Shorter tours and longer detours

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* Thanks for slides

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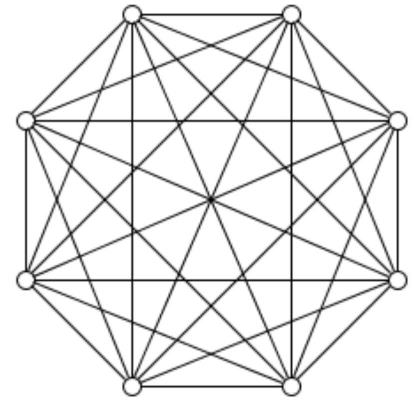
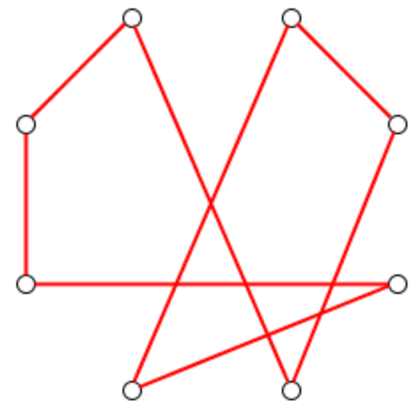
TSP and 2EC



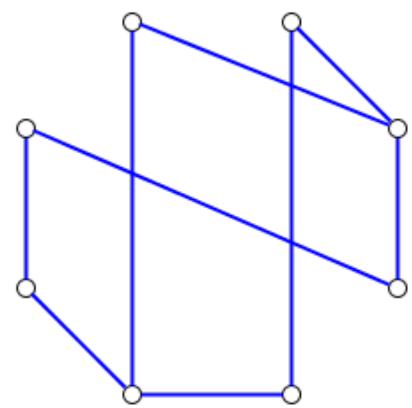
- Given graph K_n with metric weight function $w: E(G) \rightarrow \mathbb{R}^+$

TSP:



Find the min weight Hamilton cycle of G

2EC:



Find the min weight 2-edge connected subgraph of G

Subtour Elimination LP

$$z_n = \sum_{e \in E(K_n)} x_e w(e)$$

$$\sum_{e \in \delta(v)} x_e = 2 \quad for\ v \in V(K_n)$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \quad for\ \emptyset \subset S \subset V(K_n)$$

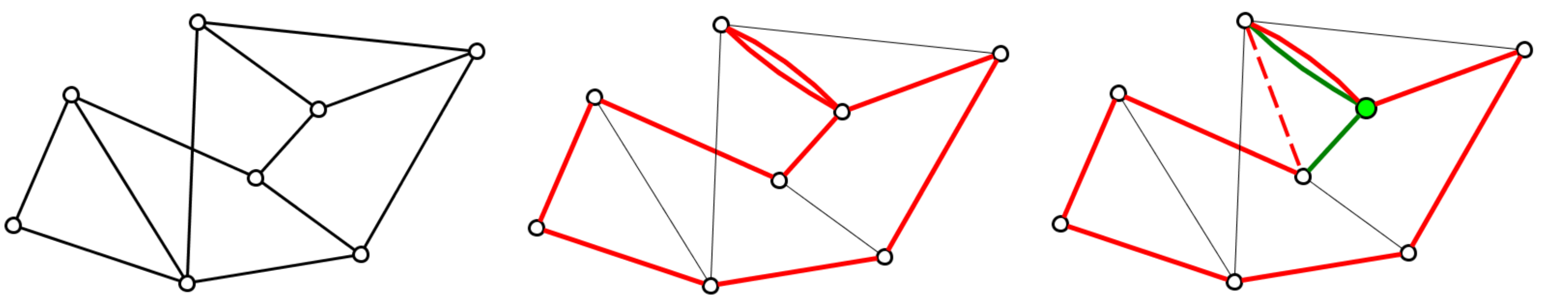
$$x_e \geq 0 \quad for\ e \in E(K_n)$$

$S(n)$

Tours and shortcuts

A tour of G

- Let G be a subgraph of K_n.
- If G has a connected Eulerian multigraph F, then K_n has a Hamilton cycle of weight at most $\sum_{e \in F} w(e)$.
- Proof: Shortcut every second visit to each node. By triangle inequality we never increase the weight, and total degree decreases.



