Optimal Control Problem for a Nonlinear Schrödinger equation

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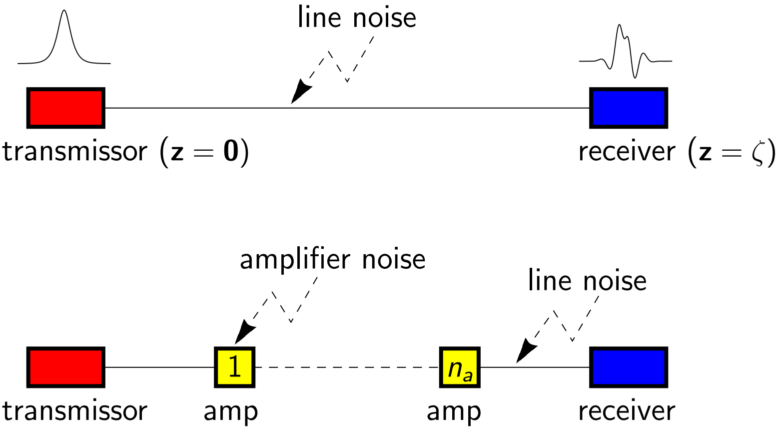
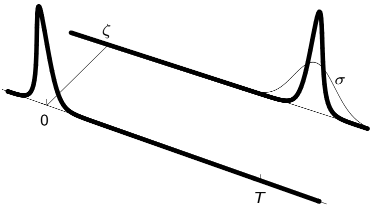
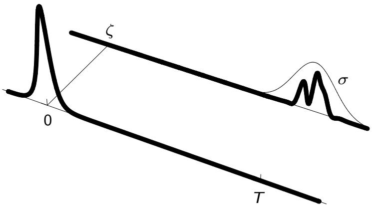
Transmission line



Pulses with or without noise



pulse without noise



pulse with noise

Transmission without noise

$$\begin{cases} \partial_z u = i\partial_t^2 u + i|u|^2 u, & z \in [0,\zeta],\ t \in \mathbb{R} \\ u(0,t) = u_0(t) \end{cases}$$

- $|u|^2$ is the signal.
- $|u_0|^2$ is the initial signal.
- Conservation of charge: $\|u(z)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}$.

Transmission with noise

$$\begin{cases} \partial_z u = i\partial_t^2 u + i|u|^2 u + g, \quad z \in [0,\zeta],\ t \in \mathbb{R} \\ u(0,t) = u_0(t) \end{cases}$$

- $g \in L^2([0,\zeta], L^2(\mathbb{R}))$.
- $u_0 \in L^2(\mathbb{R})$, $|u_0|^2$ initial signal.
- The L^2 norm is not conserved:

$$\|u(z)\|^2_{L^2(\mathbb{R})} = \|u_0\|^2_{L^2(\mathbb{R})} + \int_0^z 2\mathrm{Re}\left(\langle u(z'), g(z')\rangle_{L^2(\mathbb{R})}\right) dz'$$

Signal degradation

σ : temporal window .

$\sigma(t) = \alpha e^{-\beta(t-T)^2},\ T = \zeta/c.$

Given an initial pulse, can the noise degrade the signal?

$$\int_{\mathbb{R}} \sigma^2(t)\,|u(\zeta,t)|^2 dt \leq \eta$$

Control Problem

State equation: $\partial_z u = A(u) + g,\ u(0) = u_0$

- $A(u) = i\partial_t^2 u + i|u|^2 u$: noise free evolution operator
- $g \in L^2([0,\zeta], L^2(\mathbb{R}))$: control/noise
- u_0 : initial state
- $u[g](\zeta) \in \mathcal{U}$: state contraint at the end of the transmission, with

$$\|\sigma u[g](\zeta)\|_{L^2(\mathbb{R})}^2 \leq \eta.$$

¿Is there a minimum noise g for wich the signal is degraded

$$\int \sigma^2(t)\left|u(\zeta,t)\right|^2 dt \leq \eta,$$

where $u = u[g]$ is the solution of $\partial_z u = A(u) + g,\ u(0) = u_0$?

