Quantum/false/mock modular forms and vertex algebras

Antun Milas, SUNY-Albany

March 19, 2018

This talk

Part I: Intro: Rational and Irrational Vertex Algebras
Part II: W-algebras and their characters
Part III: **Modularity via regularization**
Part IV: **Modularity via QMFs**

Primarily based on my collaborations with: **K. Bringmann (Cologne), J. Kaszian (Cologne), T. Creutzig (Alberta)**.

Vertex algebras

Vertex operator algebras (or 2-dimensional conformal field theory) have been useful in proving concrete problems in mathematics and physics.

(a) Finite groups.

(b) Representation theory $\rightsquigarrow$ new algebraic structures and combinatorics.

(c) Modular forms.

Vertex algebras and quantum invariants

But vertex algebra haven't been used (directly) for computation of quantum invariants of knots/links.
It is very difficult to compute anything within a vertex tensor category (e.g. braiding).
There are faster cars on the road (e.g. Quantum Groups).

Vertex algebras

BUT vertex algebras have an advantage compared to quantum groups:
Characters of vertex algebra modules are (naturally) functions in the upper half-plane $\rightsquigarrow$ action of $SL(2, Z)$.

On the quantum group side it is highly nontrivial to even define an action on the center of the quantum group (Lyubashenko, etc.).

My talk

In my talk I'll present (old and new) ideas pertaining to characters of irrational vertex algebras.
We will see that certain q-series appearing in these irrational theories seem to be related to quantum knot invariants. Anything deeper going on?

A ~~man~~'s module's character is his fate.

Vertex algebras and modules are always infinite-dimensional, graded with f.d. graded subspaces, satisfy certain axioms. Therefore one can associate to a V-module M its **character** (the main protagonist of my talk):

This function is defined as

$$\mathrm{ch}_M(\tau) = \mathrm{tr}_M q^{L(0)-c/24} = q^{h-c/24} \sum_{n=0}^{\infty} dim(M_n) q^n$$

This is often a nice (e.g. modular) function.

Learn as much as we can about V and M just by studying $\mathrm{ch}[M](\tau)$.

Rational Vertex Algebras: key results

- (Zhu) Modular invariance of characters holds. In particular,

$$ch_{M_i}(-\frac{1}{\tau}) = \sum_{j=0}^{n-1} S_{i,j}\mathrm{ch}_{M_j}(\tau),$$

$$ch_{M_i}(\tau+1) = T_{i,i}\mathrm{ch}_{M_i}(\tau),$$

$\rightsquigarrow$ S, T-matrices and modular invariance.

- The (tensor) category of modules is semi-simple with finitely many irreducibles M_i, $0 \leq i \leq n-1$.

$$M_i \boxtimes M_j = \oplus_{k=0}^{n-1} N_{ij}^k M_k.$$

$\rightsquigarrow$ "fusion algebra":

$$x_i \cdot x_j = \sum_k N_{i,j}^k x_k$$

- (Huang) V-Mod is a modular tensor category

Verlinde formula

Corollary.[Huang] (Verlinde formula):

$$N_{ij}^{k} = \sum_{r=0}^{n-1} \frac{S_{ir} S_{jr} S_{k^* r}}{S_{0r}}.$$

Quantum dimensions

$$\mathrm{qdim}_{M_i} = \frac{S_{0,i}}{S_{0,0}}$$

Generalized quantum dimensions:

$$\mathrm{qdim}_{M_i}(j) = \frac{S_{j,i}}{S_{j,0}}$$

They define one-dimension representations of the Grothendieck (or fusion) ring.

$$\mathrm{qdim}_{M_i}(j) \cdot \mathrm{qdim}_{M_k}(j) = \sum_{\ell} N_{i,k}^{\ell} \mathrm{qdim}_{M_l}(j)$$

This statement is essentially the Verlinde formula.

