The saddle-shaped solution to the Allen-Cahn equation and a conjecture of De Giorgi

Xavier Cabré

ICREA and UPC, Barcelona

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Uniqueness and stability of saddle-shaped solutions to the Allen–Cahn equation

Xavier Cabré [1]

ICREA and Universitat Politècnica de Catalunya, Departament de Matemàtica Aplicada I, Diagonal 647, 08028 Barcelona, Spain

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Abstract

We establish the uniqueness of a saddle-shaped solution to the diffusion equation $-\Delta u = f(u)$ in all of $\mathbb{R}^{2m}$, where f is of bistable type, in every even dimension $2m \geqslant 2$. In addition, we prove its stability whenever $2m \geqslant 14$.

Saddle-shaped solutions are odd with respect to the Simons cone $\mathcal{C} = \{(x^1, x^2) \in \mathbb{R}^m \times \mathbb{R}^m \colon |x^1| = |x^2|\}$ and exist in all even dimensions. Their uniqueness was only known when $2m = 2$. On the other hand, they are known to be unstable in dimensions 2, 4, and 6. Their stability in dimensions 8, 10, and 12 remains an open question. In addition, since the Simons cone minimizes area when $2m \geqslant 8$, saddle-shaped solutions are expected to be global minimizers when $2m \geqslant 8$, or at least in higher dimensions. This is a property stronger than stability which is not yet established in any dimension.

Résumé

On montre l'unicité d'une solution du type selle de l'équation de diffusion $-\Delta u = f(u)$ dans tout $\mathbb{R}^{2m}$, où f est une non-linéarité bistable, dans toutes les dimensions paires $2m \geqslant 2$. De plus, on montre sa stabilité lorsque $2m \geqslant 14$.

Les solutions du type selle sont impaires par rapport au cône de Simons $\mathcal{C} = \{(x^1, x^2) \in \mathbb{R}^m \times \mathbb{R}^m \colon |x^1| = |x^2|\}$ et elles existent dans toutes les dimensions paires. Leur unicité était connue seulement quand $2m = 2$. D'autre part, il est connu qu'elles sont instables dans les dimensions 2, 4 et 6. Leur stabilité dans les dimensions 8, 10 et 12 reste une question ouverte. En outre, puisque

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Thm [Simons '68] $E \subset \mathbb{R}^n$ of minimal perimeter.
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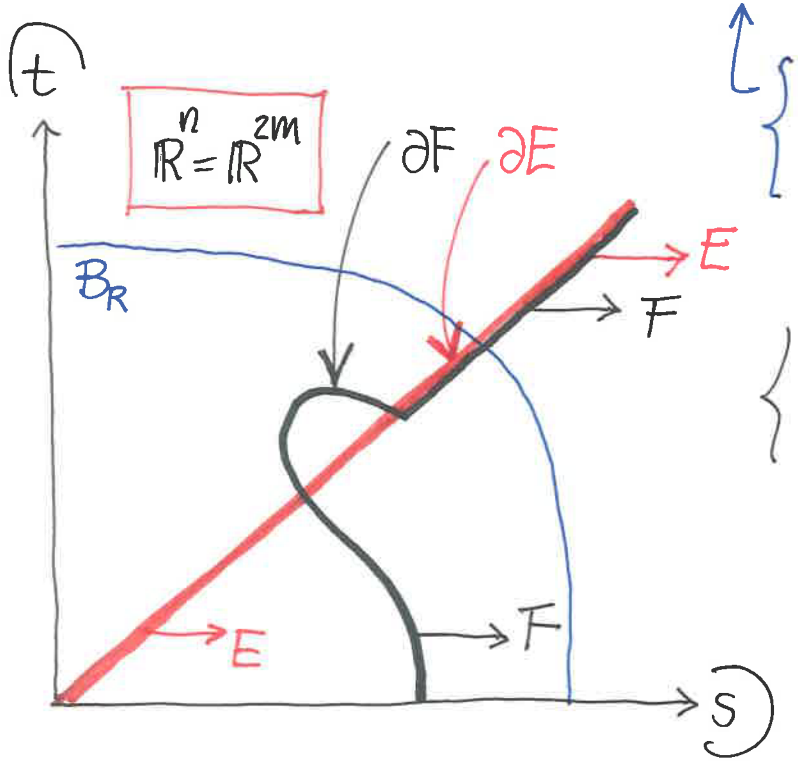

$\uparrow \left\{ \begin{array}{l} \text{Area}(B_R \cap \partial E) \\ \leq \text{Area}(B_R \cap \partial F). \end{array} \right.$

$$\left\{ \begin{array}{l} s = \sqrt{x_1^2 + \cdots + x_m^2} \\ t = \sqrt{x_{m+1}^2 + \cdots + x_{2m}^2} \end{array} \right.$$

$\partial E = \mathcal{C} := \{ s = t \}$: $\boxed{\text{Simons cone}}$
$(E = \{ s > t \})$

$\downarrow$

$\forall n = 2m$ stationary
(mean curv $= 0$)

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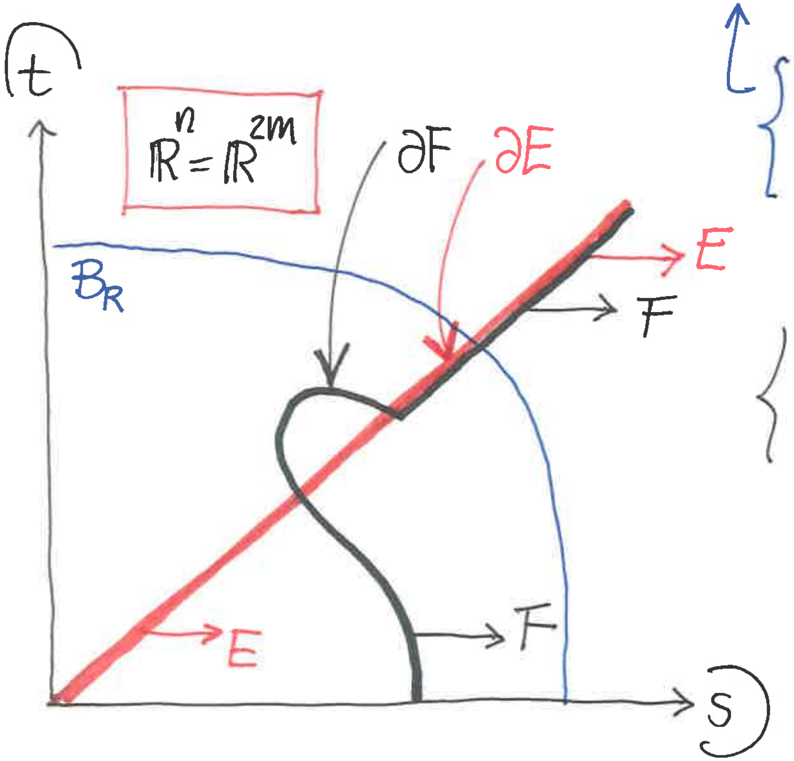

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Thm [Bombieri-DeGiorgi-Giusti '69]
Simons cone $\mathcal{C} \subset \mathbb{R}^{2m}$ minimal $\iff$ $2m \geq 8$.

