Combinatorial expansion of the Fredholm determinant representation for isomonodromic tau function and conformal field theory

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Based on works with M. Caffasso, N. Iorgov, O. Lisovyy, A. Marshakov

Tau Functions of Integrable Systems and Their Applications

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*Disclaimer: not all formulas are necessarily correct. Correct versions are in the papers. I hope.

Plan of the talk

- Definition of the determinant
- Basis in the space $L^2(S^1) \otimes \mathbb{C}^N$
- Minor expansion (von Koch formula)
- Combinatorics of minors
- Matrix elements
- Computation of minors
- More combinatorial formulas (Nekrasov functions)

CFT part:

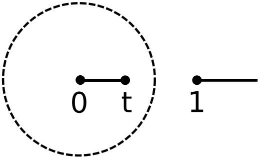
- Semi-infinite wedge and free fermions
- Heisenberg, Virasoro and W_N-algebras
- Vertex operators
- Wick theorem and appearance of determinants

Building blocks for the determinant (reminder)

Number of points: $n = 4$, regular sungularities
Rank of the system: N, arbitrary (for Painlevé VI $N = 2$)

$$\Psi(z)\left(-\frac{\overleftarrow{d}}{dz} + \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}\right) = 0$$

In the completely solvable case: $A_t = \alpha_t \cdot id + \beta_t \cdot u_t \otimes v_t$, $A_1 = \alpha_1 \cdot id + \beta_1 \cdot u_1 \otimes v_1$



3-point solutions: $\phi_+(z) \in \mathbb{C}[[z]] \otimes End(\mathbb{C}^N)$, $\phi_+(z) \in \mathbb{C}[[z^{-1}]] \otimes End(\mathbb{C}^N)$:

$$z^{\mathfrak{S}}\phi_+(z)\left(\frac{\overleftarrow{d}}{dz} - \frac{\tilde{A}_0}{z} - \frac{\tilde{A}_1}{z-1}\right) = 0\,, \qquad z^{\mathfrak{S}}\phi_-(z)\left(\frac{\overleftarrow{d}}{dz} - \frac{\tilde{A}_0}{z} - \frac{\tilde{A}_t}{z-t}\right) = 0\,.$$

Where $e^{2\pi i \mathfrak{S}} = M_0 M_t$.

$$\tau = t^{\#} \det\left(1 \pm K\right) \qquad K = \begin{pmatrix} 0 & \mathsf{a} \\ \mathsf{d} & 0 \end{pmatrix}$$

Hilbert spaces (functions on a circle $t < |z| < 1$): $L^2(S^1) \otimes \mathbb{C}^N \simeq \mathbb{C}[[z, z^{-1}]] \otimes \mathbb{C}^N$

Operators with matrix integrable kernels:

$$(\mathsf{a}g)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{a}\left(z, z'\right) g\left(z'\right) dz' , \qquad \mathsf{a}\left(z, z'\right) = \frac{\phi_+(z)\,\phi_+\left(z'\right)^{-1} - 1}{z - z'},$$

$$(\mathsf{d}g)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \mathsf{d}\left(z, z'\right) g\left(z'\right) dz' , \qquad \mathsf{d}\left(z, z'\right) = \frac{1 - \phi_-(z)\,\phi_-\left(z'\right)^{-1}}{z - z'}.$$

Action of the operators

$$\mathsf{a} : z^{-1}\mathbb{C}[[z^{-1}]] \otimes \mathbb{C}^N \to \mathbb{C}[[z]] \otimes \mathbb{C}^N , \qquad \mathsf{d} : \mathbb{C}[[z]] \otimes \mathbb{C}^N \to z^{-1}\mathbb{C}[[z^{-1}]] \otimes \mathbb{C}^N$$

Bases and minor expansion

Basis in $\mathbb{C}$: e_α – will be referred as "color"

Basis in $\mathbb{C}[[z,z^{-1}]]$: $z^{p-\frac{1}{2}}$, $p \in \mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$. Positive subspace: $p > 0$, negative: $p < 0$.
Pairing with the dual space: $\langle z^{p-\frac{1}{2}}, z^{q-\frac{1}{2}} \rangle = \oint dz \cdot z^{p-\frac{1}{2}} z^{q-\frac{1}{2}} = \delta_{p,-q}$