Quantum and asymptotic dimensions

Under certain mild conditions there is a purely analytic formula for quantum dimensions:

$$\mathrm{qdim}_M = \lim_{t \to 0^+} \frac{ch_M(it)}{ch_V(it)}$$

quantum dimension = asymptotic dimension

Asymptotic behavior of characters for rational vertex algebras

Let $F(t) = \mathrm{ch}[M](it)$. Then $(\tau = it,\ t \to 0^+)$:

$$e^{a/t} F(t) \sim b + O(t^N),\ \forall N \geq 0$$

Asymptotics

Proposition

For rational VOAs:

$$\frac{ch_M(it)}{ch_V(it)} \sim \mathrm{qdim}_M + O(t^N)$$

for every $N \geq 1$.

The quantum dimension of M is in fact the full asymptotic expansion of the quotient.

Irrational VOAs

For C_2-cofinite (or logarithmic) vertex algebra a similar picture should emerge, at least when we look at the characters.
Miyamoto proved that characters of logarithmic VOAs are sums of modular forms of non-negative weight.
Verlinde formula for logarithmic vertex algebras is still conjectural (Creutzig-Gannon , Gainuditinov-Runkel).

Modular invariance for irrational VOAs

Open Problem: to formulate modular invariance of characters. Something like

$$\mathrm{ch}[M]\left(-\frac{1}{\tau}\right)=\int_{\Omega} S_{M,\nu}\mathrm{ch}[M_\nu](\tau)d\nu+\sum_{j\in\mathcal{D}}\alpha_{M,j}\mathrm{ch}[M_j](\tau),$$

Problem: the integral part is often divergent.

Continuous Verlinde-type formula for characters

(Even harder) open problem: to formulate a continuous Verlinde-type formula. Something like

$$N_{i,j}^{k} = \int_{\Omega} \frac{S_{i\nu} S_{j\nu} S_{k^{*}\nu}}{S_{0\nu}} d\nu.$$

Problem: Badly divergent!

Asymptotics for irrational VOAs

Problem:

$$\frac{ch_M(it)}{ch_V(it)} \sim \ ??$$

can be arbitrarily bad/complicated. For example, we can have

$$\frac{ch_M(it)}{ch_V(it)} \sim \frac{1}{t} + O(1)$$

$\rightsquigarrow$ growing term $\rightsquigarrow$ $\mathrm{qdim}_M = \infty$ (!)

Asymptotics for irrational VOAs

Taming the VOA Zoo.

Conjecture

For every C_1-cofinite module M, as $t \to 0^+$:

$$\frac{ch_M(it)}{ch_V(it)} \sim a_0 + a_1 t + \cdots + a_n t^n + \ldots.$$

In particular, the quantum dimension ($= a_0$) is finite.

Toy model irrational VOA

Heisenberg (or free boson) VOA:

$$\mathcal{H}(0) = \mathbb{C}[x_{-1}, x_{-2}, ...]$$

Irreducible modules:

$$\mathrm{ch}[\mathcal{H}(\lambda)](\tau) = \mathrm{tr}_{\mathcal{H}(\lambda)} q^{L(0)-1/24} = \frac{q^{\lambda^2/2}}{\eta(\tau)} = \frac{e^{\pi i \tau \lambda^2}}{\eta(\tau)}.$$

$$\mathrm{qdim}_{\mathcal{H}(\lambda)} = \lim_{t \to 0^+} q^{\lambda^2/2} = 1$$

This is consistent with

$$\mathcal{H}(\lambda) \boxtimes \mathcal{H}(\mu) = \mathcal{H}(\lambda + \mu).$$

$$1 \cdot 1 = 1$$

Irrational theories

The category of C_1-cofinite modules for a vertex algebra is closed under a tensor product $\rightsquigarrow$ (conjecture) braided tensor category. The fusion algebra is no longer finite-dimensional so it is desirable for quantum dimensions to be functions and not just numbers. For instance, we can have many modules with $\mathrm{qdim}_M = 1$. Not distinguishable!