Minimal graphs

Thm [Simons '68] $\varphi: \mathbb{R}^n \to \mathbb{R}$ minimal graph, i.e.,

$$\operatorname{div}\left(\frac{\nabla\varphi}{\sqrt{1+|\nabla\varphi|^2}}\right)=0 \text{ in } \mathbb{R}^n.$$

If $n \leq 7$ then $\nabla\varphi \equiv ctt.$

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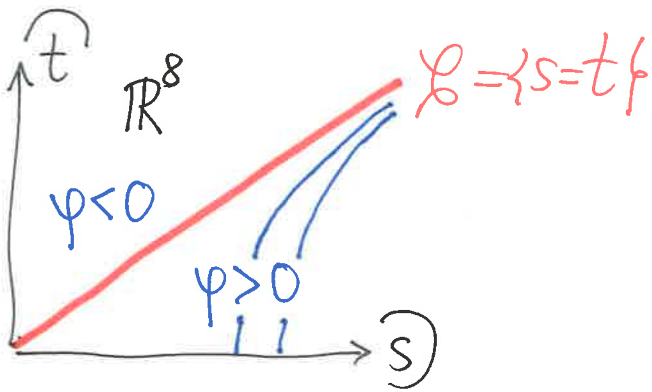
Thm [Bombieri-De Giorgi-Giusti '69]

If $n\geq 8$, there exist minimal graphs (of $\varphi:\mathbb{R}^n\to\mathbb{R}$) which are not hyperplanes.

If $n=8$, $\varphi=\varphi(s,t)$ is odd w.r.t. the Simons cone.

($\Rightarrow$ $\{\varphi=0\}=\{s=t\}\subset\mathbb{R}^8$.)

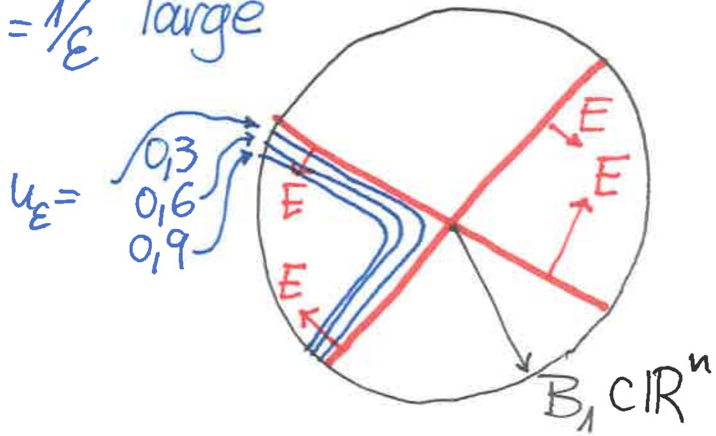


Modica-Mortola thm

$$-\Delta u = u - u^3 = f(u) \quad \text{in } \mathbb{R}^n$$

$$u_\varepsilon(x) = u(x/\varepsilon) = u(Rx) \quad , x \in B_1 \subset \mathbb{R}^n . \quad R = 1/\varepsilon \text{ large}$$

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} f(u_\varepsilon)$$



Thm [Modica-Mortola]

Minimizers u_ε in $B_1 \subset \mathbb{R}^n$.

$$u_\varepsilon \xrightarrow[\varepsilon \downarrow 0]{} \begin{cases} 1 & \text{in } E \\ -1 & \text{in } B_1 \setminus E \end{cases}$$

& E is of minimal perimeter in B_1.

"Pf."

$$E_{B_1}(u_\varepsilon) = \int_{B_1} \varepsilon \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{2} 2G(u_\varepsilon) \frac{1}{\varepsilon}$$

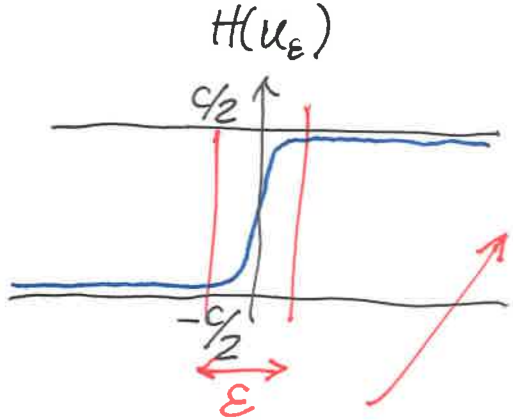
$$\geq$$

$$\int_{B_1} \sqrt{\varepsilon} |\nabla u_\varepsilon| \cdot \sqrt{2G(u_\varepsilon)} \frac{1}{\sqrt{\varepsilon}}$$

$=$ if parallel level sets

$$=$$

$$\int_{B_1} |\nabla H(u_\varepsilon)| . \xrightarrow{\varepsilon \downarrow 0} \int_{B_1} c |\nabla \mathbb{1}_E| = c \cdot \text{perimeter}_{B_1}(E)$$

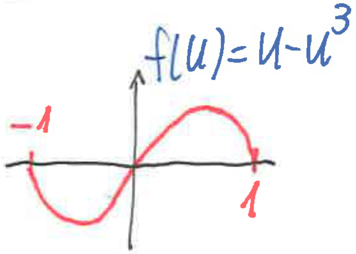


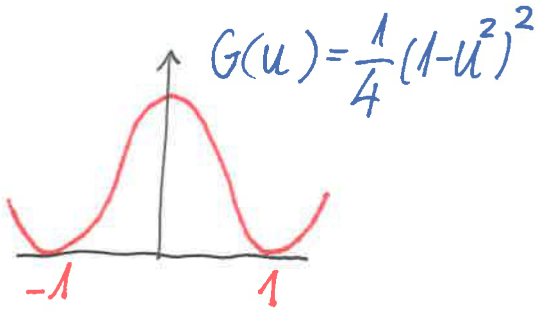
$$\int |\partial_x H(u_\varepsilon)| = \int \partial_x H(u_\varepsilon) = \text{jump} = c$$

Allen-Cahn equation. A conjecture of De Giorgi

(AC) $\quad -\Delta u = u - u^3 \quad$ in $\mathbb{R}^n$

$\hookrightarrow \quad E_{B_R}(u) = \int_{B_R} \frac{1}{2}|\nabla u|^2 + G(u)$





Thm [Savin '03→'09]

