Classes of graphs with strongly sublinear separators

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Strongly sublinear separators

Definition

$X \subseteq V(H)$ is a <u>balanced separator</u> if each component of $H - X$ has at most $|V(H)|/2$ vertices.

Definition

Class $\mathcal{G}$ has <u>f-separators</u> if for all $G \in \mathcal{G}$, each subgraph H of G has a balanced separator of size at most $f(|V(H)|)$.

<u>Strongly sublinear separators</u>: $f(n) = O(n^{1-\varepsilon})$ for some $\varepsilon > 0$.

Examples

- bounded treewidth $\Leftrightarrow$ $O(1)$-separators.
- planar (or in a fixed surface) $\Rightarrow$ $O(\sqrt{n})$-separators.
- proper minor-closed $\Rightarrow$ $O(\sqrt{n})$-separators.
- embedded in $\mathbf{R}^d$ with bounded distortion $\Rightarrow$ $O(n^{1-1/d})$-separators.

Shallow minors

Definition

An *r-shallow minor* of G is obtained from a subgraph of G by contracting vertex-disjoint subgraphs of radius at most r.

Definition

$$\nabla_r(G) = \max\{2|E(H)|/|V(H)| : H \text{ is an } r\text{-shallow minor of } G\}$$
$$\omega_r(G) = \max\{\omega(H) : H \text{ is an } r\text{-shallow minor of } G\}$$

$\nabla_0(G) =$ maximum average degree, $\omega_0(G) = \omega(G)$,
$\nabla_r(G) \geq \omega_r(G) - 1$

Shallow minors in a class

Definition

$$\nabla_r(\mathcal{G}) = \sup\{\nabla_r(G) : G \in \mathcal{G}\}$$

$$\omega_r(\mathcal{G}) = \sup\{\omega_r(G) : G \in \mathcal{G}\}$$

- $(\forall r \geq 0)\, \nabla_r(\mathcal{G})$ finite: <u>bounded expansion</u>
- $(\forall r \geq 0)\, \omega_r(\mathcal{G})$ finite: <u>nowhere-dense</u>

Separators and expansion

For a class $\mathcal{G}$, TFAE:

1. $\mathcal{G}$ has strongly sublinear separators
2. $(\forall r \ge 0)\, \nabla_r(\mathcal{G}) \le \text{poly}(r)$
3. $(\forall r \ge 0)\, \omega_r(\mathcal{G}) \le \text{poly}(r)$

1 $\Rightarrow$ 2: D. and Norin; Esperet and Raymond
2 $\Rightarrow$ 3: trivial
3 $\Rightarrow$ 1: Plotkin, Rao, and Smith

Shallow cliques and sublinear separators

Theorem (Plotkin, Rao, and Smith)

For every $a > 0$ and an n-vertex graph G, there exist disjoint $X, M \subseteq V(G)$ such that

- *$X \cup M$ is a balanced separator*
- *for $r = O(a \log n)$, $G[M]$ contains an r-shallow minor of K_b for some $b \le \omega_r(G)$ and $|M| = O(ab\omega_r(G) \log n)$*
- *$|X| \le n/a$*

If $\omega_r(G) = O(r^c)$ for some $c \ge 0$:

- $b = O\big(a^c \text{polylog}(n)\big)$ and $|M| = O\big(a^{2c+1} \text{polylog}(n)\big)$
- setting $a = \Theta\big(n^{1/(2c+2)}\big)$, we have $|X \cup M| = O\big(n^{1-1/(2c+2)} \text{polylog}(n)\big)$

Small cost separators?

Question

Let $\mathcal{G}$ have strongly sublinear separators, and for $G \in \mathcal{G}$, let $w : V(G) \to \mathbf{R}^+$ be an assignment of costs to vertices. Does there always exist a balanced separator X of small cost (e.g., $w(X)/w(V(G)) \leq \varepsilon$)?

No: $K_{1,n}$ with $w(v) = 1$ for the central vertex v and $w(x) = 1/n$ for each ray x.

Almost small cost separators

Theorem (D.)

For every $a > 0$, an n-vertex graph G, and assignment $w : V(G) \to \mathbf{R}^+$ of costs, there exist disjoint $X, M \subseteq V(G)$ such that

- $X \cup M$ *is a balanced separator*
- *for* $O(a \log n)$, $G[M]$ *contains an r-shallow minor of K_b for some $b \le \omega_r(G)$ and* $|M| = O(ab\omega_r(G) \log n)$
- $w(X) \le w(V(G))/a$

If $\omega_r(G) \le \text{poly}(r)$, then $|M| \le \text{poly}(a)\text{polylog}(n)$.

Small treewidth

Letting $a = \varepsilon / \log n$ and iterating:

Corollary

Let $\mathcal{G}$ be a class with strongly sublinear separators, and let $w : V(G) \to \mathbf{R}^+$ be an assignment of costs to an n-vertex graph $G \in \mathcal{G}$. For every $\varepsilon > 0$, there exists $X \subseteq V(G)$ such that

$$w(X)/w(V(G)) \leq \varepsilon$$

and

$$tw(G - X) \leq poly(1/\varepsilon)polylog(n).$$

Dual formulation

Corollary

Let $\mathcal{G}$ be a class with strongly sublinear separators. For every $\varepsilon > 0$ and an n-vertex graph $G \in \mathcal{G}$, there exists a probability distribution on

$$\{X \subseteq V(G) : tw(G - X) \leq poly(1/\varepsilon)polylog(n)\}$$

with support of size at most $n + 1$ such that

$$Pr[v \in X] \leq \varepsilon \text{ for all } v \in V(G).$$

Observation

Such probability distributions for $\varepsilon = 1, 1/2, 1/3, \ldots, 1/n$ certify that for some $\delta > 0$, all subgraphs with $k \geq polylog(n)$ vertices have balanced separators of size at most $O(k^{1-\delta})$.

Tw-fragility

Definition

Class $\mathcal{G}$ is fractionally tw-fragile if for some function f, every $\varepsilon > 0$, and every $G \in \mathcal{G}$, there exists a probability distribution on

$$\{X \subseteq V(G) : \mathrm{tw}(G - X) \leq f(1/\varepsilon)\}$$

such that

$$\Pr[v \in X] \leq \varepsilon$$

for all $v \in V(G)$.

Strongly fractionally tw-fragile when f is polynomial.

Tw-fragility and sublinear separators

- Strongly fractionally tw-fragile ⇒ strongly sublinear separators.
- Forbidden minor ⇒ strongly (fractionally) tw-fragile (DeVos, Ding, Oporowski, Sanders, Reed, Seymour, and Vertigan)
- Embedded in $\mathbf{R}^d$ with bounded distortion ⇒ strongly fractionally tw-fragile

Tw-fragility and sublinear separators

Theorem (D.)

Strongly sublinear separators + bounded maximum degree $\Rightarrow$ fractionally tw-fragile (even with tw replaced by component size).

Conjecture

Strongly sublinear separators $\Rightarrow$ strongly fractionally tw-fragile.

Observation

$\mathcal{G}$ *fractionally tw-fragile* $\Leftrightarrow \chi^f_{tw}(\mathcal{G}) = 1$.

Thank you for your attention.

Questions?