Idea:

$$(a_0, a_1, ...,) \rightsquigarrow \mathrm{qdim}_M^{\epsilon}$$

defined on the space Irreps, parametrized $\epsilon \in \Omega$, such that

$$\mathrm{qdim}_{M_i}^{\epsilon} \cdot \mathrm{qdim}_{M_j}^{\epsilon} = \sum_k N_{i,j}^k \, \mathrm{qdim}_{M_k}^{\epsilon}$$

More complicated model: W-algebras

$(1, p)$-singlet vertex algebra (subalgebra of $\mathcal{H}(0)$).
Atypical modules: $M_{1,s}$:

$$\mathrm{ch}[M_{1,s}](\tau) = \frac{F_{s,p}(\tau)}{\eta(\tau)}$$

$$F_{s,p}(\tau) := \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) q^{p\left(n+\frac{s}{2p}\right)^2} = \sum_{n\geq 0} q^{p\left(n+\frac{s}{2p}\right)^2} - \sum_{n\geq 0} q^{p\left(n+\frac{2p-s}{2p}\right)^2}$$

is Rogers' false theta function. There are additional characters/modules $\mathrm{ch}[M_{r,s}]$, $r \neq 1$, obtained by adding a finite q-series to $F_{s,p}$.

Problem: no good modular properties.

Modular invariance for the singlet and irrational vertex algebras

Two approaches:
(1.) (with Creutzig (2013)) Replace characters with ϵ-regularized characters $\rightsquigarrow$ modular invariance. Requires extra variables.

(2.) (with Bringmann (2014)) Extend the character to a QMF $\rightsquigarrow$ modular invariance for better behaved companions. No longer holomorphic.

Two approaches are connected via resummation of asymptotic expansion.

One picture is worth a thousand words

Method 1.

Figure: Character

Figure: "Decorated" character

One picture is worth a thousand words

Method 2.

Figure: Character

Figure: Quantum character

Method 1: Regularized characters

Let $\epsilon \in \mathbb{C}$.

$$\mathrm{ch}[M^{\epsilon}_{r,s}](\tau) = \frac{1}{\eta(\tau)} \sum_{n \geq 0} \Bigg(e^{\frac{2\pi\varepsilon}{\sqrt{2p}}(2pn-s-pr+2p)} q^{\frac{1}{4p}(2pn-s-pr+2p)^2}$$

$$- e^{\frac{2\pi\varepsilon}{\sqrt{2p}}(2pn+s-pr+2p)} q^{\frac{1}{4p}(2pn+s-pr+2p)^2} \Bigg).$$

This regularization is canonical! Categorified false theta function.

Modular invariance for the singlet: ϵ-regularization

We expect a formula like:

$$\mathrm{ch}[M^{\epsilon}]\left(-\frac{1}{\tau}\right)=\int_{\Omega}S_{M,\nu}\mathrm{ch}[{M_{\nu}}^{\epsilon}](\tau)d\nu+\sum_{j\in\mathcal{D}}\alpha_{M,j}\mathrm{ch}[{M_j}^{\epsilon}](\tau) \quad (1)$$

Modularity

Theorem (Creutzig-M. 2013)

For $\epsilon \notin i\mathbb{R}$,

$$ch[M_{r,s}]^{\epsilon}\left(-\frac{1}{\tau}\right) = \frac{1}{\eta(\tau)}\int_{\mathbb{R}} S^{\epsilon}_{(r,s),\mu+\alpha_0/2} e^{\pi i \tau \mu^2/2} d\mu + X^{\epsilon}_{r,s}(\tau)$$

with

$$S^{\epsilon}_{(r,s),\mu+\alpha_0/2} = -e^{-2\pi\epsilon((r-1)\alpha_+/2+\mu)} e^{\pi i (r-1)\alpha_+\mu} \frac{\sin\left(\pi s \alpha_-(\mu+i\epsilon)\right)}{\sin\left(\pi\alpha_+(\mu+i\epsilon)\right)}$$

and

$$X^{\epsilon}_{r,s}(\tau) = \frac{(\mathrm{sgn}(\mathrm{Re}(\epsilon))+1)}{4\eta(\tau)} \sum_{n\in\mathbb{Z}} (-1)^{rn} e^{\pi i \frac{s}{p} n} q^{\frac{1}{2}\left(\frac{n^2}{\alpha_+^2}-\epsilon^2\right)} \left(q^{-i\epsilon\frac{n}{\alpha_+}} - q^{i\epsilon\frac{n}{\alpha_+}}\right).$$

Resurgence

For $\mathrm{Re}(\epsilon) < 0$, essentially the same result appeared in Gukov-Marino-Putrov (2016).