Basis in $\mathbb{C}[[z,z^{-1}]] \otimes \mathbb{C}^N$: $e_{\alpha,p} := z^{p-\frac{1}{2}} \otimes e_\alpha$

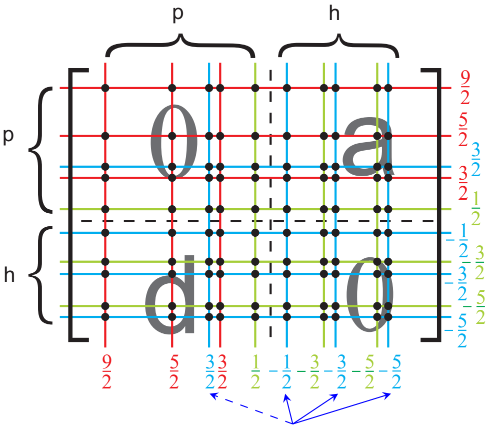
Von Koch formula:

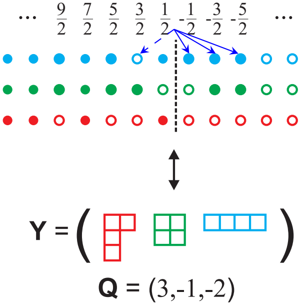
$$\det(1+K) = \sum_{k=0}^{\infty} \operatorname{tr}\left(\Lambda^k K\right) = \sum_{k=0}^{\infty} \sum_{\{i_1,\dots,i_k\}} K_{i_1\dots i_k}^{i_1\dots i_k} = \sum_{k=0}^{\infty} \sum_{\{i_1,\cdots,i_l\}} \begin{vmatrix} K_{i_1}^{i_1} & K_{i_2}^{i_1} & \dots & K_{i_k}^{i_1} \\ K_{i_1}^{i_2} & K_{i_2}^{i_2} & \dots & K_{i_k}^{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ K_{i_1}^{i_k} & K_{i_2}^{i_k} & \dots & K_{i_k}^{i_k} \end{vmatrix} =$$

$$= \sum_{i_1} K_{i_1}^{i_1} + \sum_{i_1<i_2} \begin{vmatrix} K_{i_1}^{i_1} & K_{i_2}^{i_1} \\ K_{i_1}^{i_2} & K_{i_2}^{i_2} \end{vmatrix} + \sum_{i_1<i_2<i_3} \dots + \dots$$

— sum of all possible diagonal minors.

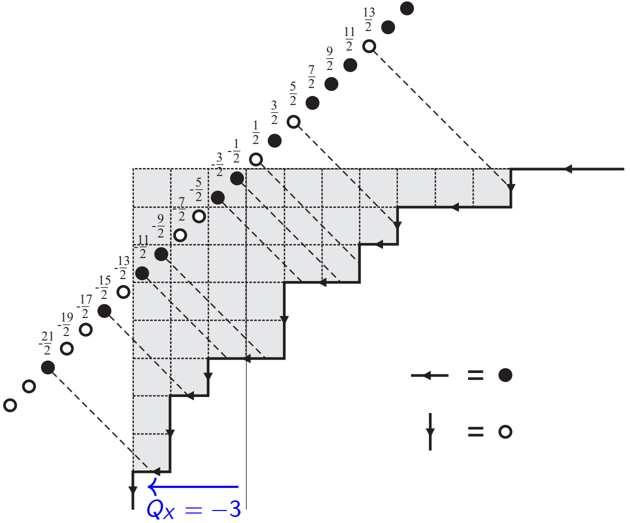
Example: $N = 3$, basis: $e_{1,q}$, $e_{2,q}$, $e_{3,q}$.
$\{i_1, \ldots, i_{2k}\} = \{(\alpha_1, p_1), \ldots, (\alpha_k, p_k)\} \sqcup \{(\alpha_1, h_1), \ldots, (\alpha_k, h_k)\}$





All minors are described by the N–tuples of Maya diagrams.