The four-thirds conjecture

Minimum weight Hamilton cycle of $K_n \leq \frac{4}{3} \cdot z_n$

- Replace tour with 2-edge-connected spanning multigraph and we call it the 2EC-four-thirds-conjecture. Similarly we can make a 2EC-six-fifths-conjecture.
- Both TSP and 2EC open for anything below $\frac{3}{2}$ for decades

Definition

An α-vector of $G = (V, E)$ is a vector $v \in \mathbb{R}^{E(G)}$ where $v_e = \alpha$ for all $e \in E$.

Example

The $\frac{2}{n-1}$-vector of K_n, (call it) $v \in S(K_n)$

Proof

$$\sum_{e \in \delta(v)} v_e = \frac{2}{n-1} |\delta(v)| = 2$$

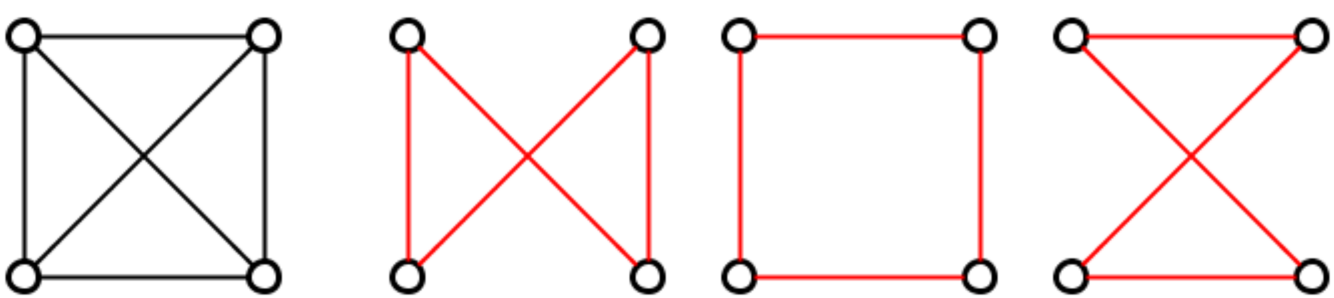
$$\sum_{e \in \delta(S)} v_e = \frac{2}{n-1} |\delta(S)| \geq 2$$

Uniform covers

- Is the α-vector for G in the convex hull of tours (or 2-edge-connected multigraphs) of G?
- If yes, we say G has an α-uniform cover for TSP (or 2EC)

Example

Graph K_4 has a $\frac{2}{3}$-uniform cover for TSP



Relation to uniform covers

Lemma

If the four-thirds conjecture holds, then for every $k \in \mathbb{Z}^+$, there is an $\frac{8}{3k}$-uniform cover for TSP on any k-edge-connected k-regular graph.

Proof

$x = \frac{2}{k}$ for every edge of the k-regular k-EC graph is in the subtour polytope

Four-thirds conjecture implies $\frac{4x}{3}$ is a convex combination of tours

A framework for approaching the conjecture

Lemma

If for every $k \in \mathbb{Z}^+$, there is an $\frac{8}{3k}$-uniform cover for TSP on any k-edge-connected k-regular graph, then the four-thirds conjecture follows.

Proof

x = optimal solution to the subtour elimination LP

$$t = \min \{m \in \mathbb{Z}^+ \colon mx \text{ is integer})$$

Lemma

If for every $k \in \mathbb{Z}^+$, there is an $\frac{8}{3k}$-uniform cover for TSP on any k-edge-connected k-regular graph, then the four-thirds conjecture follows.