The optimal control problem

$$\min_{g \in \mathcal{G}} \mathcal{J}(g), \quad \mathcal{G} = \left\{ g \in L^2\left([0,\zeta], L^2(\mathbb{R})\right) : \Lambda(g) \in \mathcal{U} \right\}, \text{ with}$$

- $\mathcal{J}(g) = \|g\|_{L^2([0,\zeta],L^2(\mathbb{R}))}$.
- $\Lambda(g) = \sigma u[g](\zeta)$ where $u[g] \in C([0,\zeta], L^2(\mathbb{R}))$ is the solution of the nonlinear Schrödinger equation

$$\begin{cases} \partial_z u = i\partial_t^2 u + i|u|^2 u + g \\ u(0,t) = u_0(t). \end{cases}$$

- $\mathcal{U} = \{ y \in L^2(\mathbb{R}) / \|y(t)\|^2_{L^2(\mathbb{R})} \leq \eta \}$.
- We assume $0 \notin \mathcal{G}$.

Integral equation

- Let $S(z)$ be the unitary group generated by $i\partial_t^2$.
- A mild solution for the state equation with noise is

$$u(z) = S(z)u_0 + \int_0^z S\left(z - z'\right)\left(i|u(z')|^2 u(z') + g(z')\right)dz'.$$

- We will need Strichartz estimates. We define: (p, q) is a pair of admisible exponents for $1 \leq p \leq \infty$, if

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{p}.$$

- $(6, 6)$ and $(2, \infty)$ are pairs of admisible exponents.

Strichartz estimates

- If (p, q) is a pairs of admissible exponents, then there exists $C_p > 0$ such that for any $u_0 \in L^2(\mathbb{R})$ it holds
$$\|S(z)\, u_0\|_{L^q(I, L^p)} \leq C_p \|u_0\|_{L^2}.$$
- Let (p, q), (r', γ') be pairs of admissible exponents, then there exists $C_{p,r} > 0$ such that for $g \in L^\gamma(I, L^r)$ it holds
$$\left\| \int_0^z S(z - z')\, g(z')\, dz' \right\|_{L^q(I, L^p)} \leq C_{p,r} \|g\|_{L^\gamma(I, L^r)},$$
where r, γ are the conjugate exponents of r', γ' respectively.

Strichartz estimates

For $u \in L^6([0,\zeta], L^6(\mathbb{R}))$, we have

$$\left\| |u|^2 u \right\|_{L^2(0,\zeta,L^2)} = \|u\|^3_{L^6(0,\zeta,L^6)} .$$

For $u, \tilde{u} \in L^6([0,\zeta], L^6(\mathbb{R}))$

$$\left\| |u|^2 u - |\tilde{u}|^2 \tilde{u} \right\|_{L^2(0,\zeta,L^2)} \leq C \left(\|u\|^2_{L^6(0,\zeta,L^6)} + \|\tilde{u}\|^2_{L^6(0,\zeta,L^6)} \right) \times \|u - \tilde{u}\|_{L^6(0,\zeta,L^6)} .$$

see Cazanave 2003.

Local existence

Space of solutions

$$\mathcal{X}_{\mathrm{z}} = C([0,z], L^2(\mathbb{R})) \cap L^6([0,z], L^6(\mathbb{R}))$$

Theorem

Given $u_0 \in L^2(\mathbb{R})$, *let* $r = \text{máx}\left\{\|u_0\|_{L^2}, \|g\|_{L^1(0,\zeta,L^2)}\right\}$. *Then, there exist* $\mathrm{z} = \mathrm{z}(r) \in (0,\zeta]$ *and* $u \in \mathcal{X}_{\mathrm{z}}$ *solution of the integral equation*

$$u(z) = S(z)u_0 + \int_0^z S(z-z')\left(\mathrm{i}|u(z')|^2 u(z') + g(z')\right) dz'.$$

The solution u *depends continuously on* u_0, g *and*

$$\|u\|_{C(0,\mathrm{z},L^2)} \leq \|u_0\|_{L^2} + 2\,\|g\|_{L^1(0,\zeta,L^2)}.$$