From Maya to Young diagrams



$$p_X = \left\{\frac{1}{2}, \frac{5}{2}, \frac{13}{2}\right\}, \; h_X = \left\{-\frac{21}{2}, -\frac{15}{2}, -\frac{11}{2}, -\frac{9}{2}, -\frac{3}{2}, -\frac{1}{2}\right\}$$

Computation of the matrix elements

We need to know the Fourier expansion:

$$a_{\beta}^{\alpha}(z,z') = \frac{\phi_{+}(z)\,\phi_{-}(z')^{-1} - \mathbb{1}}{z-z'} = \sum_{p,q\in\mathbb{Z}'_{+}} a_{\beta,-q}^{\alpha,\ p}\, z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}+q}$$

and the same for d.
Warm-up example for $N=1$:

$$a(z,w) = \frac{(1-z)^{\nu}(1-w)^{-\nu} - 1}{z-w} = \sum_{a,b=0}^{\infty} \frac{(-\nu)_{a+1}(\nu)_{b+1}}{(a+b+1)\nu a! b!} z^{a} w^{b}$$

— we see here Cauchy matrix with two diagonal factors. $(\nu)_k = \nu(\nu+1)\dots(\nu+k-1)$.

Proof: $\mathcal{L}_0 = z\partial_z + z'\partial_{z'} + 1$

$$\mathcal{L}_0(l.h.s.) = \mathcal{L}_0 a(z,w) = -\nu(1-z)^{\nu-1}(1-w)^{-\nu-1}$$

$$\mathcal{L}_0(r.h.s.) = \mathcal{L}_0 a(z,w) = \sum_{a,b=0}^{\infty} \frac{(-\nu)_{a+1}(\nu)_{b+1}}{\nu a! b!} z^{a} w^{b}$$

$$\mathcal{L}_0(l.h.s.) = \mathcal{L}_0 \mathrm{a}\left(z, z'\right) = \frac{\left(z\partial_z + z'\partial_{z'}\right)\phi_+\left(z\right)\phi_+\left(z'\right)^{-1}}{z - z'} = \text{(apply linear system)} =$$

$$= \left[\mathrm{a}\left(z, z'\right), \mathfrak{S}\right] - \frac{\phi_+\left(z\right)}{z-1}\tilde{A}_1 \frac{\phi_+\left(z'\right)^{-1}}{z'-1} \quad \text{— finite rank}$$

$$\mathcal{L}_0(r.h.s.) = \mathcal{L}_0 \mathrm{a}\left(z, z'\right) = \sum_{p,q \in \mathbb{Z}'_+} (p + q + \sigma_\alpha - \sigma_\beta)\mathrm{a}^{\alpha;\ p}_{\beta;-q}\, z^{-\frac{1}{2}+p} z'^{-\frac{1}{2}+q}$$

$$\mathrm{a}^{\alpha;\ p}_{\beta;-q} = \sum_{r=1}^{\mathrm{rk}A_1} \frac{\left(\varphi_r\right)^{p;\alpha}\left(\bar{\varphi}_r\right)_{q;\beta}}{p + q + \sigma_\alpha - \sigma_\beta} \quad \text{— for } \mathrm{rk}A_1 = 1 \text{ again Cauchy matrix}$$

Solution $\phi_+(z)$ for $\mathrm{rk}A_1 = 1$ (rigid 3-point system):

$$\phi_+(z)^\alpha_\beta = N_{\alpha\beta} z^{1-\delta_{\alpha\beta}}\, {}_N F_{N-1}\left(\begin{matrix} \left\{1 - \delta_{\alpha\beta} - \theta^\infty_\gamma + \sigma_\alpha\right\}_{\gamma=\overline{1,N}} \\ \left\{1 + \sigma_\alpha - \sigma_\gamma + \delta_{\beta\gamma} - \delta_{\alpha\beta}\right\}_{k=\overline{1,N};\gamma\neq\alpha} \end{matrix} \middle|\, z\right)$$

θ^∞_α — eigenvalues of $A_0 + A_1$, σ_α — eigenvalues of A_0, $N_{\alpha\beta}$ — simple normalization.
(*) There are possible degenerations to ${}_l F_{N-1}$. Marshakov's talk: $l = 0$.

