^{2,\infty}(L^2) \cap \mathcal{C}^{0,1}(BV)$, $\sigma \in W^{1,\infty}(L^2)$ and $p \in \mathcal{C}^{0,1}(\mathcal{M}_b)$ such that

$$u_\varepsilon \rightharpoonup u, \quad \sigma_\varepsilon \rightharpoonup \sigma, \quad p_\varepsilon \rightharpoonup p.$$

✓ Consequences :

$$\begin{cases} \ddot u - \mathrm{div}\sigma = 0 \text{ in } \Omega \times (0,T), \\ Du = \sigma + p \text{ in } \Omega \times (0,T), \\ |\sigma| \leq 1 \text{ in } \Omega \times (0,T), \\ +I.C. \end{cases}$$

✓ Energy inequality : $\forall\ t \geq 0$,

$$\frac{1}{2}\int_\Omega |\dot u_\varepsilon(t)|^2 + \frac{1}{2}\int_\Omega |\sigma_\varepsilon(t)|^2 + \int_0^t \int_\Omega |\dot p_\varepsilon| + \frac{1}{2}\int_0^t \int_{\partial\Omega} |\dot u_\varepsilon|^2 + \frac{1}{2}\int_0^t \int_{\partial\Omega} |(\sigma_\varepsilon + \varepsilon \nabla \dot u_\varepsilon)\cdot \nu|^2 \leq \frac{1}{2}\int_\Omega |v_0|^2 + \frac{1}{2}\int_\Omega |\sigma_0|^2.$$

✓ Relaxation of the B.C. if $u_k \rightharpoonup^* u$ in BV, $\sigma_k \rightharpoonup \sigma$ in L^2,

$$|Du - \sigma|(\Omega) + \int_{\partial\Omega} \psi(u) \leq \liminf_k \left\{ \int_\Omega |\nabla u_k - \sigma_k| + \frac{1}{2}\int_{\partial\Omega} |u_k|^2 \right\},$$

where $\psi(z) = \frac{|z|^2}{2}\mathbf{1}_{|z|\leq 1} + (|z| - \frac{1}{2})\mathbf{1}_{|z|>1}$ and $\psi^*(z) = \frac{|z|^2}{2} + \mathrm{I}_{|z|\leq 1}$.

✓ Energy inequality : $\forall\ t \geq 0$,

$$\frac{1}{2}\int_\Omega |\dot{u}(t)|^2 + \frac{1}{2}\int_\Omega |\sigma(t)|^2 + \int_0^t |\dot{p}(s)|(\Omega)\,ds + \int_0^t \int_{\partial\Omega} \psi(\dot{u}) + \int_0^t \int_{\partial\Omega} \psi^*(\sigma\cdot\nu) \leq \frac{1}{2}\int_\Omega |v_0|^2 + \frac{1}{2}\int_\Omega |\sigma_0|^2.$$

✓ Energy equality : we use here that

- on $\partial\Omega \times (0,T)$: $\psi(\dot{u}) + \psi^*(\sigma\cdot\nu) \geq -(\sigma\cdot\nu)\dot{u}$;
- on $\Omega \times (0,T)$: $|\dot{p}| \geq [\sigma\cdot\dot{p}]$ (distributional duality of Anzellotti) because $|\sigma| \leq 1$.

We obtain the other inequality $\forall\ t \geq 0$,

$$\frac{1}{2}\int_\Omega |\dot{u}(t)|^2 + \frac{1}{2}\int_\Omega |\sigma(t)|^2 + \int_0^t |\dot{p}(s)|(\Omega)\,ds + \int_0^t \int_{\partial\Omega} \psi(\dot{u}) + \int_0^t \int_{\partial\Omega} \psi^*(\sigma\cdot\nu) = \frac{1}{2}\int_\Omega |v_0|^2 + \frac{1}{2}\int_\Omega |\sigma_0|^2.$$

✓ Flow rule and boundary condition :

$$|\dot{p}| = [\sigma\cdot\dot{p}] \text{ in } \Omega\times(0,T);$$

$$\psi(\dot{u}) + \psi^*(\sigma\cdot\nu) = -(\sigma\cdot\nu)\dot{u} \iff \sigma\cdot\nu + \psi'(\dot{u}) = 0 \text{ on } \partial\Omega\times(0,T).$$

Theorem (B.-Mifsud)

$\exists\,!$ $u \in W^{2,\infty}(L^2) \cap \mathcal{C}^{0,1}(BV)$, $\sigma \in W^{1,\infty}(L^2)$ and $p \in \mathcal{C}^{0,1}(\mathcal{M}_b)$ such that

$$
\begin{cases}
\ddot{u} - \mathrm{div}\sigma = 0 \text{ in } \Omega \times (0,T), \\
Du = \sigma + p \text{ in } \Omega \times (0,T), \\
|\sigma| \leq 1 \text{ in } \Omega \times (0,T), \\
[\sigma \cdot \dot{p}] = |\dot{p}| \text{ in } \Omega \times (0,T), \\
\sigma \cdot \nu + \psi'(\dot{u}) = 0 \text{ on } \partial\Omega \times (0,T), \\
+I.C.
\end{cases}
$$

Theorem (B.-Mifsud)

(u, σ, p) is a "variational" solution if and only if $U = (\dot{u}, \sigma) \in W^{1,\infty}(L^2)$ is an entropic solution : $\forall\ \kappa \in K(=: \mathbb{R} \times B)$ and $\forall\ \varphi \geq 0$,

$$
\int_0^T \int_\Omega |U - \kappa|^2 \partial_t \varphi + \sum_{i=1}^{2} A_i(U-\kappa)\cdot(U-\kappa)\partial_i \varphi + \int_\Omega |U_0 - \kappa|^2 \varphi(0) + \int_0^T \int_{\partial\Omega} M\kappa^+ \cdot \kappa^+ \varphi \geq 0.
$$

Regularity of solutions

✓ Explicit examples of singular solutions of jump or Cantor type (Suquet, Demyanov)

✓ For smooth compactly supported initial data, the solution remains smooth in short time :

- Finite speed propagation property : if $\mathrm{Supp}(u_0, v_0, \sigma_0, p_0) \subset \Omega$, then there exists $T^* > 0$ such that $\mathrm{Supp}(u(t), \sigma(t), p(t)) \subset \Omega$ for all $t \leq T^*$;
- Kato inequality : if (u_i, σ_i, p_i) $(i = 1, 2)$ are two solutions associated to initial data $(u_{0i}, v_{0i}, \sigma_{0i}, p_{0i})$ then

$$\int_0^T \int_\Omega [(\dot{u}_1 - \dot{u}_2)^2 + |\sigma_1 - \sigma_2|^2] \leq T \int_\Omega [(v_{01} - v_{02})^2 + |\sigma_{01} - \sigma_{02}|^2].$$

✓ Generalization by Mifsud to the general vectorial model of perfect plasticity for any closed and convex elasticity set.

Thank you !