Proof of the tree packing conjecture for bounded degree trees

Daniela Kühn

University of Birmingham

joint work with
Felix Joos, Jaehoon Kim, Deryk Osthus and Mykhaylo Tyomkyn

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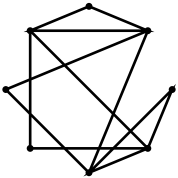
General theme

Decomposition of large/dense object into small/sparse objects.

Graph decompositions

G has a decomposition into $H_1, \dots, H_s$ if there exist pairwise edge-disjoint copies of $H_1, \dots, H_s$ in G which cover all edges of G.



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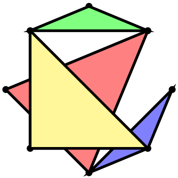

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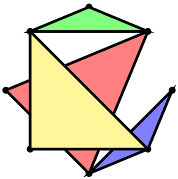


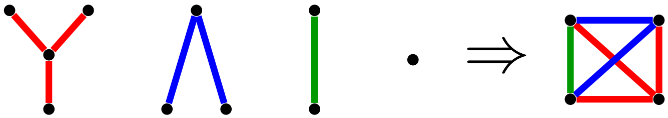
Graph packings

$H_1, \dots, H_s$ pack into G if there exist pairwise edge-disjoint copies of $H_1, \dots, H_s$ in G.

Tree packing conjecture

Conjecture (Gyárfás & Lehel, 1976)

Given trees $T_1, \dots, T_n$ such that T_i has i vertices, K_n has a decomposition into $T_1, \dots, T_n$.



Note that $\sum_{i=1}^{n} e(T_i) = \binom{n}{2}$.

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Results on packing the smallest trees or the largest trees or very special families of trees

- **Gyárfás & Lehel:** $T_1, \ldots, T_n$ pack into K_n if each T_i is either a path or star
- **Bollobás:** $T_1, \ldots, T_{\frac{n}{\sqrt{2}}}$ pack into K_n
- **Balogh & Palmer:** $T_{n-n^{1/4}/10}, \ldots, T_n$ pack into K_{n+1}
- **Zak:** $T_{n-4}, \ldots, T_n$ pack into K_n
- ...

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approximate version for bounded degree trees:

Theorem (Böttcher, Hladkỳ, Piguet & Taraz, 2016)

If $1/n \ll \alpha, 1/\Delta$ and $T_1, \ldots, T_t$ are trees such that

- $\Delta(T_i) \leq \Delta$ *and* $|T_i| \leq (1-\alpha)n$,
- $\sum_{i=1}^{t} e(T_i) \leq (1-\alpha)\binom{n}{2}$,

then $T_1, \ldots, T_t$ pack into K_n.

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Messuti, Rödl & Schacht (2016): generalization to embeddings of $H_1, \dots, H_t$ belonging to minor-closed family $\mathcal{H}$

Approximate decompositions into spanning trees

next result allows for packing of spanning graphs:

Theorem (Ferber, Lee & Mousset, 2016+)

If $\mathcal{H}$ is minor-closed, $1/n \ll \alpha, 1/\Delta$ and $H_1, \ldots, H_t \in \mathcal{H}$ are s.t.

- $\Delta(H_i) \leq \Delta$ *and* $|H_i| \leq n$,
- $\sum_{i=1}^{t} e(H_i) \leq (1-\alpha)\binom{n}{2}$,

then $H_1, \ldots, H_t$ pack into K_n.

Approx. decompositions into general bdd degree graphs

Theorem (Kim, Kühn, Osthus & Tyomkyn, 2016+)

If $1/n \ll \alpha, 1/\Delta$ *and* $H_1, \dots, H_t$ *are such that*

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then $H_1, \dots, H_t$ *pack into* K_n.

in this general setup cannot ask for a decomposition into $H_1, \dots, H_t$

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can replace host graph K_n by any quasi-random graph:

n-vertex graph G is (ϵ, d)-quasi-random if

- $d_G(v) = (1 \pm \epsilon)dn$ for every vertex v and
- $d_G(u, v) = (1 \pm \epsilon)d^2 n$ for every pair $u \neq v$ of vertices.