Since all matrices are essentially of Cauchy type, minors are computable:

$$\det_{ij} \frac{1}{X_i - Y_j} = \frac{\prod_{i<j}(X_i - X_j)(Y_j - Y_i)}{\prod_{ij}(X_i - X_j)}$$

$$Z^{IJ}[\phi_+] = \mathrm{a}^I_J = \pm \prod_{(p,\alpha)\in I} \varphi^{p;\alpha} \prod_{(-p,\alpha)\in J} \bar{\varphi}_{p;\alpha} \times \frac{\prod\limits_{\imath,\jmath\in I;\imath<\jmath}(x_\imath - x_\jmath) \prod\limits_{\imath,\jmath\in J_{k-1};\imath<\jmath}(y_\jmath - y_\imath)}{\prod\limits_{\imath\in I}\prod\limits_{\jmath\in J}(x_\imath - y_\jmath)}. \quad (1)$$

$$x_\imath = p + \sigma_\alpha, \quad \imath \equiv (\alpha, p) \in I, \qquad y_\jmath = -q + \sigma_\beta, \quad \jmath \equiv (\beta, q) \in I$$

$$Z^{IJ}[\phi_+] = Z^{\vec{Y},\vec{Q}}[\phi_+] = \pm e^{\vec{\eta}_0\cdot\vec{Q}} \frac{C(\vec{\sigma}+\vec{Q},\vec{\theta}^\infty)}{C(\vec{\sigma},\vec{\theta}^\infty)} \sqrt{Z_{vec}(\vec{\sigma}+\vec{Q}|\vec{Y})} Z_{bif}(\vec{\sigma}+\vec{Q},\vec{\theta}^\infty|\vec{Y},\emptyset)$$

$$Z_{\vec{Y},\vec{Q}}[\phi_-] = \pm e^{\vec{\eta}\cdot\vec{Q}} \frac{C(\vec{\theta}^0,\vec{\sigma}+\vec{Q})}{C(\vec{\theta}^0,\vec{\sigma})} Z_{bif}(\vec{\theta}^0,\vec{\sigma}+\vec{Q}|\vec{Y}) \sqrt{Z_{vec}(\vec{\sigma}+\vec{Q}|\vec{Y})} \times t^{(\vec{\sigma}+\vec{Q})^2-\vec{\theta}^2+|\vec{Y}|}$$

$$C(\vec{\theta}^0, \vec{\sigma} + \vec{Q}) C(\vec{\sigma} + \vec{Q}, \vec{\theta}^\infty) = \frac{\prod_{\alpha\beta} G(1 + \theta^0_\alpha - \sigma_\beta - Q_\beta) G(1 + \sigma_\beta + Q_\beta - \theta^\infty_\alpha)}{\prod_{\alpha\beta} G(1 + \sigma_\alpha - \sigma_\beta + Q_\alpha - Q_\beta)}$$

$G(x)$ — Barnes G-function: $G(x+1) = \Gamma(x) G(x)$

$$Z^{\vec{Y},\vec{Q}}[\phi_+] Z_{\vec{Y},\vec{Q}}[\phi_-] = t^{(\vec{\sigma}+\vec{Q})^2 - \vec{\theta}^2 + |\vec{Y}|} e^{\vec{\eta}_{tot} \cdot \vec{Q}} \times$$

$$\times \frac{\prod_{\alpha\beta} Z_b(1 + \theta^0_\alpha - \sigma_\beta - Q_\beta | \emptyset, Y_\alpha) Z_b(1 + \sigma_\beta + Q_\beta - \theta^\infty_\alpha | Y_\beta, \emptyset)}{\prod_{\alpha\beta} Z_b(1 + \sigma_\alpha - \sigma_\beta + Q_\alpha - Q_\beta | Y_\alpha, Y_\beta)}$$

So *the generic* isomonodromic tau-function for 4-point Fuchsian system with $\mathrm{rk} A_t = \mathrm{rk} A_1 = 1$ is given by

