The Kelmans-Seymour Conjecture

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1. Kuratowski's Theorem

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- Every 3-connected nonplanar graph (except K_5) contains $TK_{3,3}$.
- Graphs containing no $TK_{3,3}$ have good structure.
- Conjecture (Kelmans 1979, Seymour 1977): Every 5-connected nonplanar graph contains TK_5.

2. Hajós Conjecture for $k = 4$

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- Y. and Zickfeld (2006). Any minimum counterexample to Hajós' conjecture must be 4-connected.
- Y. and Sun (2014): Suppose G is a minimum counterexample to Hajós' conjecture and S is a 4-cut in G. Then $G - S$ has exactly two components.

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- Mader (1998): Dirac's conjecture is true.

- Question (Mader 1998): Does every simple graph on $n \geq 4$ vertices with more than $12(n-2)/5$ edges contain a K_4^-, a $K_{2,3}$, or a TK_5?

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- Question (Mader 1998): Does every simple graph on $n \geq 4$ vertices with more than $12(n-2)/5$ edges contain a K_4^-, a $K_{2,3}$, or a TK_5?
- Kawarabayashi, Ma and Y. (2012): The Kelmans-Seymour conjecture holds if the answer to Mader's questions is affirmative.

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- Ma and Y. (2013): Kelmans-Seymour conjecture holds for graphs containing K_4^-.
- Let G be a 5-connected nonplanar graph and let $x_1, x_2, y_1, y_2 \in V(G)$ induce a K_4^- with $y_1 y_2 \notin E(G)$. Then there is an induced path P in $G - x_1 x_2$ between x_1 and x_2 such that
  - $\{y_1, y_2\} \not\subseteq V(P)$, and
  - $G - V(P)$ is 2-connected.

5. Lovász conjecture

- Conjecture (Lovász 1975) For each positive integer k, there exists a (minimum) integer $c(k) > 0$ with the following property: For any two vertices u and v in a $c(k)$-connected graph G, there is a path P from u to v in G such that $G - V(P)$ is k-connected.

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- Open for $k \geq 3$.

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- Let z denote the vertex representing the contraction of M, and let $H = G/M$.

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- H does not contain K_4^-, and there exists $T \subseteq H$ such that $z \in V(T)$, $T \cong K_2$ or $T \cong K_3$, and H/T is 5-connected and planar.

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- H does not contain K_4^-, and there exists $T \subseteq H$ such that $z \in V(T)$, $T \cong K_2$ or $T \cong K_3$, and H/T is 5-connected and planar.
- H does not contain K_4^-, and for any $T \subseteq H$ with $z \in V(T)$ and $T \cong K_2$ or $T \cong K_3$, H/T is not 5-connected.

Case 1. Degree 2

Let $x_1, x_2, y_1, z \in V(H)$ be distinct such that $H[\{x_1, x_2, y_1, z\}] \cong K_4^-$ and $y_1 z \notin E(H)$. Then one of the following holds:

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- H contains a TK_5 in which z is not a branch vertex.
- $H - z$ contains K_4^-.
- H has a 5-separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{z, a_1, a_2, a_3, a_4\}$, and H_2 is the graph obtained from the edge-disjoint union of the 8-cycle $a_1 b_1 a_2 b_2 a_3 b_3 a_4 b_4 a_1$ and the 4-cycle $b_1 b_2 b_3 b_4 b_1$ by adding z and the edges $z b_i$ for $i \in [4]$.

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- For any distinct $z_1, z_2, z_3 \in N(y_2) - \{x_1, x_2\}$, $H - \{y_2 v : v \notin \{z_1, z_2, z_3, x_1, x_2\}\}$ contains TK_5.

Case 2. Degree 3

Let $z, x_2, y_1, y_2 \in V(H)$ be distinct such that $H[\{z, x_2, y_1, y_2\}] \cong K_4^-$ and $y_1y_2 \notin E(H)$. Then one of the following holds:

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- H contains a TK_5 in which z is not a branch vertex.
- $H - z$ contains K_4^-, or H contains K_4^- in which z is of degree 2.
- x_2, y_1, y_2 may be chosen so that for any distinct $z_1, z_2 \in N(z) - \{x_2, y_1, y_2\}$, $H - \{zv : v \notin \{z_1, z_2, x_2, y_1, y_2\}\}$ contains TK_5.

Case 3. Planarity

Suppose H does not contain K_4^-, and there exists $T \subseteq H$ such that $z \in V(T)$, $T \cong K_2$ or $T \cong K_3$, and H/T is 5-connected and planar

Then $H - z$ contains K_4^- (by a discharging argument).

Case 4. Special separations

Suppose H has a 5-separation (H_1, H_2) such that $|V(H_i)| \geq 7$ for $i = 1, 2$ and there exists $z \in V(H_1 \cap H_2)$ with $(H - z, V(H_1 \cap H_2) - \{z\})$ planar. Then one of the following holds:

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- H contains a TK_5 in which z is not a branch vertex.
- $H - z$ contains K_4^-, or H contains K_4^- in which z is of degree 2.
- For any $u_1, u_2, u_3 \in N(z) - \{z_1, z_2\}$, $H - \{zv : v \notin \{z_1, z_2, u_1, u_2, u_3\}\}$ contains TK_5.

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H has a 6-separation in (H_1, H_2) such that $z \in V(H_1 \cap H_2)$, $H[V(H_1 \cap H_2)]$ contains a triangle zz_1z_2z, $|V(H_i)| \geq 7$ for $i = 1, 2$ (and we then minimize H_1). Then $N(x) \cap V(H_1 - H_2) \neq \emptyset$, or one of the following holds:

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- There exists $z_3 \in N(z)$ such that for any distinct $y_1, y_2 \in N(z) - \{z_1, z_2, z_3\}$, $G - \{xv : v \notin \{z_1, z_2, z_3, y_1, y_2\}\}$ contains TK_5.

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- For some $i \in [2]$ and some $j \in [3]$, $N(z_i) \subseteq V(H_1 - H_2) \cup \{z, z_{3-i}\}$, and any three independent paths in $H_1 - z$ from $\{z_1, z_2\}$ to v_1, v_2, v_3, respectively, with two from z_i and one from z_{3-i}, must contain a path from z_{3-i} to v_j.

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