Proof

Consider the graph $H = (V, E)$, where E contains tx_e copies of each $e \in E$

Graph H is $2t$-edge-connected and $2t$-regular:

$$\deg_H(v) = \sum_{e \in \delta_H(v)} tx_e = 2t$$

$$|\delta_H(S)| = \sum_{e \in \delta_H(S)} tx_e \geq 2t$$

Lemma

If for every $k \in \mathbb{Z}^+$, there is an $\frac{8}{3k}$-uniform cover for TSP on any k-edge-connected k-regular graph, then the four-thirds conjecture follows.

Proof

G is $2t$-edge-connected and $2t$-regular

The $\frac{8}{3(2t)}$-vector of G is in the convex hull of tours of G

For any weight function w, there is a tour with weight

$$\leq \frac{4}{3t} \sum_{e \in E(H)} t x_e w(e) = \frac{4}{3} \cdot z_{LP}$$

What is known?

General k	$\frac{3}{k}$-uniform cover for TSP [Christofides ’76, Wolsey ‘90]
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$k = 3$ and G Hamiltonian	$\frac{6}{7}$-uniform cover for TSP [Boyd, Sebő '17]

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$k = 3$	$\frac{15}{17}$-uniform cover for 2EC [polytime] [This talk]
$k = 3$ and G bipartite	$\frac{12}{13}$-uniform cover for TSP [polytime]
$k = 3$ and G bipartite	$\frac{7}{8}$-uniform cover for 2EC [polytime]

Theorem

There is an $\frac{18}{19}$-uniform cover for TSP on 3-edge-connected cubic graphs.

Theorem [Boyd, Iwata, Takazawa '13]

Let G be bridgeless and cubic, then G has a cycle cover C that covers all 3-edge and 4-edge cuts of G.

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Proof

Pick C as above. Let $M = E(G/C)$, and $F = E \setminus M \cup C$

$$\begin{cases} 1 & C \\ 4/5 & M \\ 0 & F \end{cases} \qquad \begin{cases} 3/4 & C \\ 3/2 & M \\ 3/2 & F \end{cases}$$

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$$\mathbf{\frac{15}{19}} \times \begin{cases} 1 & C \\ 4/5 & M \\ 0 & F \end{cases} \qquad \mathbf{\frac{4}{19}} \times \begin{cases} 3/4 & C \\ 3/2 & M \\ 3/2 & F \end{cases} \qquad = \qquad \begin{cases} 18/19 & C \\ 18/19 & M \\ 6/19 & F \end{cases}$$

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v is in the connector polytope of G/C

Proof

$$\sum_{e \in \delta(\mathcal{P})} x_e \geq |\mathcal{P}| - 1 \qquad for\ \mathcal{P} \in \Pi_n$$

$$x_e \geq 0 \qquad for\ e \in E$$

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v is in the connector polytope of G/C

$2v =$ conv comb of doubled connected subgraphs of G/C

$u + 2v =$ convex combination of tours

Proof

Pick C as above. Let $M = E(G/C)$, and $F = E \setminus M \cup C$

$$\frac{\mathbf{15}}{\mathbf{19}}\times\begin{cases}1 & C\\ 4/5 & M\\ 0 & F\end{cases} + \frac{\mathbf{4}}{\mathbf{19}}\times\begin{cases}\mathbf{3/4} & \boldsymbol{C}\\ \mathbf{3/2} & \boldsymbol{M}\\ \mathbf{3/2} & \boldsymbol{F}\end{cases} = \begin{cases}18/19 & C\\ 18/19 & M\\ 6/19 & F\end{cases}$$

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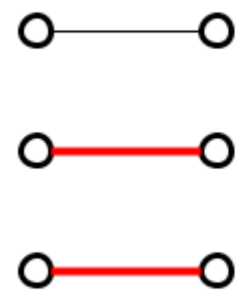
u is in the subtour polytope of G