> u global minimizer of (AC) in $\mathbb{R}^n$. If $\boldsymbol{n \leq 7}$, then u is **1-D**,
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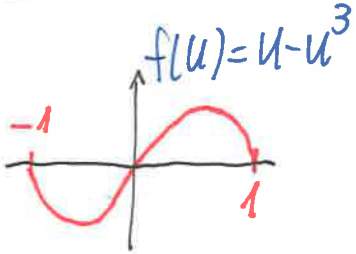

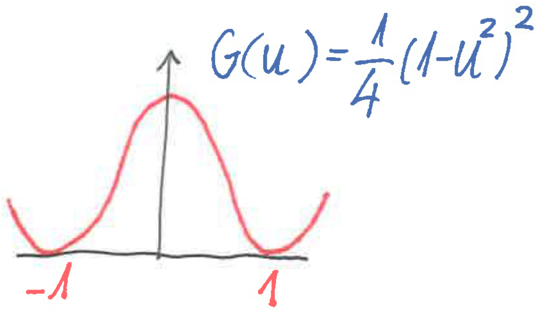

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$\exists$ u global minimizer of (AC) in $\mathbb{R}^9$, u not 1-D, with $u_{x_9} > 0$. Its level sets, when blown-down, converge to the Bombieri-De Giorgi-Giusti minimal graph $\mathbb{R}^8 \to \mathbb{R}$.

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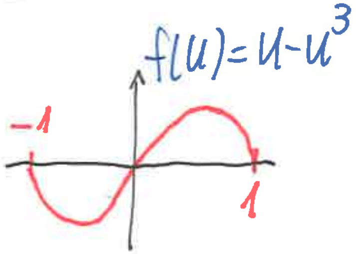

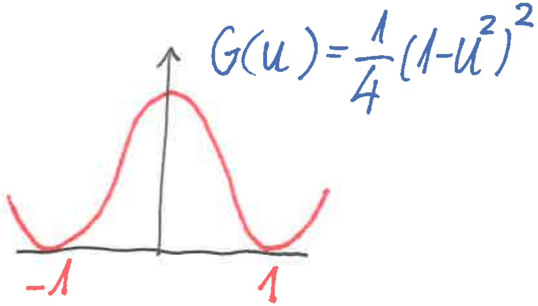

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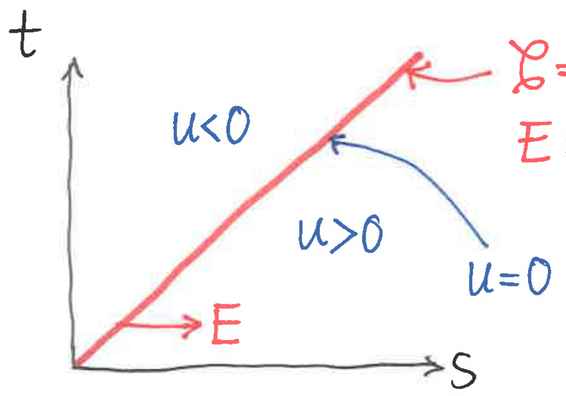
Open pb 1. $n = 8$? $\longrightarrow$ Saddle-shaped solution of (AC): is it a minimizer ?

Saddle-shaped sol'ns of (AC)



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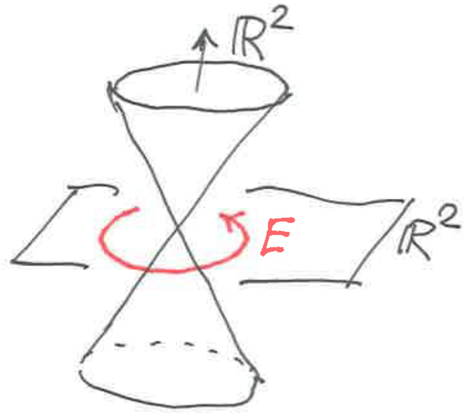

$$u = u(s,t) \qquad \begin{cases} |u| < 1 \\ u(t,s) = -u(s,t) \end{cases}$$

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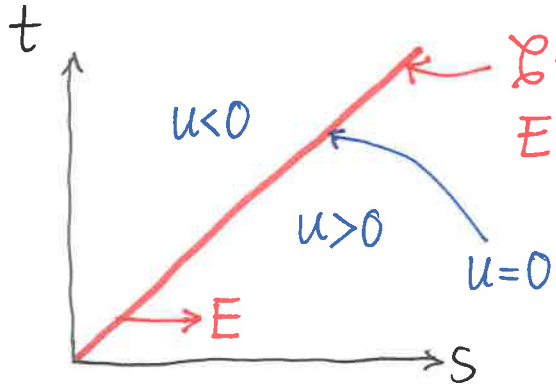

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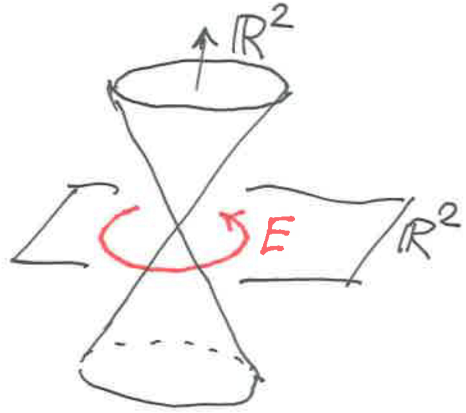

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$\exists$ saddle sol'n in $\mathbb{R}^{2m}$ $\forall m \geq 1$. It is **unstable** if $2m = \mathbf{2, 4, 6}$.

Its Morse index $= 1$ in $\mathbb{R}^2$ ← [Schatzman] (?)

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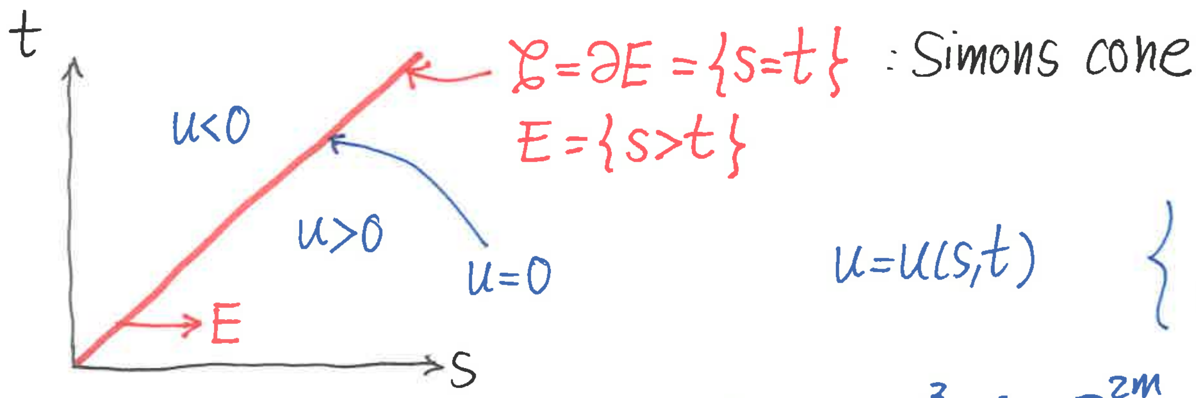

