Approximation of fracture problems via Gamma-convergence: state of art and new results

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Outline

- A variational model for fracture mechanics: Griffith's energy;
- Fracture problems as Γ-limit of damage problems;
- Hydraulic fracking and generalization;
- Some ideas for an existence result;

$$\mathcal{G}(u) := \int_\Omega \mathbb{C}e(u) : e(u)\,\mathrm{d}x + \kappa\mathcal{H}^{n-1}(J_u).$$

Braides - Dal Maso ('97) observed (for the scalar case) that such an energy cannot be (Γ)*-approximate by functional on* H^1 *of the form*

$$\int_\Omega f_\varepsilon(\nabla\phi)\,\mathrm{d}x$$

so no "Modica-Mortola"-type functional for such an energy!

Let's mention that there are (non)-local approximation

scalar case - Braides, Dal Maso ('97)

$$\int_\Omega f\left(\varepsilon \int_{B_\varepsilon} |\nabla u(y)|^2 \, \mathrm{d}y\right) \mathrm{d}x$$

extended to Griffith's by Negri ('05)

scalar case - De Giorgi, Gobbino, Mora ('96 - '01)

$$\frac{1}{\varepsilon}\int_{\Omega\times\Omega} \arctan\left(\frac{(u(x+\varepsilon\xi)-u(x))^2}{\varepsilon}\right) e^{-|\xi|^2} \, \mathrm{d}\xi \, \mathrm{d}y$$

extended to Griffith's by Alicandro, Focardi, Gelli ('01)

For a *local* approximation

Mumford-Shah: Ambrosio - Tortorelli ('90)

$$\int_\Omega \left[(|\nabla u|^2 + |\nabla v|^2)(1 - v^2)^{2h} + \frac{1}{4}(\alpha^2 h^2) v^2 \right] \mathrm{d}x$$

Fracture mechanics: Francfort, Marigò, Bourdin, Dal Maso, Iurlano, Focardi, Chambolle, Crismale, Conti... ('93 - '18)

The idea is to replace the crack with a *damaged region*. In particular with the introduction of a damage variable $v \in \mathrm{Lip}(\Omega; [0,1])$ representing the state of the material:

- the regions where $v = 1$ are the healthy regions;
- the regions where $v << 1$ are the damaged regions;

Idea: Introducing a damage variable $v \in [0, 1]$.

The energy can be thought as

$$\int_\Omega v\mathbb{C}e(u) : e(u)\,\mathrm{d}x + k|\omega|$$

where ω is the damaged region.

With the introduction of a non-increasing function ψ such that $\psi(1) = 0$ we can force ω to have size $\approx \varepsilon\mathcal{H}^{n-1}(J_u)$

$$\int_\Omega v\mathbb{C}e(u) : e(u)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega \psi(v)\,\mathrm{d}x$$

Griffith's energy as a damage limit

In order not to lose control on $e(u)$ in the damage region we do not allow the material to fully damage

$$AT_\varepsilon(u,v) = \int_\Omega (\eta_\varepsilon + v)\mathbb{C}e(u) : e(u)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega \psi(v)\,\mathrm{d}x + \int_\Omega \varepsilon|\nabla v|^2\,\mathrm{d}x$$

$(\eta_\varepsilon \to 0)$.

Or by adding a constraint of the form $|\nabla v| \leq 1/\varepsilon$

Variant of AT_ε has been introduced to approximate the cohesive fracture by Conti, Focardi, Iurlano ('15)

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two different regimes!

Case - $\eta_\varepsilon/\varepsilon \to \alpha > 0$:

$$AT_\varepsilon \longrightarrow_\Gamma \mathcal{G}(u) + \int_{J_u} \sqrt{\mathbb{C}([u] \odot \nu) : ([u] \odot \nu)}\, \mathrm{d}\mathcal{H}^{n-1}$$

Focardi, Iurlano ('14)

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Chambolle, Crismale ('17)

Scalar case: both regimes analyzed by Dal Maso, Iurlano ('13)

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The non interpenetration condition reads as $[u] \cdot \nu \geq 0$ on J_u. Namely

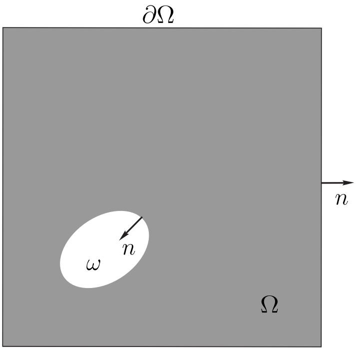
$$\mathcal{G}_{NI}(u) := \begin{cases} \mathcal{G}(u) & \text{if } [u] \cdot \nu \geq 0 \ \mathcal{H}^{n-1}\text{-a.e. on } J_u \\ +\infty & \text{otherwise} \end{cases}$$

Phase field approximation in $d = 2$: Conti, Chambolle, Francfort ('18).

$$\int_\Omega (\eta_\varepsilon + v)^2 [|e_d(u)|^2 + (\mathrm{div}(u))^2] \, \mathrm{d}x + \int_\Omega \left[\frac{\psi(v)}{\varepsilon} + \varepsilon |\nabla v|^2 \right] \mathrm{d}x.$$

A phase field approximation is still missing in $d \geq 3$ (more precisely is missing the Γ-convergence statement of the approximant candidate!).