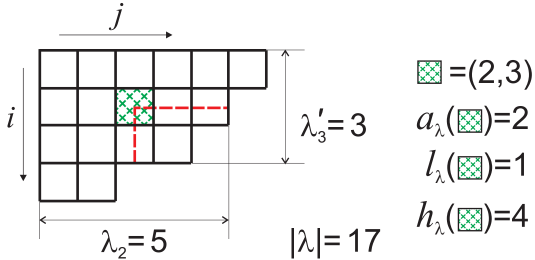
$$\tau(t) = t^{\#} \cdot \sum (-1)^{|I|} \mathrm{a}^I_J \mathrm{d}^J_I = t^{\#} \cdot \sum_{\vec{Q}: \sum Q_\alpha = 0} \sum_{\vec{Y}} Z^{\vec{Y},\vec{Q}}[\phi_+] Z_{\vec{Y},\vec{Q}}[\phi_-] =$$

$$= \sum_{\vec{Q}: \sum Q_\alpha = 0} e^{\vec{\eta}_{tot} \cdot \vec{Q}} t^{(\vec{\sigma}+\vec{Q})^2 - \vec{\theta}^2 + |\vec{Y}|} \times (\cdots) \text{ — Fourier transformation in } \vec{\sigma}$$

Definition of Z_b is on the next slide.

$a_Y(s)$ and $l_Y(s)$ — arm and leg lengths in the diagram Y. They depend on box s on the plane (s not necessarily in Y). $h_Y(s) = a_Y(s) + l_Y(s) + 1$ is a hook length.

$$Z_b(\nu|Y',Y) = \prod_{s\in Y'}(\nu+1+a_{Y'}(s)+l_Y(s))\prod_{t\in Y}(\nu-1-a_Y(t)-l_{Y'}(t))$$



Special function

$$\mathcal{B}(\vec{\theta}^{\infty},\vec{\sigma},\vec{\theta}_0|t)=t^{\vec{\sigma}^2-\vec{\theta}^2}\cdot\sum_{\vec{Y}}t^{|\vec{Y}|}\frac{\prod_{\alpha\beta}Z_b(1+\theta^0_\alpha-\sigma_\beta|\emptyset,Y_\alpha)Z_b(1+\sigma_\beta-\theta^\infty_\alpha|Y_\beta,\emptyset)}{\prod_{\alpha\beta}Z_b(1+\sigma_\alpha-\sigma_\beta|Y_\alpha,Y_\beta)}$$

is called 4-point conformal block of the W-algebra. This exact formula was found from the Alday-Gaiotto-Tachikawa relation between $D=2$ CFT and $\mathcal{N}=2$ $D=4$ supersymmetric Yang-Mills (or Seiberg-Witten) theory.
Will be defined alternatively below.

$N = 1$, $n = 4$. From the combinatorial formula:

$$\tau(t) = \sum_{Y} t^{|Y|} \prod_{(i,j)\in Y} \frac{(\nu + i - j)(-\tilde{\nu} + i - j)}{h_Y^2(i,j)}$$

From explicit computation, e.g., via Toeplitz determinant:

$$\tau(t) = \det(1 - \mathsf{a} \circ \mathsf{d}) = (1 - t)^{\nu\tilde{\nu}}$$

$$\mathsf{a}(z, z') = \frac{(1-z)^{\nu}(1-z')^{-\nu} - 1}{z - z'} = -\nu + \frac{\nu(\nu-1)}{2} z - \frac{\nu(\nu+1)}{2} z' + \ldots$$

$$\mathsf{d}(z, z') = \frac{1 - (1 - t/z)^{\tilde{\nu}}(1 - t/z')^{-\tilde{\nu}}}{z - z'} = \ldots$$

Free-fermionic CFT

Consider $\Lambda^{\frac{\infty}{2}}\left(\mathbb{C}[[\zeta,\zeta^{-1}]]\otimes\mathbb{C}^N\right)$. There are two sets of operators:

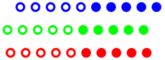
$$\psi_{\alpha,p} = e_{\alpha,p}\wedge, \qquad \psi^*_{\alpha,p} = \imath_{e^{\alpha,-p}}.$$

Anticommutation relations:

$$\{\psi_{\alpha,p},\psi^*_{\beta,q}\} = \delta_{\alpha\beta}\delta_{p,-q}$$

Vacuum vector (picture for $N=3$):

$$|0\rangle = e_{1,\frac{1}{2}}\wedge e_{2,\frac{1}{2}}\wedge\wedge\ldots e_{N,\frac{1}{2}}\wedge e_{1,\frac{3}{2}}\wedge\ldots$$