Logarithmic open Hopf link invariants

(with Creutzig and Rupert) These regularized quantum dimensions capture logarithmic open Hopf link invariants (after J. Murakami) for the unrolled quantum of sl_2 at $2p$-th root of unity (computed by Christian Blanchet, Francesco Costantino, Nathan Geer, Bertrand Patureau-Mirand).

False theta functions: q-series identities

q-series identities for false theta functions have been studied from many standpoints: tails of $(2, 2p)$-links, representation theory, Bailey pairs, etc.

Example (Ramanujan)

$$\sum_{n\in\mathbb{Z}} sgn(n) q^{2n^2+n} = (q;q)_\infty \sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n^2}.$$

Dasbach, Garoufalidis, Garvan, Lovejoy, Folsom, Warnaar, Osburn, Hajij, Yuasa, Bringmann-M., etc.

Higher rank W-algebras and "higher" false theta functions

There are "higher rank" generalizations of the singlet vertex algebra whose characters are (what I call) higher rank false theta functions. Introduced around (2012) and studied by several people. Let

(i) $p \in \mathbb{N}_{\geq 2}$
(ii) Q, ADE root lattice and $L = \sqrt{p}Q$.
(iii) $\mu \in (\sqrt{p}Q)^0$ and $\mu = \lambda + \sqrt{p}\beta$, $\beta \in Q$
(iv) $\lambda = \hat{\lambda} + \frac{1}{\sqrt{p}}\bar{\lambda}$.

Characters of atypical $W^0(p)_Q$-modules

$$\mathrm{ch}[W(p,\mu)_Q](\tau) = \sum_{\alpha\in Q\cap P^+} \dim\left(V\left(\widehat{\lambda}+\alpha\right)_{\beta+\hat{\lambda}}\right)$$

$$\cdot\left(\sum_{w\in W}(-1)^{\ell(w)}\frac{q^{\frac{1}{2}||\sqrt{p}w\left(\alpha+\rho+\widehat{\lambda}\right)+\overline{\lambda}-\frac{1}{\sqrt{p}}\rho||^2}}{\eta(\tau)^{\mathrm{rank}(Q)}}\right)$$

Here $V(\gamma)$ is the f.d. irreducible $\mathfrak{g}$-module of h.w. γ.

The character of $W^0(p)_Q$

Here $\mu = \hat{\lambda} = \bar{\lambda} = \beta = 0$.

$$\eta(\tau)^{\mathrm{rank}(Q)} \mathrm{ch}[W(p)_Q](\tau) = \sum_{\alpha \in Q \cap P^+} \dim(V(\alpha)_0) \sum_{w \in W} (-1)^{\ell(w)} q^{\frac{1}{2}||\sqrt{p} w(\alpha+\rho) - \frac{1}{\sqrt{p}}\rho||^2}$$

For $Q = A_1$ this recovers Rogers' false theta function $F_{p-1,p}$.
Important: Multivariable character $\mathrm{ch}[W(p)_Q](\mathbf{z};\tau)$ s.t.

$$\mathrm{ch}[W(p)_Q](\tau) = \mathrm{Const.Term.}_{\mathbf{z}}\{\mathrm{ch}[W(p)_Q](\mathbf{z};\tau)\}$$

ϵ-regularized $W^0(p)_Q$ characters

For $\epsilon \in \mathbb{C}^n$ (space of parameters of Irreps of W).

$$\mathrm{ch}[W(p)_Q](\tau) \rightsquigarrow \mathrm{ch}[W(p)_Q^{\epsilon}](\tau)$$

Modular invariance

Theorem (Creutzig-M.)