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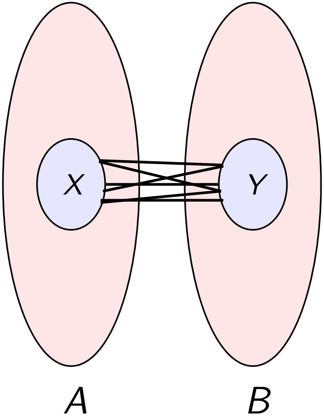
Theorem (Kim, Kühn, Osthus & Tyomkyn, 2016+)

If $1/n \ll \epsilon \ll \alpha, d, 1/\Delta$, if G is (ϵ, d)-quasi-random and if $H_1, \ldots, H_t$ are such that $\Delta(H_i) \leq \Delta$ and $|H_i| \leq n$ and $\sum_{i=1}^{t} e(H_i) \leq (1-\alpha)e(G)$, then $H_1, \ldots, H_t$ pack into G.

Actually, consider setting of ϵ-regularity:

$|A| = |B| = n$



(A, B) is ϵ-regular if

$$\frac{e(X,Y)}{|X||Y|} = (1 \pm \epsilon)\frac{e(A,B)}{|A||B|}$$

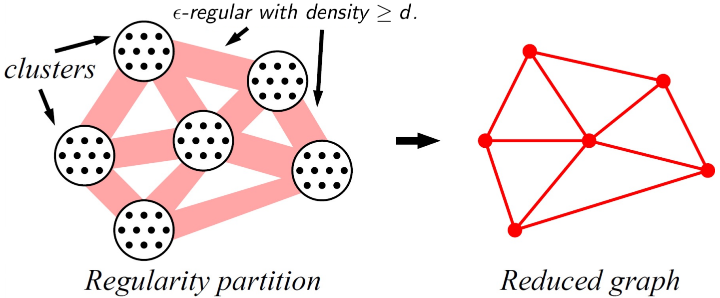
for not too small X, Y.

(A, B) is (ϵ, d)-super-regular if

- (A, B) ϵ-regular, density $d \pm \epsilon$,
- $d(a), d(b) = (d \pm \epsilon)n$.

Theorem (Szemerédi's regularity lemma, 1976)

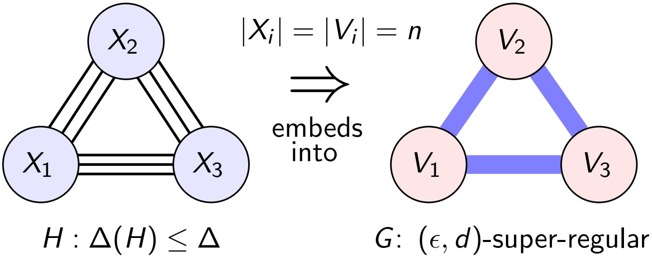
We can partition any large dense graph G into a bounded number of clusters so that almost all pairs are ϵ-regular.



Blow-up lemma

Theorem (Komlós, Sárközy & Szemerédi, 1997)

If $1/n \ll \epsilon \ll 1/\Delta, d$, then the following embedding exists:



Important tool to find spanning structures, e.g.

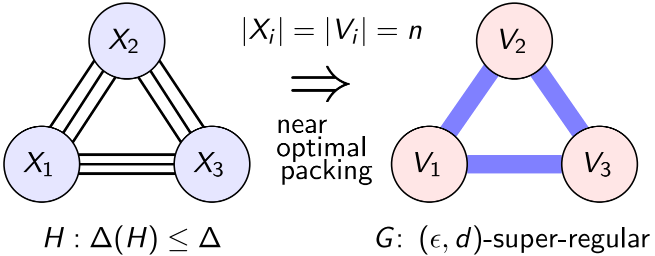
- powers of Hamilton cycles (Komlós, Sárközy & Szemerédi)
- H-factors (Komlós, Sárkőzy & Szemerédi, Kühn & Osthus)
- ...

Main result

can almost decompose G into copies of H:

Theorem (Kim, Kühn, Osthus & Tyomkyn, 2016+)

If $1/n \ll \epsilon \ll 1/\Delta, d$, then the following near-optimal packing exists:



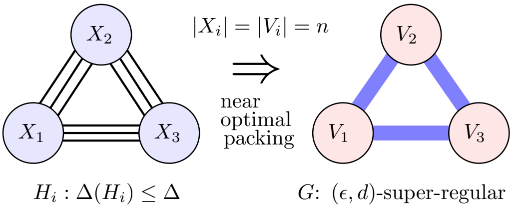
Main result

Theorem (Kim, Kühn, Osthus & Tyomkyn, 2016+)

Suppose $1/n \ll \epsilon \ll d, \alpha, 1/\Delta, 1/r$ and that

- *each of $H_1, \dots, H_s$ has vertex classes $X_1, \dots, X_r$ of size n and $\Delta(H_i) \leq \Delta$,*
- *G has vertex classes $V_1, \dots, V_r$ of size n such that all pairs (V_i, V_j) are (ϵ, d)-super-regular,*
- *$\sum_{\ell=1}^{s} e(H_\ell) \leq (1-\alpha)e(G)$.*

Then $H_1, \dots, H_s$ pack into G.