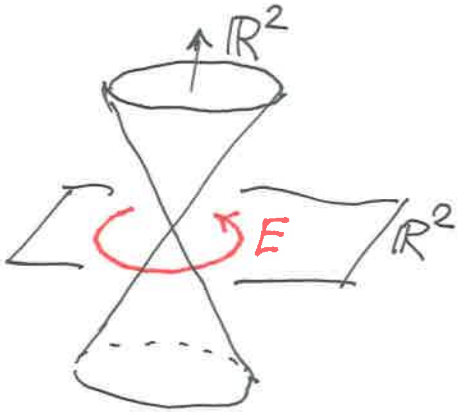

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Thm [C.'12] In $\mathbb{R}^{2m}$, $\forall 2m \geq 2$, the saddle solution is **unique**.

Thm [C'12] In $\mathbb{R}^{2m}$, the saddle sol'n is **stable** if $2m \geq 14$.

Open pb. 1. The saddle-shaped solution in $\mathbb{R}^{2m}$,
is it a minimizer for $2m \geq 8$?
(or, at least, for some $2m$ large enough)
Is it stable in dimensions $2m = 8, 10, 12$?

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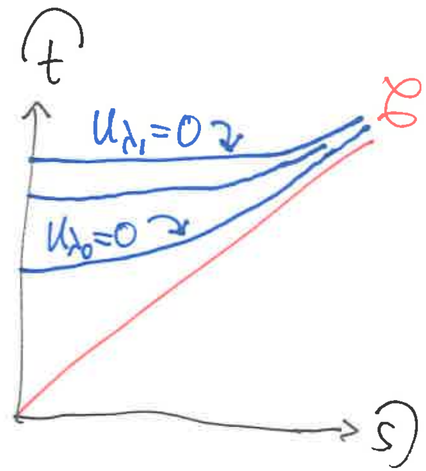
Thm [Pacard-Wei '13 & Yong Liu-Kelei Wang-Juncheng Wei '16]

In $\mathbb{R}^{2m}$, $2m \geq 8$, $\exists$ a family of solutions $\{u_\lambda\}_{\lambda \geq \lambda_0}$ of AC, $|u_\lambda| < 1$,
with $u_\lambda \longrightarrow 1$ as $\lambda \to +\infty$, $u_\lambda = u_\lambda(s,t)$, $\{u_\lambda = 0\}$ is **not**
a hyperplane and converges to the Simons cone $\mathscr{C}$ at ∞,

and $\begin{cases} u_\lambda \text{ is stable} & \text{[P-W '13]} \\ u_\lambda \text{ is a global minimizer} & \text{[L-W-W '16]} \end{cases}$



Open pb. Does this family include the saddle-shaped solution ?

A conjecture of De Giorgi '78

$-\Delta u = u - u^3$ in $\mathbb{R}^n$, $u_{x_n} > 0$ in $\mathbb{R}^n$.

Then $\{u = \lambda\}$ are hyperplanes $\forall \lambda$, at least if $n \leq 8$.

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True in dim. $\begin{cases} n=2 & \text{[Ghoussoub-Gui '98]} \\ n=3 & \text{[Ambrosio-C. '00]} \end{cases}$ > Previous work of [Berestycki-Caffarelli-Nirenberg '97]

Open pb. 2. $4 \leq n \leq 8$?

[Savin]: true if $n \leq 8$ and $u(x', x_n) \xrightarrow[x_n \to \pm\infty]{} \pm 1$.

$\Longrightarrow$ u minimizer

[AAC]

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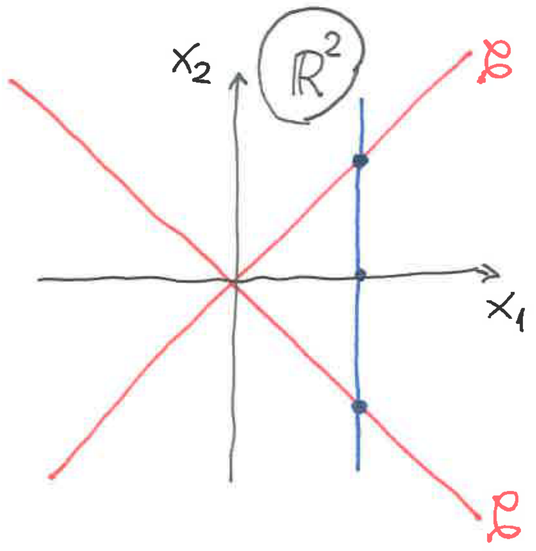
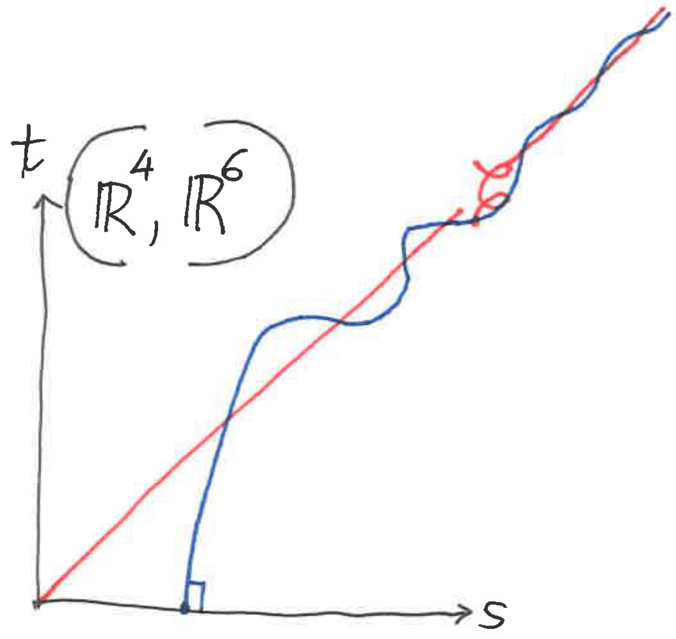
$\xrightarrow[\text{[AAC]}]{} u$ minimizer

Open pb. 3. $-\Delta u = u - u^3$ in $\mathbb{R}^n$, $|u| < 1$, u stable.