For $\epsilon_i \notin i\mathbb{R}$,

$$\mathrm{ch}[W^0(p,\mu)_Q^\epsilon]\left(-\frac{1}{\tau}\right) =$$

$$\frac{C_Q}{\eta(\tau)^n}\int_{\mathbb{R}^n}\frac{q^{\frac{1}{2}\left(w,A^{-1}w\right)}e^{2\pi i(\hat{\lambda}+\mu,w+i\epsilon)}\mathrm{num}_{(-\sqrt{p}\bar{\lambda})}\left(-\frac{w+i\epsilon}{p}\right)}{\prod\limits_{\alpha\in\Delta^+}\sin(\alpha,w+i\epsilon)}d^nw$$

$$+\ \mathrm{lower}\ \ \text{"degree"}\ \ \mathrm{iterated}\ \ \mathrm{integrals},$$

where lower terms are $(i<n)$-fold iterated integrals multiplied with theta functions (wall-crossings). If $\mathrm{Re}(\epsilon_i)<0$, there are no lower degree terms.

Modular invariance

Now we can study

$$\mathrm{qdim}^{\epsilon}_{W^0(p,\mu)_Q}$$

It is extremely complicated to compute this for all values of ϵ. We did show that in a specific range this compute quantum dimensions of the level $p - h^\vee \geq 0$ WZW model.
We expect that these are related to logarithmic open Hopf link invariants of unrolled quantum group at $2p$-th root of unity.

Method 2: Quantum modular forms

Introduced by Zagier in his 2010 Clay lectures. Studied by many people in the audience: Lawrence, Hikami, Folsom, Lovejoy, Garoufalidis, Osburn, Zwegers.

First definition. We say that $f : \mathbb{Q} \setminus S \to \mathbb{C}$ (here S is an appropriate subset of $\mathbb{Q}$) is a quantum modular form of weight k and multiplier ϵ for $\Gamma \subset SL(2,\mathbb{Z})$ if for all $\gamma \in \Gamma$ the functions $h_\gamma : \mathbb{Q} - (S \cup \gamma^{-1}(\infty))$ defined by

$$h_\gamma(x) := f(x) - \epsilon(\gamma)(cx+d)^{-k} f(\frac{ax+b}{cx+d})$$

satisfies a "suitable" property of continuity or analyticity (with respect to the real topology).

Strong QMF

Definition. A strong qmf is a function f on $\mathbb{H} \cup \overline{\mathbb{H}} \cup \mathcal{Q}$ holomorphic in $\mathbb{H}$ and holomorphic or real analytic in $\bar{\mathbb{H}}$ such that:

1. (as before) h_γ is "sufficiently nice".

2. Let $\tilde{f} := f_{\mathbb{H}}$ and $f^* := f_{\bar{\mathbb{H}}}$. Then $\tilde{f}(\tau)$ and $f^*(\tau)$ "agree" to infinite order, that is : at every rational number $\frac{d}{c} \in \mathbb{Q}$ as $t \to 0^+$:

$$\tilde{f}\left(\frac{d}{c} + \frac{it}{2\pi}\right) \sim \sum_{n \geq 0} \beta(n) t^n$$

$$f^*\left(\frac{d}{c} - \frac{it}{2\pi}\right) \sim \sum_{n \geq 0} \beta(n)(-t)^n$$

Higher depth QMF

After Zagier-Zwegers' higher depth mock modular forms.

Definition (Bringmann, Kaszian, M.)