Fermionic currents:

$$\psi_\alpha(z) = \sum_p \frac{\psi_{\alpha,p}}{z^{p+\frac{1}{2}+\sigma_\alpha}}, \qquad \psi^*_\alpha(z) = \sum_p \frac{\psi^*_{\alpha,p}}{z^{p+\frac{1}{2}-\sigma_\alpha}}.$$

Operator product expansion:

$$\psi^*_\alpha(z)\psi_\beta(w) = \frac{\delta_{\alpha\beta}}{z-w} + reg.$$

Regular product $(AB)(w)$:

$$A(z)B(w) = \sum_{k=-l}^{\infty}(AB)_k(w)(z-w)^k, \quad \text{then} \quad (AB)(w) := (AB)_0(w).$$

Generators of $W_N \oplus H$ algebra:

$$U_k(z) = \sum_{\alpha=1}^{N} (\psi_\alpha^*(z)\partial^{k-1}\psi_\alpha(z)) = \sum_{n\in\mathbb{Z}} \frac{U_{k,n}}{z^{k+n}}$$

Properties:

- There are always two sub-algebras: Heisenberg $U_1(z)$ and Virasoro $T(z) = U_2(z) - (\dots)$ with central charge $c = N - 1$.
- Only $U_1(z), \dots, U_N(z)$ are independent, others are expressed through them in some non-linear way.
- $U_{k,n}|0\rangle = 0$ for $n > 0$, $W_{k,0}|0\rangle = u_k(\vec{\sigma})|0\rangle$, u_k — some symmetric functions.
- $U_k(z)$ do not change charges Q_α. All space is decomposed into W_N modules with charges $\vec{\sigma} + \vec{Q}$.

Examples:

- $N = 1$ Heisenberg algebra: $U_1(z)U_1(w) = \frac{1}{(z-w)^2} + reg.$, $[U_{1,k}, U_{1,m}] = k\delta_{k,-m}$.
- $N = 2$ Virasoro + Heisenberg: $T(z)T(w) = \frac{1/2}{(z-w)^2} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg.$
- $N = 3$: W_3 algebra + Heisenberg — already not a Lie algebra: $U_3 \times U_3 \sim (T^2) + \dots$

Defining relation (cf. with $W_{k,n>0}|0\rangle = 0$):

$$U_k(z)V_{\boldsymbol{\nu}}(t) = \left(\frac{u_k(\boldsymbol{\nu})}{(z-t)^k} + \text{less singular}\right) V_{\boldsymbol{\nu}}(t)$$

$N = 1$ example. If $U_1(z) = i\partial\varphi(z)$, then

$$V_\nu(t) = e^{i\nu\varphi(t)}$$

Back to the toy Fredholm determinant:

$$\langle 0|V_{\tilde{\nu}}(1)V_\nu(t)|0\rangle = (1-t)^{\nu\tilde{\nu}}$$

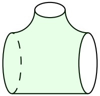
4-point conformal block by definition:

$$\mathcal{B}(\vec{\theta}^\infty, \vec{\sigma}, \vec{\theta}^0|t) := \langle\vec{\theta}_\infty|V_1(1)\mathcal{P}_{\vec{\sigma}}V_t(t)|\vec{\theta}^0\rangle$$

$\mathcal{P}_{\vec{\sigma}}$ is a projector onto W_N module with charge $\vec{\sigma}$.

Dictionary of the Isomonodromy-CFT correspondence

Isomonodromy	**CFT**
$N \times N$ Fuchsian system with regular singularities	W_N algebra with $c = N - 1$
monodromy $M \sim e^{2\pi i \,\mathrm{diag}\, \boldsymbol{\theta}}$	W-charge $\boldsymbol{\theta}$, like field $: e^{i(\boldsymbol{\theta}, \vec{\phi}(z))} :$
tau function	holomorphic correlation function of primary fields: $\tau(z_1, ..., z_n) = \langle V_1(z_1) ... V_n(z_n) \rangle$
solution of the linear problem	expectation value of the fermion ψ^*: $\Psi(z) = \frac{1}{\tau} \langle \psi^*(z) V_1(z_1) ... V_n(z_n) \rangle$
$\mathrm{tr}\, A(z)^2$	expectation value of the energy-momentum tensor $\mathrm{tr}\, A(z)^2 = \frac{1}{\tau} \langle T(z) O_1(z_1) ... O_n(z_n) \rangle$
regular singularities $(z - z_k)^{-1}$	W_N primaries
irregular singularities $(z - z_k)^{-p}$	Gaiotto states