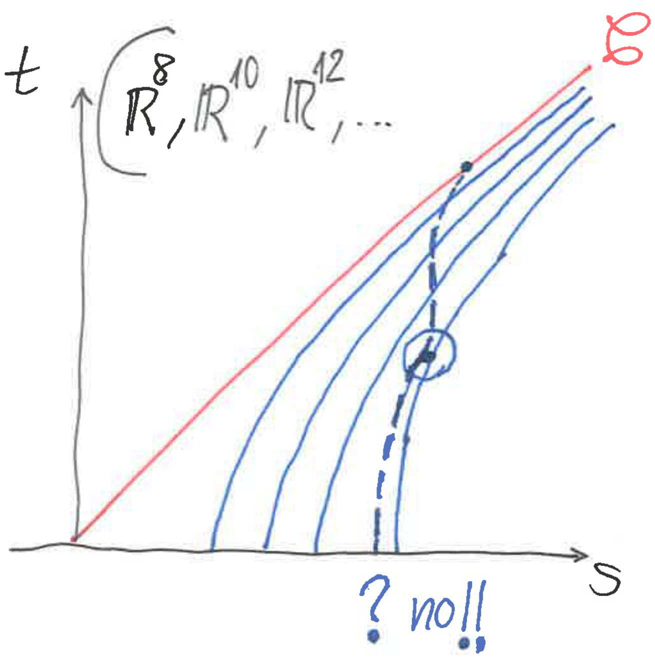
$\underline{n \leq 3} \Rightarrow \underline{u \text{ is 1d} \ ?}$ (or $n \leq 8$) $\longleftrightarrow$ stable minimal surfaces

Simons cone. Foliations.





Minimizers $\left\langle \begin{array}{c} \text{are not} \\ \text{are} \end{array} \right\rangle$ singular



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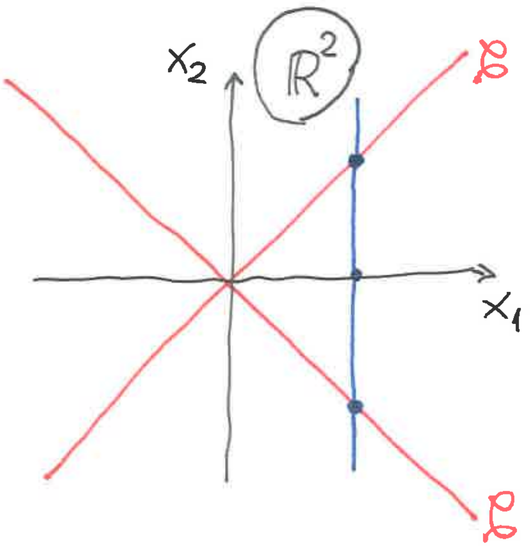

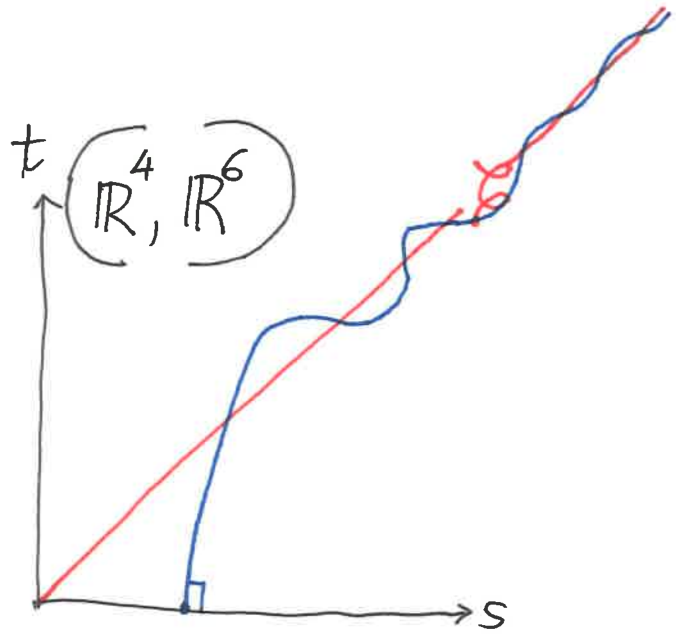

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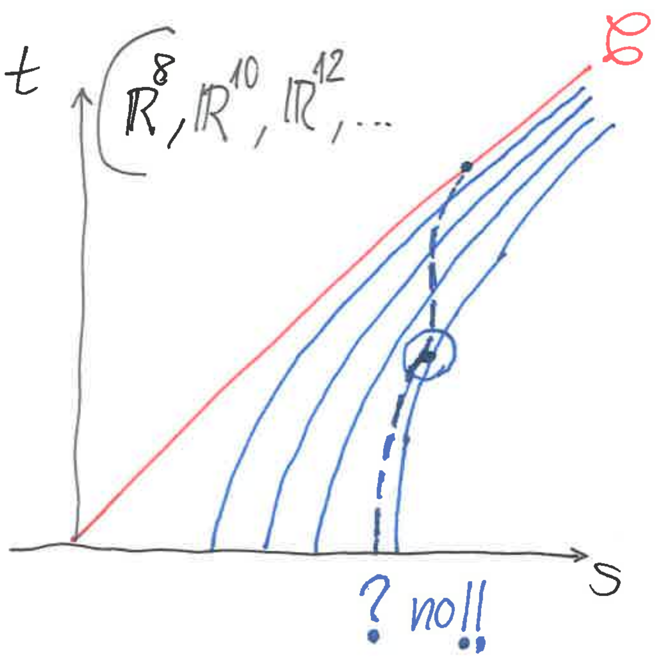


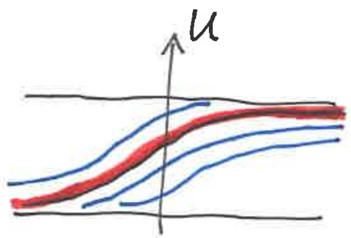
Foliation by stationary

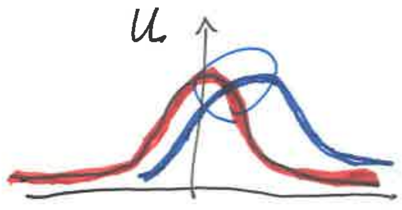
$\Downarrow$

Minimizers

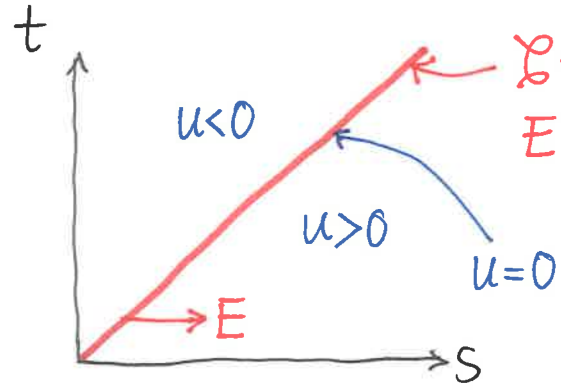
[Alberti-Ambrosio-Cabré '01]