$3u/2$ is a convex combination of tours of G



Tree Augmentation Problem (WTAP)

Given a tree T and non tree-edges (links), find a minimum cost set of links whose addition makes the tree 2-edge-connected

Theorem [Cheriyan, Jordan, Ravi '99]

Let y be a half-integral feasible solution to the cut LP, then $\frac{4}{3}y$ can be decomposed into integral feasible solutions.

$$\min \sum_{\ell \in L} y_\ell c(\ell)$$
$$y(\delta(e)) \geq 1 \quad \text{for } e \in T$$
$$y \geq 0$$

Theorem

There is an $\frac{15}{17}$-uniform cover for 2EC on 3-edge-connected cubic graphs.

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$$u = \begin{cases} 1 & C \\ 0 & M \\ 0 & F \end{cases} \qquad v = \begin{cases} 0 & C \\ 2/5 & M \\ 0 & F \end{cases}$$

v is in the subtour polytope for G/C

$$\frac{3}{2} v = \text{convex combination of tours of } G/C$$

$$u + \frac{3}{2} v = \text{convex combination of 2ECS's of } G$$

Proof

Pick C as above. Let $M = E(G/C)$, and $F = E \setminus M \cup C$

$$\frac{5}{17} \times \begin{cases} 1 & C \\ 3/5 & M \\ 0 & F \end{cases} \quad \frac{12}{17} \times \begin{cases} \mathbf{5/6} & \boldsymbol{C} \\ \mathbf{1} & \boldsymbol{M} \\ \mathbf{1} & \boldsymbol{F} \end{cases} \quad = \quad \begin{cases} 15/17 & C \\ 15/17 & M \\ 12/17 & F \end{cases}$$

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$u =$ convex combination of connected subgraphs of G

$$u = \sum_{i=1}^{k} \lambda_i T_i \ , \lambda \in \mathbb{R}^+, \|\lambda\|_1 = 1$$

Proof

Pick C as above. Let $M = E(G/C)$, and $F = E \setminus M \cup C$

$$\frac{5}{17} \times \begin{cases} 1 & C \\ 3/5 & M \\ 0 & F \end{cases} \quad \frac{12}{17} \times \begin{cases} 5/6 & C \\ 1 & M \\ 1 & F \end{cases} \quad = \begin{cases} 15/17 & C \\ 15/17 & M \\ 12/17 & F \end{cases}$$

$$\sum_{i=1}^{k} \lambda_i T_i = \begin{cases} 1/2 & C \\ 1 & M \\ 1 & F \end{cases} \qquad u_i = \begin{cases} 0 & T_i \\ 1/2 & \text{not in } T_i \end{cases}$$

By CJR: $\frac{4}{3} u_i =$ convex combination of 1-covers of T_i

$T_i + \frac{4}{3} u_i =$ convex comb of 2-edge-conn multigraphs of G

$\sum_{i=1}^{k} \lambda_i (T_i + \frac{4}{3} u_i) =$ convex comb of 2-edge-conn multigraphs of G

Theorem

There is an $\frac{12}{13}$-uniform cover for TSP on 3-edge-connected **bipartite** cubic graphs.

Theorem

There is an $\frac{7}{8}$-uniform cover for 2EC on 3-edge-connected **bipartite** cubic graphs.

Lemma

Let G be bridgeless, cubic and **bipartite** graph, then G has a cycle cover C that covers all 3-edge, 4-edge, and **5-edge** cuts of G.

What is known?