$= $ Fermionic vertex operator

Axioms:

1. V is a group-like element: $V_{\boldsymbol{\theta}_k}(z_k)\psi_{\alpha,p}V_{\boldsymbol{\theta}_k}(z_k)^{-1} = \sum_{\beta,q} C_{\alpha,p;\beta,q}\psi_{\beta,q}$
2. Its radially-ordered 3-point function with two fermions is
$$\langle 0|\mathcal{R}V_{\boldsymbol{\theta}_k}(z_k)\psi^*_\alpha(x)\psi_\beta(y)|0\rangle = \frac{\left[\Phi^{[k]}(x)\Phi^{[k]}(y)^{-1}\right]_{\alpha\beta}}{x-y}\langle 0|V_{\boldsymbol{\theta}_k}(z_k)|0\rangle$$

Here $\Phi^{[k]}(x)$ is a solution of k-th auxiliary 3-point problem. Consequences (can be checked straightforwardly under general correlation function):

1. $V_{\boldsymbol{\theta}_k}(z_k)$ is W-primary.
2. $\tau(z_1,\ldots,z_n) = \langle 0|V_{\boldsymbol{\theta}_1}(z_1)\ldots V_{\boldsymbol{\theta}_n}(z_n)|0\rangle$
3. Matrix elements of $V_{\boldsymbol{\theta}_k}(z_k)$ — determinants $\Longrightarrow$ tau function can be given by single *Fredholm determinant*

$$\langle\vec{\theta}|V_{\boldsymbol{\nu}}(1)|\{p_{\alpha,i}\},\{q_{\alpha,i}\};\vec{\sigma}\rangle =$$

$$= \langle\vec{\theta}|V_{\boldsymbol{\nu}}(1)\prod_{\alpha=1}^{N}\prod_{i=1}^{d_\alpha}\psi^*_{\alpha,-p_{\alpha,i}}\psi_{\alpha,-q_{\alpha,i}}|\vec{\sigma}\rangle = \det_{\alpha i,\beta j}\langle\vec{\theta}|V_{\boldsymbol{\nu}}(1)\psi^*_{\alpha,-p_{\alpha,i}}\psi_{\beta,-q_{\beta,j}}|\vec{\sigma}\rangle = \det_{\alpha i,\beta j}\mathsf{a}^{\alpha p_{i,\alpha}}_{\beta q_{j,\beta}}$$

Derivation of the Fredholm determinant:

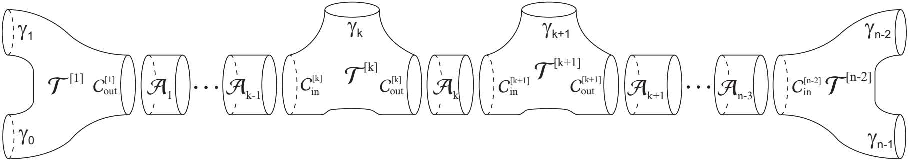
$$\tau(t) = \langle\vec{\theta}_\infty|V_{\boldsymbol{\nu}_1}(1)V_{\boldsymbol{\nu}_t}(t)|\vec{\theta}_0\rangle =$$

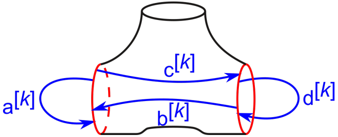
$$= \sum_{\{\{p_{\alpha,i}\},\{q_{\alpha,i}\}\}}\langle\vec{\theta}_\infty|V_{\boldsymbol{\nu}_1}(1)|\{p_{\alpha,i}\},\{q_{\alpha,i}\};\vec{\sigma}\rangle\langle\{q_{\alpha,i}\},\{p_{\alpha,i}\};\vec{\sigma}|V_{\boldsymbol{\nu}_t}(t)|\vec{\theta}_0\rangle =$$

$$= \sum_{I,J}\det\mathsf{a}^I_J\cdot\det(-\mathsf{d})^J_I(t) = \sum_{n=0}^{\infty}\sum_{|I|=|J|=n}\det\mathsf{a}^I_J\cdot\det(-\mathsf{d})^J_I(t) =$$

$$= \sum_{n=0}^{\infty}\mathrm{Tr}(-1)^n\Lambda^n(\mathsf{a})\Lambda^n(\mathsf{d(t)})\sum_{n=0}^{\infty}\mathrm{Tr}\Lambda^n(-\mathsf{a}\circ\mathsf{d(t)}) = \det(1-\mathsf{ad}(t)) = \det\begin{pmatrix}1 & \mathsf{a}\\ \mathsf{d} & 1\end{pmatrix}$$