[Weierstrass - Caratheodory]





Saddle-shaped sol'ns of (AC)



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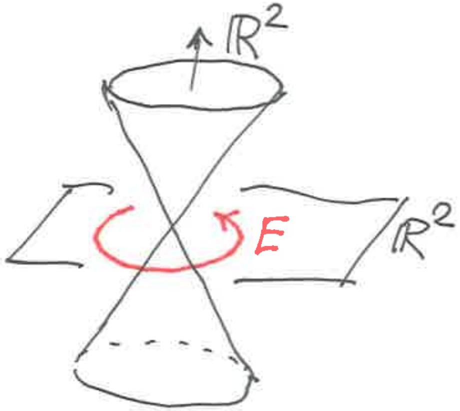

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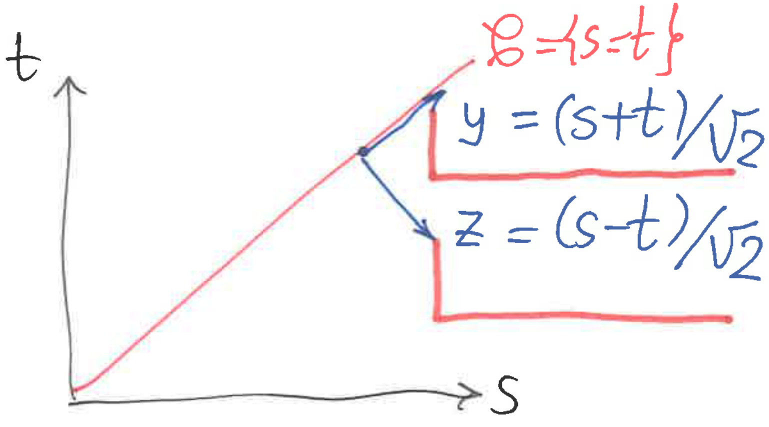
Instability in $\mathbb{R}^4$ & $\mathbb{R}^6$: [C-Terra '09, '10]

(in $\mathbb{R}^2$: [Dang-Fife-Peletier '92] [Schatzman '95])



(AC):

$$u_{yy}+u_{zz}+\frac{2(m-1)}{y^2-z^2}(yu_y-zu_z)+f(u)=0$$

$$\Downarrow$$

$$0=\{\Delta_{2m}+f'(u)\}u_z-\frac{2(m-1)}{y^2-z^2}u_z+\frac{4(m-1)z}{(y^2-z^2)^2}(yu_y-zu_z).$$

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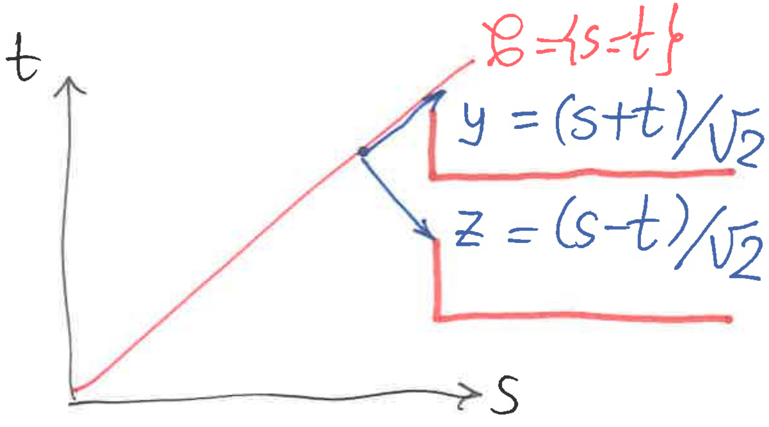


(AC):

$$u_{yy}+u_{zz}+\frac{2(m-1)}{y^2-z^2}\left(y u_y - z u_z\right)+f(u)=0$$

$$\Downarrow$$

$$0=\left\{\Delta_{2m}+f'(u)\right\} u_z-\frac{2(m-1)}{y^2-z^2}\, u_z+\frac{4(m-1)z}{(y^2-z^2)^2}\left(y u_y - z u_z\right).$$

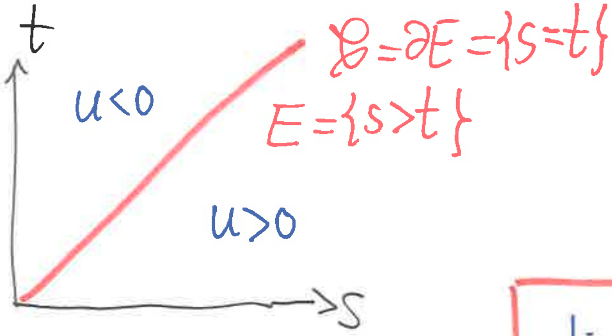
$$\Downarrow$$

$$\boxed{D^2E(u)(\xi,\xi)=\int_{\mathbb{R}^{2m}} |\nabla\xi|^2-f'(u)\,\xi^2<0} \quad \text{for}$$

$\xi(y,z)=\eta\left(\frac{y}{a}\right) u_z(y,z)$

& let $a\to+\infty$: HARDY ineq.

Saddle-shaped soln's to (AC)



$u = u(s,t)$

$$u_{ss} + u_{tt} + (m-1)\left\{ \frac{u_s}{s} + \frac{u_t}{t} \right\} + u - u^3 = 0$$

$$\boxed{\operatorname{dist}_{\mathbb{R}^{2m}}(x, \mathscr{C}) = \frac{s-t}{\sqrt{2}}}$$

+

LIOUVILLE THMS in $\mathbb{R}^{2m}$ & $\mathbb{R}^{2m}_+$ for (AC)

Asymptotic behaviour at ∞:

Thm [C.-Terra '10] Let $\boxed{V(x) := u_0\left(\frac{s-t}{\sqrt{2}}\right) = \tanh\left(\frac{s-t}{2}\right)}$.

u saddle sol'n in $\mathbb{R}^{2m}$, $\forall m \Longrightarrow$

$$\| \, |u - V| + |\nabla u - \nabla V| \, \|_{L^\infty(\mathbb{R}^{2m} \setminus B_R(0))} \longrightarrow 0 \text{ as } R \to \infty.$$

Towards uniqueness in $\mathbb{R}^{2m}$ & stability in $\mathbb{R}^{14}$.