Global existence

Theorem

Given $u_0 \in L^2(\mathbb{R})$ and $g \in L^1([0,\zeta], L^2(\mathbb{R}))$, there exists a unique $u \in \mathcal{X}_\zeta$ solution of the state equation which verifies

$$\|u\|_{\mathcal{X}_\zeta} \leq C\left(\zeta, \|u_0\|_{L^2}, \|g\|_{L^1(0,\zeta,L^2)}\right).$$

Furthermore, $u \in W^{1,1}([0,\zeta], H^{-2}(\mathbb{R}))$,

$$\|u\|_{W^{1,1}(0,\zeta,H^{-2})} \leq C\left(\zeta, \|u_0\|_{L^2}, \|g\|_{L^1(0,\zeta,L^2)}\right)$$

and the state equation is posed in H^{-2} for a.e. $z \in [0,\zeta]$.

Λ es Fréchet differentiable

Recall $u[g]$ is the solution of the integral equation

$$u(z) = S(z)u_0 + \int_0^z S(z-z')\left(\mathrm{i}|u(z')|^2 u(z') + g(z')\right) dz'.$$

Proposition

Let $u_0 \in L^2(\mathbb{R})$ and $g \in L^1\left([0,\zeta], L^2(\mathbb{R})\right)$, then $u[g]$ es Fréchet differentiable and $D_g u[g](\delta g) \in \mathcal{X}_\zeta$ is the solution of the linear integral equation

$$y(z) = \int_0^z S(z-z')\left(2\mathrm{i}\mathrm{Re}\left(\overline{u[g]}\, y\right) u[g] + \mathrm{i}\,|u[g]|^2\, y + \delta g\right)(z')dz'.$$

The optimal control problem

$$\min_{g \in \mathcal{G}} \mathcal{J}(g), \quad \mathcal{G} = \left\{ g \in L^2\left([0,\zeta], L^2(\mathbb{R})\right) : \Lambda(g) \in \mathcal{U} \right\} \text{ with}$$

- $\mathcal{J}(g) = \|g\|_{L^2([0,\zeta],L^2(\mathbb{R}))}$.
- $\Lambda(g) = \sigma u[g](\zeta)$ where $u[g] \in C([0,\zeta], L^2(\mathbb{R}))$ is the solution of the nonlinear Schrödinger equation

$$\begin{cases} \partial_z u = i\partial_t^2 u + i|u|^2 u + g \\ u(0,t) = u_0(t). \end{cases}$$

- $\mathcal{U} = \{ y \in L^2(\mathbb{R}) / \|y(t)\|^2_{L^2(\mathbb{R})} \leq \eta \}$.

The set $\mathcal{G}$ is not empty

Consider

$$\partial_z \psi = \mathrm{i}\partial_t^2 \psi + \mathrm{i}|\psi|^2\psi - \lambda\psi$$

with $\psi(0) = u_0$ and $\lambda > 0$. Then

$$\frac{d}{dz}\|\psi(z)\|^2_{L^2(\mathbb{R})} = -2\lambda\|\psi(z)\|^2_{L^2(\mathbb{R})}$$

$$\|\psi(z)\|^2_{L^2(\mathbb{R})} = \|u_0\|^2_{L^2(\mathbb{R})}\, e^{-2\lambda z}$$

$g = \lambda\psi$ with λ big enough makes that

$$\|\sigma u[g](\zeta)\|^2_{L^2(\mathbb{R})} \leq \|\sigma\|^2_{L^\infty(\mathbb{R})}\, \|u_0\|^2_{L^2(\mathbb{R})}\, e^{-2\lambda z} \leq \eta$$

Minimizing sequence

$$g_k \in \mathcal{G},\ \|g_k\|_{L^2([0,\zeta],L^2(\mathbb{R}))} \to \inf_{g\in\mathcal{G}} \|g\|_{L^2([0,\zeta],L^2(\mathbb{R}))}$$

- Then $g_k \rightharpoonup g^*$ en $L^2([0,\zeta],L^2(\mathbb{R}))$.
- From the estimates for solution u_k associated to g_k,

$$\|u_k\|_{\mathcal{X}_\zeta} \le C.$$

- There exists $u^* \in \mathcal{X}_\zeta$

$$u_k \to u^* \text{ in } L^2([0,\zeta],L^2_{loc}(\mathbb{R})).$$

- Then

$$|u_k|^2 u_k \rightharpoonup |u^*|^2 u^* \text{ in } L^2([0,\zeta],L^2(\mathbb{R})).$$

- Is u^* the associated solution to the control g^*?