Hydraulic damage



$$\mathcal{G}_\varepsilon(u,v) := \int_\Omega v\mathbb{C}e(u) : e(u)\,\mathrm{d}x + \kappa|\omega| - \int_\omega p\mathrm{div}(u)\,\mathrm{d}x.$$

$$\mathcal{G}_\varepsilon(u,v) := \int_\Omega (\eta_\varepsilon + v)\mathbb{C}e(u) : e(u)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega \psi(v)\,\mathrm{d}x - \int_\Omega \phi(v)p(x)\mathrm{div}(u)\,\mathrm{d}x.$$

Novotny, Van Goethem, Xavier ('17)

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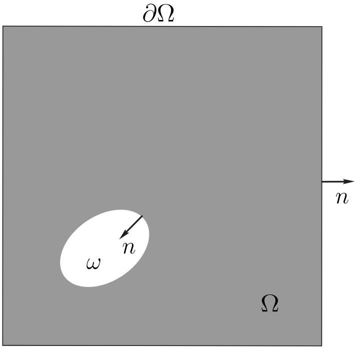

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In general we considered a potential F with linear growth at $+\infty$ and we analyzed

$$\mathcal{G}_\varepsilon(u,v) := \int_\Omega (\eta_\varepsilon + v)\mathbb{C}e(u) : e(u)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega \psi(v)\,\mathrm{d}x + \int_\Omega F(x, e(u), v)\,\mathrm{d}x.$$

with

$$-\sigma_1|M| \leq F(x, M, v) \leq \sigma_2|M|.$$

The constant σ_1 seems to be strongly dependent from the asymptotic behavior of $\eta_\varepsilon/\varepsilon$. In particular if $\eta_\varepsilon/\varepsilon \to 0$ then (with our analysis) we are forced to consider $\sigma_1 = 0$.

$\eta_\varepsilon/\varepsilon \approx \alpha > 0$ plus the contribution of potential F:

$$\mathcal{G}_\varepsilon(u,v) \to_\Gamma \mathcal{G}(u) + b\int_{J_u} \sqrt{\mathbb{C}([u]\odot\nu):([u]\odot\nu)}\,\mathrm{d}\mathcal{H}^{n-1} + c\int_{J_u} F_\infty(x,[u]\odot\nu,0)\,\mathrm{d}\mathcal{H}^{n-1} + \int_\Omega F(x,e(u),1)\,\mathrm{d}x = \mathcal{G}_0(u,1)$$

C. - Van Goethem ('18)

In particular with $F(x,M,v) = \phi(v)\mathrm{tr}\,(M)p(x)$ we recover the model of hydraulic fracking.

Existence of weak minimizers with Dirichlet boundary condition

$$\gamma_\varepsilon := \inf\left\{ \mathcal{G}_\varepsilon(u,v) \;\middle|\; \begin{array}{c} u = f,\ v = 1 \text{ on } \partial\Omega, \\ (u,v) \in H^1(\Omega;\mathbb{R}^n) \times V_\varepsilon, \\ \|u\|_{L^\infty} \le d \end{array} \right\}$$

$$\gamma_0 := \inf\left\{ \mathcal{G}_0(u,1) + \mathcal{R}(u,f) \mid u \in SBD^2(\Omega),\ \|u\|_{L^\infty} \le d \right\}$$

where

$$\begin{aligned} \mathcal{R}(u,f) := &\int_{\partial\Omega} F_\infty(z,[u-f]\odot\nu)\,\mathrm{d}\mathcal{H}^{n-1}(z) \\ &+ b\mathcal{H}^{n-1}(\{x \in \partial\Omega \mid u(x) \neq f(x)\}) \\ &+ a\int_{\partial\Omega} \sqrt{\mathbb{C}[u-f]\odot\nu\cdot[u-f]\odot\nu}\,\mathrm{d}\mathcal{H}^{n-1}(z). \end{aligned}$$

Then $\gamma_\varepsilon \to \gamma_0$ and minimizers of γ_ε converges to minimizers of γ_0 (C. Van Goethem ('18)).

Thank you for your attention!