Prop'n [C.'12] u saddle sol'n in $\mathbb{R}^{2m}$ $\Longrightarrow$

$L_u := \Delta + f'(u(x))$ satisfies the maximum principle in $\mathcal{O} = \{s > t\}$.

(i.e., $L_u v \geq 0$ in $\mathcal{O}$, $v \leq 0$ on $\partial\mathcal{O}$ & $\limsup_{x \in \mathcal{O}, |x| \to \infty} v(x) \leq 0$

$\Longrightarrow v \leq 0$ in $\mathcal{O}$)

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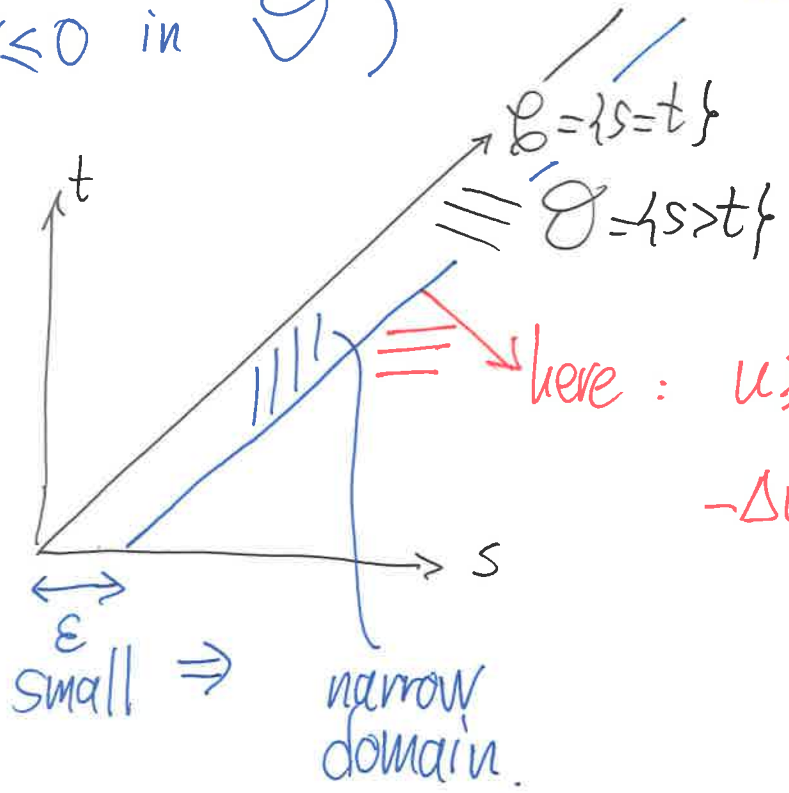
Proof. uses



here : $u \geq \delta > 0$ &

$-\Delta u = f(u) \geq f'(u)\,u$

$-L_u u \geq 0$

(supersol'n, $\geq \delta > 0$) $\square$

Maximum principle in $\mathcal{O}$ for L_u

$\oplus$

Asymptotics of saddle sol'ns at ∞

$\oplus$

$\exists$ of smallest saddle in $\mathcal{O}$

$\Rightarrow$

Thm [C'12] (Uniqueness) The saddle sol'n in $\mathbb{R}^{2m}$ is unique, $\forall 2m \geq 2$.

Pf. $\underline{u} \leq u$ in $\mathcal{O}$

$\uparrow$ smallest saddle in $\mathcal{O}$

$\searrow$ $-\Delta(u-\underline{u}) = f(u) - f(\underline{u}) \leq f'(u)(u-\underline{u})$ in $\mathcal{O}$

$\Downarrow$ Asymptotics + Max. Pr.

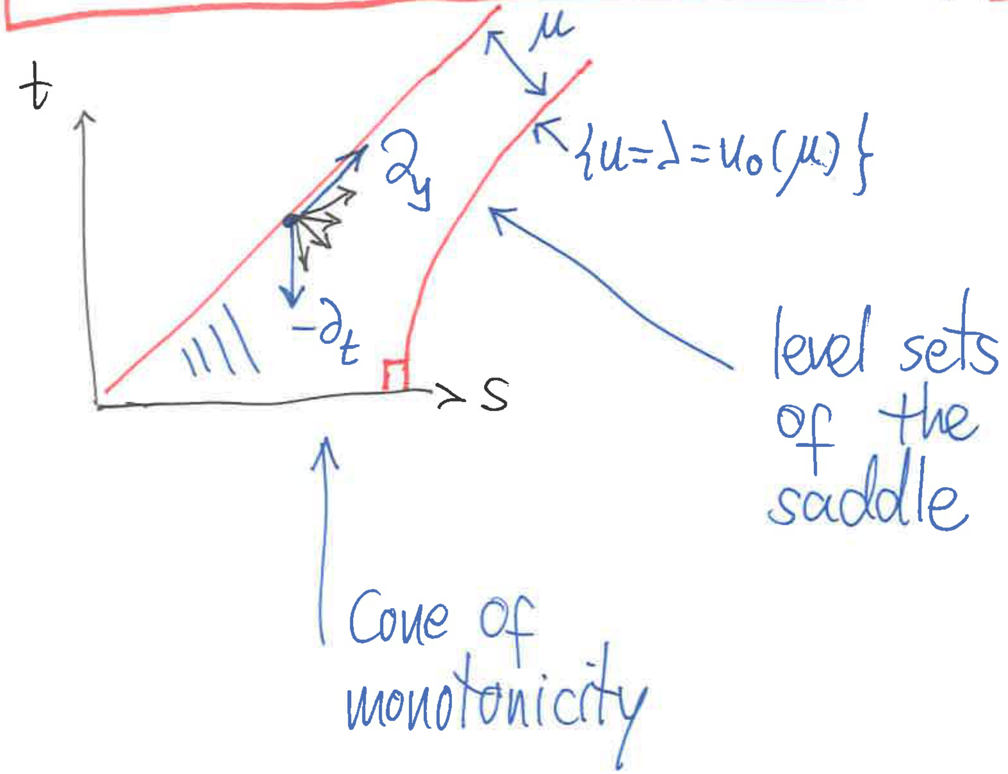
$u - \underline{u} \leq 0$ in $\mathcal{O}$. $\square$

Maximum principle in $\mathcal{O}$ $\oplus$ Asymptotics at ∞ $\Downarrow$

$\hookrightarrow$ Monotonicity & convexity properties of saddles.

Thm [C'12] u saddle sol'n in $\mathbb{R}^{2m}$, $2m \geq 2$. Then:

in $\mathcal{O} \setminus \{t=0\} = \{s>t>0\}$: $u_y > 0$, $-u_t > 0$, $u_{st} > 0$.