Convergence

- From the existence theorem, $u_k \in H^1([0,\zeta], H^{-2}(\mathbb{R}))$ and is bounded.
- Then $u^* \in H^1([0,\zeta], H^{-2}(\mathbb{R}))$ and

$$\partial_z u_k \rightharpoonup \partial_z u^* \text{ in } L^2([0,\zeta], H^{-2}(\mathbb{R})).$$

- For all k, $\partial_z u_k = \mathrm{i}\partial_t^2 u_k + \mathrm{i}|u_k|^2 u_k + g_k$.
- For $\theta \in C_0^1([0,\zeta], H^2(\mathbb{R}))$

$$\int_0^\zeta \langle \partial_z u_k, \theta \rangle_{H^{-2},H^2} dz = \int_0^\zeta \langle i\partial_t^2 u_k, \theta \rangle_{H^{-2},H^2} dz + \int_0^\zeta \langle i|u_k|^2 u_k, \theta \rangle_{L^2} dz + \int_0^\zeta \langle g_k, \theta \rangle_{L^2} dz$$

- Then $\partial_z u^* = \mathrm{i}\partial_t^2 u^* + \mathrm{i}|u^*|^2 u^* + g^*$ and $u^* \in C([0,\zeta], L^2(\mathbb{R}))$.

The limit is optimal

Since

$$\int_{\mathbb{R}} \sigma^2(t)|u_k(t,\zeta)|^2 dt \leq \eta$$

and

$$\sigma u_k(\zeta) \rightharpoonup \sigma u^*(\zeta) \text{ en } L^2(\mathbb{R})$$

then

$$\|\sigma u^*(\zeta)\|_{L^2(\mathbb{R})} \leq \text{lím inf}\, \|\sigma u_k(\zeta)\|_{L^2(\mathbb{R})} \leq \eta$$

and since $g_k \rightharpoonup g^*$ in $L^2(\mathbb{R})$

$$\mathcal{J}_* \leq \|g_*\|^2_{L^2(0,\zeta,L^2)} \leq \liminf_{n\to\infty} \|g_n\|^2_{L^2(0,\zeta,L^2)} = \mathcal{J}_*$$

Then g^* is optimal and $\|\sigma u[g^*](\zeta)\|_{L^2(\mathbb{R})} = \|\sigma u^*(\zeta)\|_{L^2(\mathbb{R})} = \eta$.

Abstract theorem

Theorem (Casas 1993)

Given G and Z Banach spaces and $\mathcal{U} \subset Z$ a convex subspace with nonempty interior. Let g_* be a solution of the problem

$$\begin{cases} \text{mín}\, \mathcal{J}(g) \\ g \in G, \Lambda(g) \in \mathcal{U} \end{cases}$$

where $\mathcal{J} : G \to (-\infty, +\infty]$ and $\Lambda : G \to Z$ are Gateuax differentiable. Then, there exist $\lambda \geq 0$ and $\mu_* \in Z'$ such that

- $\lambda + \|\mu_*\|_{Z'} > 0$
- $\langle \mu_*, z - \Lambda(g_*) \rangle \leq 0$ *for all* $z \in \mathcal{U}$
- $\langle \lambda \mathcal{J}'(g_*) + [D\Lambda(g_*)]^\star \mu_*, g - g^* \rangle_{G^*,G} \geq 0.$