A function $f : \mathcal{Q} \to (\mathcal{Q} \subset \mathbb{Q})$ is called a *quantum modular form of depth* $N \in \mathbb{N}$, *weight* $k \in \frac{1}{2}\mathbb{Z}$, *multiplier* χ, *and quantum set* $\mathcal{Q}$ *for* Γ if for $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$

$$f(\tau) - \chi(M)^{-1}(c\tau + d)^{-k} f(M\tau) \in \sum_j \mathcal{Q}^{N-1}_{\kappa_j}(\Gamma, \chi_j)\mathcal{O}(R),$$

where j runs through a finite set, $\kappa_j \in \frac{1}{2}\mathbb{Z}$, the χ_j are characters, $\mathcal{O}(R)$ is the space of real-analytic functions on $R \subset \mathbb{R}$ which contains an open subset of , $\mathcal{Q}^1_k(\Gamma, \chi) := \mathcal{Q}_k(\Gamma, \chi)$, $\mathcal{Q}^0_k(\Gamma, \chi) := 1$, and $\mathcal{Q}^N_k(\Gamma, \chi)$ denotes the space of quantum modular forms of weight k, depth N, multiplier χ for Γ.

Similarly we define vector-valued higher depth QMF.

Back to rank one: $(1, p)$-singlet algebra

Let us recall atypical characters $\mathrm{ch}[M_{r,s}](\tau)$. They involve $(1 \leq j \leq p-1)$ $F_{j,p}(\tau)$ and $\eta(\tau)$ (which we ignore),

$$F_{j,p}(\tau) := \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n) q^{p\left(n+\frac{j}{2p}\right)^2} = \sum_{n\geq 0} q^{p\left(n+\frac{j}{2p}\right)^2} - \sum_{n\geq 0} q^{p\left(n+\frac{2p-j}{2p}\right)^2}$$

Rogers' false theta function.

False theta series $F_{j,p}$ and quantum knots invariants

These q-series already appeared in the literature on quantum knot/link invariants.
Hikami's work (Kashaev's invariants for torus link $T(2, 2p)$, for q $2p$-th root of unity).
For $j = p - 1$ this is the tail of the $(2, 2p)$ torus knot (Garoufalidis?).
Also studied (in some special cases) by Zagier and Lawrence-Zagier.

Quantum modularity of $F_{j,p}$

Theorem (Hikami,..., Bringmann-M.)

$$F_{j,p}(\tau) \textit{ and } \mathcal{E}_{j,p}(\tau) := \int_{-\bar{\tau}}^{i\infty} \frac{\partial\Theta_{j,p}(z)}{\sqrt{-i(\tau+z)}} dz, \quad \tau \in \mathbb{H}.$$

agree at all orders at all roots of unity (here $\partial\Theta_{j,p}$ is a unary theta function of weight 3/2).

We call $\mathcal{E}_{j,p}(\tau)$ the "companion" of $F_{j,p}(\tau)$.

Vector valued quantum modularity

They combine into a vector-valued QMF:

$$\mathcal{E}_{j,p}(\tau) - \frac{1}{\sqrt{-i\tau}}\sqrt{\frac{2}{p}}\sum_{k=1}^{p-1}\sin\left(\frac{\pi kj}{p}\right)\mathcal{E}_{k,p}\left(-\frac{1}{\tau}\right) = i\sqrt{2p}\cdot r_{f_{j,p}}(\tau),$$

where, for f a holomorphic modular form of weight k,

$$r_{f_{j,p}}(\tau) := \int_0^{i\infty} \frac{\partial\Theta_{j,p}(w)}{\sqrt{-i(w+\tau)}}dw. \tag{2}$$

From Eichler to Mordell with Zwegers

Proposition

$$\int_0^{i\infty} \frac{\partial\Theta_{j,p}(z)}{\sqrt{-i(\tau+z)}}dz = \frac{1}{\sqrt{2}}\int_{\mathbb{R}} \cot\left(\pi i\left(x - i\frac{j}{2p}\right)\right) e^{2\pi i\tau x^2}dx.$$