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$k = 3$ and G bipartite	$\frac{12}{13}$-uniform cover for TSP [polytime]
$k = 3$ and G bipartite	$\frac{7}{8}$-uniform cover for 2EC [polytime]

Node-weighted w

- Given graph $G = (V, E)$, function $f: V \to \mathbb{R}^+$

  define $w: E \to \mathbb{R}^+$

$$\text{for } e = (u, v) \in E: w(e) = f(u) + f(v)$$

TSP:

Find the min weight Eulerian connected multigraph of G

2EC:

Find the min weight 2-edge connected multigraph of G

A "BIT" beyond: A $\frac{7}{5}$-approximation for node weight TSP on cubic 3EC graphs

Edge weights w(u,v) = w(u) + w(v). Let $W = \sum_v w(v)$

- Subtour bound Z = 2W (Assign $\frac{2}{3}$ everywhere)
- BIT cycle cover C costs 2W
- G/C is 5-edge connected so putting $\frac{2}{5}$ on these edges dominates a convex combination of spanning trees. Double and add to tree for a cost of $\frac{4}{5}$ on the edges of G/C. Additional cost = $\frac{4}{5}$ W
- Total cost = $(2 + \frac{4}{5})$ W $\leq (1 + \frac{2}{5})$ Z

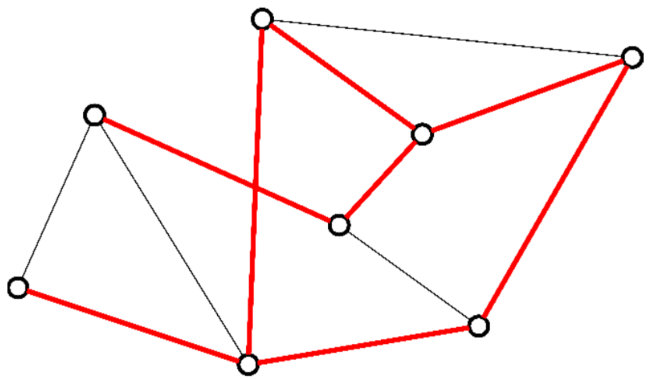
Other Results (see arXiv)

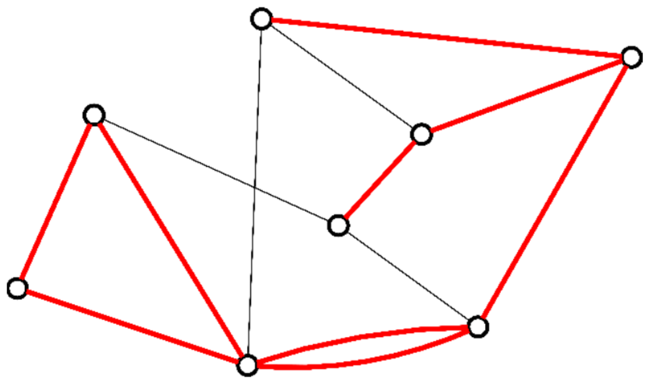
- Node Weighted on 3-edge-connected, cubic
  - $\frac{7}{5}$ for TSP
  - $\frac{13}{10}$ for 2EC
  - Refinements for bipartite cases
- Node Weighted on 2-edge-connected, cubic
  - $\frac{4}{3}$ for 2EC

Connector

- A connector F of G is a connected spanning multigraph of G, where F has at most 2 copies of every edge in G.

Example: A spanning tree is a connector.





Theorem: There are connectors $F_1, \dots, F_k$ of G such that

1. $x^* \geq \sum_{i=1}^{t} \lambda_i F_i$, where $\lambda_i > 0$, $\sum_{i=1}^{t} \lambda_i = 1$
2. Every F_i has an even number of edges crossing a 2-edge cut in G

Implication: 4/3 approximation for node-weighted 2EC in subcubic graphs

Open Problems

- Get a better than $\frac{3}{4}$ - uniform cover for TSP on 4-edge-connected 4-regular graphs.
- Improve $\frac{18}{19}$ - uniform cover of TSP on 3-edge-connected cubic graphs.
- Find a $(\frac{3}{2} - \epsilon)$-approximation for TSP or 2EC.

Paper at https://arxiv.org/pdf/1707.05387.pdf