Main result

Actually prove a stronger version with 'bells and whistles' added, e.g.:

- allowed to specify 'target sets' for some of the vertices,
- allowed to have a bounded degree reduced graph with many clusters (i.e. much more than $1/\epsilon$),
- super-regular pairs in G allowed to have different densities,
- clusters allowed to have slightly different sizes.

Proof sketch: strategy

Strategy: pack $H_1, \dots, H_s$ into G successively,
i.e. embed H_i into $G_i := G - H_1 \cdots - H_{i-1}$

Naive approach: Choose H_i 'uniformly at random' in G_i.

Aim to show:

(a) each edge of G_i equally likely to be chosen.

(b) G_{i+1} is ϵ_{i+1}-regular, where $\epsilon_{i+1} \sim \epsilon$.

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Problems:

(a) is impossible (e.g. if H_i a triangle factor, G_i may have edges not in any triangle)
(b) seems infeasible, as ϵ_i increases too rapidly.

Proof sketch: using many rounds

To maintain ϵ-regularity of G: use bounded number of rounds

- choose embedding $\phi(H_i)$ of H_i independently for all i within the same round,
- update G only after each round

i.e. allow overlaps within a round

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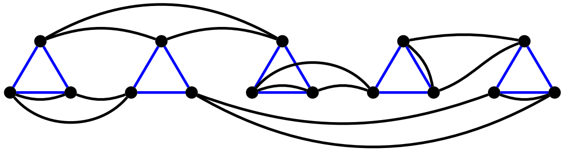
Example: each H_i is a triangle-factor

$\phi(H_i) \cup \phi(H_j)$



$\Rightarrow$ $\phi(H_i)$ and $\phi(H_j)$ are

- almost edge-disjoint if embedded in the same round,
- edge-disjoint if embedded in different rounds

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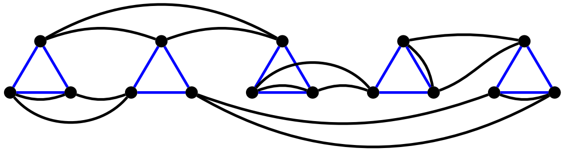

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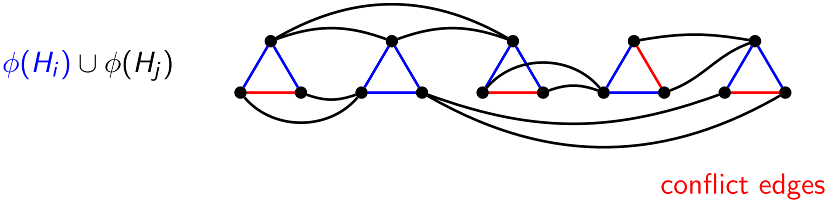
Aim: repair packing at the end to achieve edge-disjointness

Proof sketch: repairing the packing

Patching:

- set aside patching graph $P \subseteq G$ at the beginning of proof, ($P=$ thin edge-slice of G)
- use P to patch each H_i in turn



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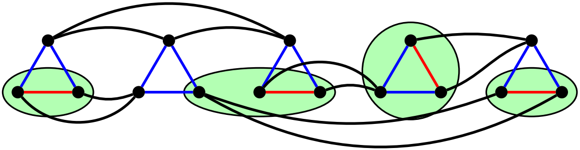
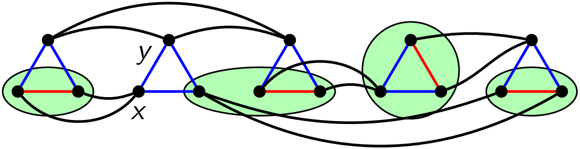


X= green set

Add small random vertex set to the vertices incident to conflict edges and re-embed $H_i[X]$ using P

$\phi(H_i) \cup \phi(H_j)$



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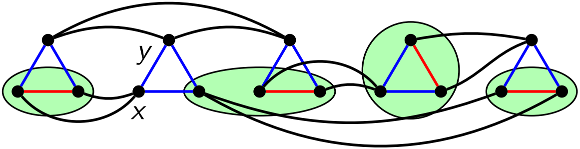
Problem: if x and y have common neighbours in X, they need to have many common neighbours in $P[X]$