Proposition

$$\mathcal{E}_{j,p}\left(-\frac{1}{\tau}\right)\frac{1}{\sqrt{-2i\tau}} + \sum_{m=1}^{p-1} \underbrace{\sin\left(\frac{\pi mj}{p}\right)}_{S-matrix} \mathcal{E}_{m,p}(\tau)$$

$$= -i\sum_{m=1}^{p-1}\sin\left(\frac{\pi mj}{p}\right)\int_{\mathbb{R}} \underbrace{\cot\left(\pi i\left(x - i\frac{j}{2p}\right)\right)}_{S-kernel} \cdot \underbrace{e^{2\pi i\tau x^2}}_{generic\quad characters} dx.$$

Weight $\frac{3}{2}$ quantum modular forms

Slightly different Eichler integrals and "weight" 3/2 false thetas.
Hikami's work on Kashaev's invariants of (p, q)-torus knots (and in Lawrence-Zagier).
Remarkably, these q-series are essentially characters of modules for another family of vertex algebras called (p, q)-singlet vertex algebras (Adamovic-M., Creutzig-M. Wood, Bringmann-M.).
In particular, for $(2, 3)$-torus knot (Kontsevich "strange" function), the relevant vertex algebra is highly degenerate and has central charge 0.

Back to characters of W-algebras $W^0(p)_Q$

Motivated by the rank one case.

Conjecture

$$F_{p,Q}(q) = \sum_{\alpha \in Q \cap P^+} \dim(V(\alpha)_0) \sum_{w \in W} (-1)^{\ell(w)} q^{\frac{1}{2}||\sqrt{p}w(\alpha+\rho) - \frac{1}{\sqrt{p}}\rho||^2}$$

extends to a depth $n = rank(Q)$ quantum modular form. Moreover, it is a component of a vector-valued QMF (of depth n).

(with Bringmann and Kaszian) This is true for A_2!

Explicit formula for $W^0(p)_{A_2}$ characters

Let $1 \leq s_1, s_2 \leq p$:

$$\mathbb{F}_{s_1,s_2}(q) := \sum_{\substack{m_1,m_2 \geq 1 \\ m_1 \equiv m_2 \pmod 3}} \min(m_1, m_2) q^{\frac{p}{3}\left(\left(m_1 - \frac{s_1}{p}\right)^2 + \left(m_2 - \frac{s_2}{p}\right)^2 + \left(m_1 - \frac{s_1}{p}\right)\left(m_2 - \frac{s_2}{p}\right)\right)}$$
$$\times \left(1 - q^{m_1 s_1} - q^{m_2 s_2} + q^{m_1 s_1 + (m_1+m_2)s_2} + q^{m_2 s_2 + (m_1+m_2)s_1} - q^{(m_1+m_2)(s_1+s_2)}\right).$$

Remarkable double series!

Quantum modularity for $W^0(p)_{A_2}$ characters

We first decompose

$$\mathbb{F}_{s_1,s_2}(q) = F_1(q) + F_2(q)$$

into "weight" one and "weight" two components.

Higher depth quantum modular forms

Theorem (Bringmann, Kaszian, M. 2018)

For every $p \geq 2$, every $1 \leq s_1, s_2 \leq p$, series F_1 and F_2 extend to depth two quantum modular forms on $\mathbb{Q}$ of weight 1 and 2, respectively.

We constructed explicit mock companions $\mathcal{E}_1(\tau)$ and $\mathcal{E}_2(\tau)$ (in the upper half-plane).

Companions of F_1 and F_2

Theorem (Bringmann, Kaszian, M. 2018)

(a) $\mathcal{E}_1$ *is a sum of iterated Eichler integrals*

$$I_{f,g}(\tau) := \int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{f(w_1)g(w_2)}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1$$

where f *and* g *have weight* $3/2$.

(b) $\mathcal{E}_2$ *is a sum of iterated Eichler integrals*

$$I_{\tilde{f},g}(\tau) := \int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{\tilde{f}(w_1)g(w_2)}{\sqrt{-i(w_1+\tau)^3}\sqrt{-i(w_2+\tau)}} dw_2 dw_1$$

where $\tilde{f}$ *has weight* $1/2$ *and* g *weight* $3/2$.