Generalization to higher number of points (sphere)





$$\left(\mathsf{a}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_{\mathrm{in}}^{[k]}} \frac{\left[\Phi_+^{[k]}(z)\, \Phi_+^{[k]}(z')^{-1} - \mathbb{1}\right] g(z')\, dz'}{z - z'}$$

$$\left(\mathsf{b}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_{\mathrm{out}}^{[k]}} \frac{\Phi_+^{[k]}(z)\, \Phi_+^{[k]}(z')^{-1} g(z')\, dz'}{z - z'}$$

$$\left(\mathsf{c}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_{\mathrm{in}}^{[k]}} \frac{\Phi_+^{[k]}(z)\, \Phi_+^{[k]}(z')^{-1} g(z')\, dz'}{z - z'}$$

$$\left(\mathsf{d}^{[k]} g\right)(z) = \frac{1}{2\pi i} \oint_{\mathcal{C}_{\mathrm{out}}^{[k]}} \frac{\left[\Phi_+^{[k]}(z)\, \Phi_+^{[k]}(z')^{-1} - \mathbb{1}\right] g(z')\, dz'}{z - z'}$$

Generalization to higher number of points (sphere)

$$K_{\vec{I},\vec{J}} := \begin{pmatrix} 0 & (\mathsf{a}^{[2]})^{I_1}_{J_1} & (\mathsf{b}^{[2]})^{I_1}_{I_2} & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ (\mathsf{d}^{[1]})^{J_1}_{I_1} & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & (\mathsf{a}^{[3]})^{I_2}_{J_2} & (\mathsf{b}^{[3]})^{I_2}_{I_3} & 0 & \cdot & \cdot & 0 & 0 \\ 0 & (\mathsf{c}^{[2]})^{J_2}_{J_1} & (\mathsf{d}^{[2]})^{J_2}_{I_2} & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\mathsf{a}^{[4]})^{I_3}_{J_3} & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & (\mathsf{c}^{[3]})^{J_3}_{J_2} & (\mathsf{d}^{[3]})^{J_3}_{I_3} & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & (\mathsf{b}^{[n-3]})^{I_{n-2}}_{I_{n-3}} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 & 0 & (\mathsf{a}^{[n-2]})^{I_{n-3}}_{J_{n-3}} \\ 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & (\mathsf{c}^{[n-3]})^{J_{n-3}}_{J_{n-4}} & (\mathsf{d}^{[n-3]})^{J_{n-3}}_{I_{n-3}} & 0 \end{pmatrix}$$

Permutation of rows:

$$\tilde{K}_{\vec{I},\vec{J}} = (\mathsf{d}^{[1]})^{J_1}_{I_1} \oplus \begin{pmatrix} (\mathsf{a}^{[2]})^{I_1}_{J_1} & (\mathsf{b}^{[2]})^{I_1}_{I_2} \\ (\mathsf{c}^{[2]})^{J_2}_{J_1} & (\mathsf{d}^{[2]})^{J_2}_{I_2} \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} (\mathsf{a}^{[n-3]})^{I_{n-2}}_{J_{n-4}} & (\mathsf{b}^{[n-3]})^{I_{n-2}}_{I_{n-3}} \\ (\mathsf{c}^{[n-3]})^{J_{n-3}}_{J_{n-4}} & (\mathsf{d}^{[n-3]})^{J_{n-3}}_{I_{n-3}} \end{pmatrix} \oplus (\mathsf{d}^{[n-2]})^{I_{n-3}}_{J_{n-3}}$$

Punctured Torus case: work in progress with Fabrizio Del Monte, Giulio Bonelli and Alessandro Tanzini.

Thank you for your attention!