Proof sketch: repairing the packing

$\phi(H_i) \cup \phi(H_j)$



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Problem: if x and y have common neighbours in X, they need to have many common neighbours in $P[X]$

Solution: ensure that $\phi(H_i)$ behave well with respect to patching graph P already when choosing $\phi(H_i)$

Proof sketch: summary of strategy

Step 1: embed the H_i one by one using bounded number of rounds

- embed the H_i independently from each other within the same round, choosing a 'uniform' embedding of each H_i
- update G only after each round

i.e. allow overlaps within a round

Step 2: deal with overlaps using patching graph

G will still be super-regular because

- choose 'uniform' embedding of each H_i
- perform only bounded number of updates of G

Optimal packings

Conjecture (Gyárfás & Lehel, 1978)

Given trees $T_1, \ldots, T_n$ such that T_i has i vertices, K_n has a decomposition into $T_1, \ldots, T_n$.

Tree packing conjecture holds for bounded degree trees:

Theorem (Joos, Kim, Kühn & Osthus, 2016+)

Suppose $1/n \ll 1/\Delta$. For each $i \in [n]$, let T_i be a tree with i vertices and $\Delta(T_i) \leq \Delta$. Then K_n decomposes into $T_1, \ldots, T_n$.

can omit condition $\Delta(T_i) \leq \Delta$ for first εn trees

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Proof uses

- 'bells and whistles' version of blow-up lemma for packings
- even-regular robust expanders have Hamilton decompositions (Kühn & Osthus)
- iterative absorption method

Optimal packings

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Theorem (Joos, Kim, Kühn & Osthus, 2016+)

Suppose $1/n \ll \alpha, 1/\Delta$. Let $\mathcal{H}$ be collection of graphs such that

- $|H| \leq n$ *and* $\Delta(H) \leq \Delta$ *for each* $H \in \mathcal{H}$,
- $\mathcal{H}$ *contains at least* $(1/2 + \alpha)n$ *trees* T *with* $\alpha n \leq |T| \leq (1 - \alpha)n$,
- $\sum_{H \in \mathcal{H}} e(H) = \binom{n}{2}$.

Then the graphs in $\mathcal{H}$ *pack into* K_n.

also prove a version with K_n replaced by any quasi-random graph

Optimal packings

Conjecture (Ringel 1963)

Let T be an $(n+1)$-vertex tree. Then K_{2n+1} decomposes into $2n+1$ copies of T.

Joos, Kim, Kühn & Osthus:
conjecture holds for bounded degree trees

Proof sketch of tree packing conjecture for bdd degree trees

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Proof approach via absorption:

(1.) Remove sparse absorbing graph A from K_n,

(2.) use Blow-up lemma for approx. decompositions to find almost optimal packing of trees into of $K_n - E(A)$, call leftover L,

(3.) hope that $L \cup A$ has decomposition into remaining trees.

Proof sketch of tree packing conjecture for bdd degree trees

Theorem (Joos, Kim, Kühn & Osthus, 2016+)

Suppose $1/n \ll 1/\Delta$. *For each* $i \in [n]$, *let* T_i *be a tree with* i *vertices and* $\Delta(T_i) \leq \Delta$. *Then* K_n *decomposes into* $T_1, \ldots, T_n$.

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Use **iterative absorption** approach:
Split up the absorbing process into many steps which gradually make leftover smaller and smaller.

Proof sketch

Let $G = K_n$ and γ be a small constant.
Step 1: consider sequence $V(G) = U_0 \supseteq U_1 \supseteq \ldots \supseteq U_k$ with

$$|U_{i+1}| = \gamma |U_i| \quad \text{and} \quad |U_k| \approx n^{1/3}$$

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Step 2: split $\{T_1, \ldots, T_n\}$ into sets $\mathcal{T}_1, \ldots, \mathcal{T}_k$ such that:
$\mathcal{T}_i$ contains $\approx |U_{i-1}|$ trees, each of order at most $|U_{i-1}|$.

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Assume **after i iterations** we have packed $\mathcal{T}_1 \cup \ldots \cup \mathcal{T}_i$ such that

- all edges not inside U_i are covered,
- most edges inside U_i are not covered.

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- all edges not inside U_i are covered,
- most edges inside U_i are not covered.