Maximum principle in $\mathcal{O}$ ⊕ Asymptotics at ∞ $\Downarrow$

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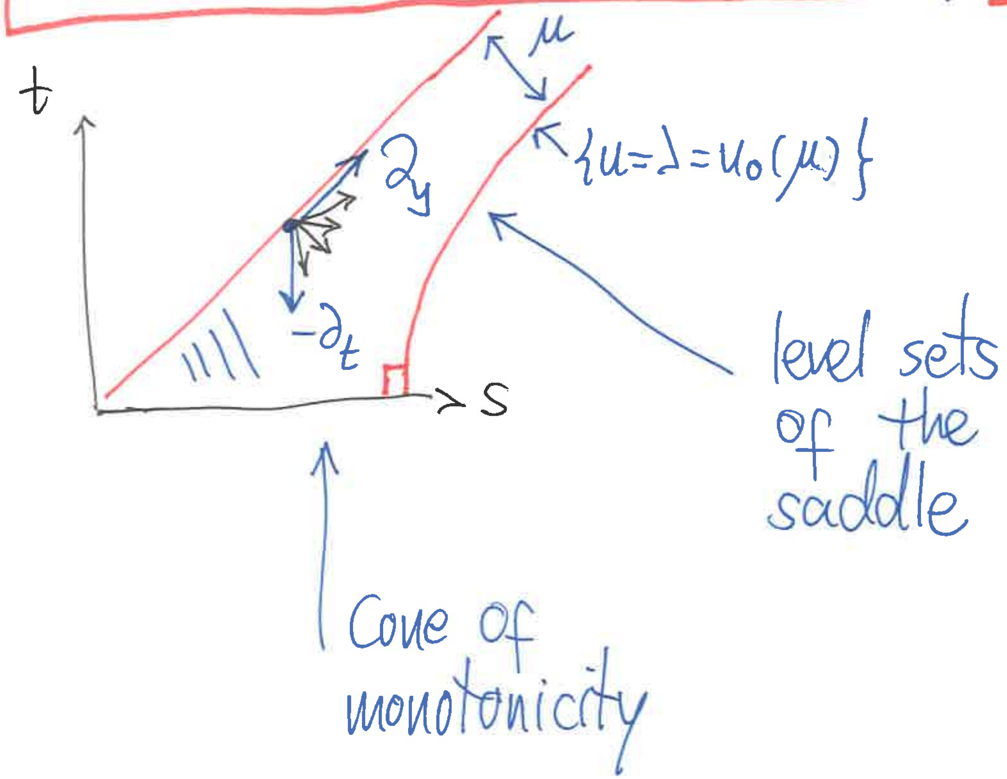

Pf: MPrinciple ⊕ asympt. ∞

⊕

$$\{\Delta + f'(u)\} u_y = \frac{m-1}{s^2} u_y + \frac{(m-1)(s^2-t^2)}{\sqrt{2}\, s^2 t^2} u_t$$

$$\{\Delta + f'(u)\} u_t - \frac{m-1}{t^2} u_t = 0$$

$$\{\Delta + f'(u)\} u_{st} - (m-1)\left(\frac{1}{s^2} + \frac{1}{t^2}\right) u_{st} < 0$$

$\square$

Thm [C'12] (stability in $\mathbb{R}^{14}$, $\mathbb{R}^{2m}$ for $2m \geq 14$.)

$2m \geq 14 \iff \exists b \in \mathbb{R}$ s.t. $b(b-m+2)+m-1 \leq 0$. Then:

$\varphi = \varphi(s,t) := t^{-b} u_s - s^{-b} u_t$ $(b>0)$

satisfies

$$\left.\begin{aligned} \varphi &> 0 \\ \{\Delta + f'(u)\}\varphi &\leq 0 \end{aligned}\right\} \text{ in } \mathbb{R}^{2m} \setminus \{st=0\}.$$

$\Longrightarrow$ Stability of the saddle in $\mathbb{R}^{2m}$, $2m \geq 14$.

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$\Longrightarrow$ Stability of the saddle in $\mathbb{R}^{2m}$, $2m \geq 14$.

Pf: φ is even w.r.t. $\mathcal{C}$ $\Longrightarrow$

$\varphi > 0$ in $\mathcal{O}$ $\quad (\Rightarrow$ in $\mathbb{R}^{2m} \setminus \{st=0\})$

$$\Delta u_s + f'(u) u_s - \frac{m-1}{s^2} u_s = 0 \quad ; \quad \Delta u_t + f'(u) u_t - \frac{m-1}{t^2} u_t = 0$$

$$\Delta t^{-b} = b(b-m+2)\, t^{-b-2} \quad ; \quad \Delta s^{-b} = b(b-m+2)\, s^{-b-2}$$

$$\varphi = t^{-b} u_s - s^{-b} u_t$$

$$\longrightarrow \underline{\Delta\varphi + f'(u)\varphi} = u_s\, t^{-b}\left\{(m-1)s^{-2} + b(b-m+2)t^{-2}\right\}$$
$$+ (-u_t)\, s^{-b}\left\{(m-1)t^{-2} + b(b-m+2)s^{-2}\right\}$$
$$+ u_{st}\, 2b\left\{s^{-b-1} - t^{-b-1}\right\}$$

$u_{st} > 0$ in $\mathcal{O}$

$$\le u_s\, t^{-b}\left\{(m-1)s^{-2} + b(b-m+2)t^{-2}\right\}$$
$$+ (-u_t)\, s^{-b}\left\{(m-1)t^{-2} + b(b-m+2)s^{-2}\right\} \quad \text{in } \mathcal{O}$$

$$\downarrow$$

$$
\begin{aligned}
\{\Delta + f'(u)\}\varphi \;&\le\; t^{-b}(u_s + u_t)\{(m-1)s^{-2} + b(b-m+2)t^{-2}\} \\
&\quad -s^{-b}u_t\{(m-1)t^{-2} + b(b-m+2)s^{-2}\} \\
&\quad -t^{-b}u_t\{(m-1)s^{-2} + b(b-m+2)t^{-2}\} \\
&= u_y\sqrt{2}t^{-b}\{(m-1)s^{-2} + b(b-m+2)t^{-2}\} \\
&\quad +(-u_t)\,(m-1)(s^{-b}t^{-2} + t^{-b}s^{-2}) \\
&\quad +(-u_t)\,b(b-m+2)(s^{-2-b} + t^{-2-b}) \\
&\le u_y\sqrt{2}t^{-b}(m-1)\{s^{-2} - t^{-2}\} \\
&\quad +(-u_t)\,(m-1)(s^{-b}t^{-2} + t^{-b}s^{-2} - s^{-2-b} - t^{-2-b}) \\
&\le (-u_t)\,(m-1)(s^{-b}t^{-2} + t^{-b}s^{-2} - s^{-2-b} - t^{-2-b})
\end{aligned}
$$