Higher depth vector-valued examples

For general p it gets messy but for $p = 2$, the relevant space is 2-dimensional: Vector-valued QMF of depth two (with Bringmann and Kaszian):

$$\int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{\eta(w_1)^3 \eta(3w_2)^3}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1$$

$$\int_{-\overline{\tau}}^{i\infty} \int_{w_1}^{i\infty} \frac{\eta(w_1)^3 \eta\left(\frac{w_2}{3}\right)^3}{\sqrt{-i(w_1+\tau)}\sqrt{-i(w_2+\tau)}} dw_2 dw_1.$$

Completion

Theorem (Bringmann, Kaszian, M. 2017-18)

In some cases, $\mathcal{E}_1(\tau)$ and $\mathcal{E}_2(\tau)$ can be completed to higher depth harmonic Maass forms (after Zagier and Zwegers).

This result uses generalized "double-error" integrals introduced by Sergei Alexandrov, Sibasish Banerjee, Jan Manschot, Boris Pioline (2016).

More precise conjecture

Consider r-fold Eichler integral:

$$I_{f_1,\ldots,f_r} := \int_{-\overline{\tau}}^{i\infty} \int_{w_{r-1}}^{i\infty} \cdots \int_{w_2}^{i\infty} \prod_{j=1}^{r} \frac{f_j(w_j)}{(-i(w_j+\tau))^{2-k_j}} dw_1 \cdots dw_r,$$

where f_j are of weight $1/2$ or $3/2$. Then

Conjecture

For every $p \geq 2$, $F_{Q,p}(q)$ is a component of a quantum modular form of depth $\mathrm{rank}(Q)$ whose companion in the upper half-plane is a linear combination of $I_{f_1,\ldots,f_r}$.

Modular invariance: higher Mordell Integral

Error of quantum modularity for $\mathbb{F}_{s_1,s_2}(q)$.

Theorem (Bringmann, Kaszian, M.)

Component of the error term for $\mathcal{E}_1$:

$$\int_0^{i\infty}\int_{w_1}^{i\infty} \frac{\theta_1\left(\boldsymbol{\alpha};\boldsymbol{w}\right)+\theta_2\left(\boldsymbol{\alpha};\boldsymbol{w}\right)}{\sqrt{-i\left(w_1+\tau\right)}\sqrt{-i\left(w_2+\tau\right)}}dw_2dw_1$$

$$\doteq \int_{\mathbb{R}^2}\cot\left(\pi i w_1+\pi\alpha_1\right)\cot\left(\pi i w_2+\pi\alpha_2\right)e^{2\pi i\tau\left(3w_1^2+3w_1w_2+w_2^2\right)}dw_1dw_2.$$

Similar formula for the $\mathcal{E}_2$ error term. This only applies to non-integral α values!

Future directions

1. Depth three is highly non-trivial. For $\mathfrak{sl}_4$, $\dim(V(m_1\omega_1 + m_2\omega_2 + m_3\omega_3)_0)$ is very messy!
2. Automorphic forms and higher depth.
3. Asymptotics properties of iterated Eichler integrals and values of L-functions $\rightsquigarrow$ multiple L-values and shuffle relations for iterated integrals.
4. Cohomological interpretation (Manin's work on iterated integrals and non-abelian cohomology).

Questions for the audience

(a) Is $\mathbb{F}_{1,1}(q)$ the tail of the $(2, 2p)$ torus knot colored by sl_3 representations $V(n\rho)$, where $V(\rho)$ is the adjoint representation?

(b) Can you write a q-hypergeometric representation for $\frac{\mathbb{F}_{s_1,s_2}(q)}{(q;q)_\infty^2}$?

(c) General false theta functions

$$F_{p,Q,\lambda}(it) = q^a + ...$$

$$F_{p,Q,\lambda}(it) \sim \frac{dim(V(\lambda))}{p^{|\Delta_+|}} + O(t)$$

Does this remind you of something?

THANK YOU!