$(i+1)$th iteration:
Step a: use approx. decomposition blow-up lemma to pack most of $\mathcal{T}_{i+1}$ into $G[U_i] - E(G[U_{i+1}])$, obtain leftover L_{i+1}
Step b: use 'unpacked' trees in $\mathcal{T}_{i+1}$ and some edges of $G[U_{i+1}]$ to cover leftover L_{i+1} greedily

Proof sketch

1st iteration: (Recall $|U_1| = \gamma n$.)
Step a: use approx. decomposition blow-up lemma to pack most of $\mathcal{T}_1$ into $G - E(G[U_1])$
$\Rightarrow$ obtain leftover L_1 of density α

Aim: use 'unpacked' trees in $\mathcal{T}_1$ and some edges of $G[U_1]$ to cover leftover L_1 greedily

- set aside quasi-random bipartite graph B of density $\beta \gg \alpha$ between U_1 and $V(G) \setminus U_1$
- use edges edges in $B \cup G[U_1]$ to greedily cover all edges in L_1 lying outside U_1
- use edges edges in $G[U_1]$ cover **all** remaining edges of $B \cup L_1$

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Need: $\gamma \gg \beta \gg \alpha$

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Need: $\gamma \gg \beta \gg \alpha$
Problem: in 2nd iteration will get leftover of density $\alpha' \gg \beta$
i.e. error terms explode

Proof sketch

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$\Rightarrow$ obtain leftover L_1 of density α

Aim: use 'unpacked' trees in $\mathcal{T}_1$ and some edges of $G[U_1]$ to cover leftover L_1 greedily

- set aside quasi-random bipartite graph B of density $\beta \gg \alpha$ between U_1 and $V(G) \setminus U_1$
- use edges edges in $B \cup G[U_1]$ to greedily cover all edges in L_1 lying outside U_1
- use edges edges in $G[U_1]$ cover **all** remaining edges of $B \cup L_1$

Need: $\gamma \gg \beta \gg \alpha$
Solution: apply Regularity lemma with tiny ε before applying approx. decomposition blow-up lemma to suitable reduced graph cycles $\Longrightarrow$ leftover of density α

Proof sketch

after k iterations:

- are left with almost complete graph H on U_k
- but are now seeking a **decomposition**!

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$\Rightarrow$ **introduce new Step 0:**

- choose collection $\mathcal{T}^*$ of m small trees (where $m \approx \binom{|U_k|}{2} \approx n^{2/3}$)
- remove a leaf ℓ_{T^*} from each $T^* \in \mathcal{T}^*$
- let z_{T^*} be neighbour of ℓ_{T^*} in T^*
- before 1st iteration, embed each $T^* - \ell_{T^*}$ so that
  - z_{T^*} is embedded into U_k
  - no other vertex of T^* is embedded into U_k
  - each vertex in U_k is image of exactly d vertices z_{T^*} (where $d = m/|U_k|$)

Proof sketch

New situation after k iterations:

- are left with almost complete graph H on U_k, where $e(H) = m = d|U_k|$
- have embedded everything apart from the leaves ℓ_{T^*} of the T^*

Can now complete decomposition by adding the remaining 'leaf edge' to each 'incomplete' tree T^*:

- show that H has an orientation of outdegree d
- use this orientation to embed all the ℓ_{T^*}

Open problems

Problem

Relax condition on maximum degree in tree packing conjecture and Ringel's conjecture.

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Relax condition on maximum degree in tree packing conjecture and Ringel's conjecture.

possible for approximate tree packings:

Theorem (Ferber & Samotij, 2016+)

If $1/n \ll \alpha$ *and* $T_1, \ldots, T_t$ *are trees s.t.*

- $\Delta(T_i) \leq n^{1/6}/(\log n)^6$ *and* $|T_i| = n$,
- $\sum_{i=1}^{t} e(T_i) \leq (1-\alpha)\binom{n}{2}$,

then $T_1, \ldots, T_t$ *pack into* K_n.

Open problems: Graceful labellings

Graceful labelling ϕ of n-vertex G:
labelling of $V(G)$ with $[n]$ so that all resulting edge-labels are distinct, where edge label of uv is $|\phi(u) - \phi(v)|$

Conjecture

Every tree has a graceful labelling.

- Implies Ringel's conjecture
- Adamaszek, Allen, Grosu, Hladky (2016^+)
  proved approximate version of graceful labelling conjecture for trees T with $\Delta(T) = O(n/\log n)$
  (approximate = needs $(1+\epsilon)n$ labels)

Open problems

Problem (Oberwolfach Problem)

Suppose that n is odd and F is a vertex-disjoint union of cycles with total length n. Then does K_n have an F-decomposition?

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Theorem (Bryant & Scharaschkin, 2009)

Answer is yes for infinitely many n.

Our results imply an approximate F-decomposition for every large n.