Optimal control problem

Applied to our problem

- $\mathcal{J} : L^2([0,\zeta], L^2(\mathbb{R})) \to [0,\infty)$, $\mathcal{J}(g) = \|g\|^2_{L^2[0,\zeta],L^2(\mathbb{R})}$ is Gateaux differentiable and $\mathcal{J}'(g) = 2g$.
- $\Lambda : L^2([0,\zeta], L^2(\mathbb{R})) \to L^2(\mathbb{R})$, $\Lambda(g) = \sigma u[g](\zeta)$ is Gateaux differentiable and $D_g\Lambda(g) = \sigma D_g u[g](\zeta)$.
- $\mathcal{U} = \{y \in L^2(\mathbb{R}) / \|y\|_{L^2(\mathbb{R})} \leq \eta\} \subset L^2(\mathbb{R})$.
- The problem is

$$\begin{cases} \text{mín } J(g) = \|g\|_{L^2([0,\zeta],L^2(\mathbb{R}))} \\ g \in G = L^2([0,\zeta], L^2(\mathbb{R})), \Lambda(g) = \sigma u[g](\zeta) \in \mathcal{U} \end{cases}$$

Dual problem

We compute $(D\Lambda(g))^\star : L^2(\mathbb{R}) \to L^2([0,\zeta], L^2(\mathbb{R}))$:

- Given $g \in L^2\left([0,\zeta], L^2\left(\mathbb{R}\right)\right)$, $u[g] \in \mathcal{X}_\zeta$ the associated state, and $\mu_\zeta \in L^2(\mathbb{R})$, let $\mu \in \mathcal{X}_\zeta$ be the solution of the dual equation

$$\begin{aligned}\partial_z \mu &= \mathrm{i}\partial_t^2 \mu + 2\mathrm{i}|u|^2\mu - \mathrm{i}u^2\overline{\mu},\\ \mu\left(\zeta\right) &= \sigma\mu_\zeta.\end{aligned}$$

- From

$$\langle \mu_\zeta, D\Lambda(g)(\delta g)\rangle_{L^2} = \langle \mu_\zeta, \sigma D_g u[g](\delta g)(\zeta)\rangle_{L^2}$$

$$= \int_0^\zeta \frac{d}{dz}\langle \mu(z), D_g u[g](\delta g)(z)\rangle_{L^2} = \int_0^\zeta \langle \mu, \delta g\rangle_{L^2}.$$

- Then $(D\Lambda(g))^\star \mu_\zeta = \mu$.

Necessary conditions

From Casas's Theorem, if g_* is an optimal control

- There exists $\lambda \geq 0$ and $\mu_\zeta \in L^2(\mathbb{R})$ such that
- $\lambda + \|\mu_\zeta\|_{L^2(\mathbb{R})} > 0$
- $\langle \mu_\zeta, u - \sigma u[g^*](\zeta) \rangle \leq 0$ for all $u \in \mathcal{U}$.
- $\langle \lambda 2g^* + \mu, g - g^* \rangle_{L^2([0,\zeta],L^2(\mathbb{R}))} \geq 0$, for all $g \in L^2([0,\zeta], L^2(\mathbb{R}))$.
- $\mu_\zeta = \alpha\sigma u[g^*](\zeta)$, with $\alpha \geq 0$.
- $\lambda = 1$, $\mu_\zeta \neq 0$, then $\alpha > 0$ and $g^* = -\frac{1}{2}\mu$.
- $g^*(\zeta) = -\frac{1}{2}\mu(\zeta) = -\frac{1}{2}\sigma\mu_\zeta = -\frac{1}{2}\alpha\sigma^2 u[g^*](\zeta)$.

Necessary conditions

Let g^* be an optimal control and $u^* = u[g^*]$ its associated state

$$
\begin{aligned}
&\partial_z u^* = \mathrm{i}\partial_t^2 u^* + \mathrm{i}|u^*|^2 u^* + g^*,\\
&\partial_z g^* = \mathrm{i}\partial_t^2 g^* + 2\mathrm{i}|u^*|^2 g^* - \mathrm{i}(u^*)^2 \overline{g^*},\\
&u\,(0) = u_0,\\
&g^*\,(\zeta) = \beta\sigma^2 u^*(\zeta) \text{ with } \beta < 0\\
&\|\sigma u^*(\zeta)\|_{L^2}^2 = \eta
\end{aligned}
$$