A priori Lipschitz estimates for unbounded solutions of local and nonlocal Hamilton-Jacobi viscous equations with Ornstein-Uhlenbeck Operator

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Mostly Maximum Principle
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Statement of the problem : Equation (HJ)

$$\boxed{\lambda u^\lambda - \mathcal{F}(x,[u^\lambda]) + \langle b(x), Du^\lambda\rangle + H(x,Du^\lambda) = f(x), \quad x \in \mathbb{R}^N}$$

$\lambda > 0$

b Ornstein-Uhlenbeck drift : $\langle b(x)-b(y), x-y\rangle \geq \alpha |x-y|^2$

H sublinear Hamiltonian : $|H(x,p)| \leq C_H(1+|p|)$
(no further assumptions)

Higher order term either :
local : $\mathcal{F}(x,[u]) = \mathrm{tr}(A(x)D^2u)$, $A = \sigma(x)\sigma(x)^T$
nonlocal : $\mathcal{F}(x,[u])$ integro-differential

$x \in \mathbb{R}^N$ unbounded set
$u^\lambda(x)$ unbounded solution

Goal

For every continuous solution u^λ of (HJ),

$$\boxed{|u^\lambda(x) - u^\lambda(y)| \leq C(\phi_\mu(x) + \phi_\mu(y))|x - y|} \quad \text{(Lip)}$$

when $f \in C(\mathbb{R}^N)$ satisfies the same condition.

- ϕ_μ growth function (needed in this unbounded setting)

- We look for $C > 0$, ϕ_μ **independent** of λ and solution u^λ

➪ Applications to existence & uniqueness of solutions, ergodic problems, large time behavior of solutions
[T.T.Nguyen 2017]

Some References

Extensive literature in bounded domains/periodic domains/for bounded solutions

Local case :
Bernstein method : [Gilbarg-Trudinger],[Barles 1991], [Lions-Souganidis 2005], [Capuzzo Dolcetta-Leoni-Porretta 2010],...
Elliptic equations : [Ishii-Lions 1990], [Barles-Souganidis 2001],...

Nonlocal case : [Barles-Chasseigne-Ciomaga-Imbert 2012], [Barles-Topp 2016], [Barles-OL-Topp 2017]

Unbounded setting (local, with Ornstein-Uhlenbeck op.) : [Fujita-Ishii-Loreti 2006], [Fujita-Loreti 2009], [Ghilli 2016]

[Fujita-Ishii-Loreti 2006]

$$\lambda u^\lambda - \Delta u + \alpha\langle x, Du^\lambda\rangle + H(Du^\lambda) = f(x), \quad x \in \mathbb{R}^N$$

(Lip) with $\phi_\mu(x) = e^{\mu|x|^2}$, $\mu < \alpha$,
for every continuous solution u^λ in

$$\mathcal{E}_\mu = \left\{ g : \mathbb{R}^N \to \mathbb{R} : \lim_{|x|\to+\infty} \frac{g(x)}{\phi_\mu(x)} = 0 \right\}, \qquad (1)$$

- "Pure" Laplacian case
- $H(Du)$ Lipschitz continuous, independent of x
- Natural growth condition
- Ellipticity not really needeed

Assumptions : datas

$|H(x,p)| \leq C_H(1+|p|)$

➪ H merely sublinear without further assumptions, depends on x

Assumptions on $\mathcal{F}(x,[u])$:

local case : $\mathcal{F}(x,[u]) = \mathrm{tr}(A(x)D^2u)$,
$A = \sigma(x)\sigma(x)^T$, $\sigma \in W^{1,\infty}(\mathbb{R}^N; \mathcal{M}_N)$

➪ general diffusion, brings "bad" first-order terms

nonlocal case :

$$\mathcal{F}(x,[u]) = \int_{\mathbb{R}^N} \{u(x+z) - u(x) - \langle Du(x), z\rangle \mathbf{1}_B(z)\}\nu(dz)$$

with $\displaystyle\int_B |z|^2\nu(dz), \ \int_{B^c} \phi_\mu(z)\nu(dz) \ \leq C_\nu.$

➪ well-defined for $u \in \mathcal{E}_\mu$ which is C^2 near x

Typical example : $\nu(dz) = \frac{e^{-\mu|z|}}{|z|^{N+\beta}}dz$, $\beta \in (0,2)$ (fract. Laplacian type)

Assumptions : growth

$$\mathcal{E}_\mu = \left\{ g : \mathbb{R}^N \to \mathbb{R} : \lim_{|x|\to+\infty} \frac{g(x)}{\phi_\mu(x)} = 0 \right\},$$

with

$$\phi_\mu(x) = e^{\mu\sqrt{1+|x|^2}} \quad \text{for all } \mu > 0$$

We assume that :

- $f \in \mathcal{E}_\mu$ satisfies (Lip) : $|f(x) - f(y)| \leq C(\phi_\mu(x) + \phi_\mu(y))|x - y|$
- we consider solutions $u^\lambda \in \mathcal{E}_\mu$

➪ restriction of growth comparing to [Fujita-Ishii-Loreti 2006] due to bad nonlinearities coming from H and $\mathcal{F}$. we do not know if it is optimal.

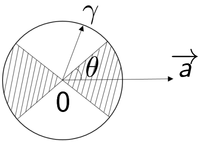
Assumptions : nondegeneracy of the equation

local case, nondegenerate : $A \geq \rho I$ for some $\rho > 0$
➪ classical ellipticity

nonlocal case : There exists $\beta \in (0,2)$ such that
for every $a \in \mathbb{R}^N$ there exist $0 < \eta < 1$ and $C_\nu > 0$ such that,

for all $\gamma > 0$, $\displaystyle\int_{\mathcal{C}_{\eta,\gamma}(a)} |z|^2 \nu(dz) \geq C_\nu \eta^{\frac{N-1}{2}} \gamma^{2-\beta}$

cone $\mathcal{C}_{\eta,\gamma}(a)$
aperture $\cos\theta = 1 - \eta$



nondegenerate : $\boxed{\beta \in (1,2)}$ ➪ kind of ellipticity condition
[Barles-Chasseigne-Ciomaga-Imbert 2012]

Results : nondegenerate equations

Theorem. For any $\mu, \alpha > 0$.
Let $u^\lambda \in C(\mathbb{R}^N) \cap \mathcal{E}_\mu$ be a solution of (HJ).
Under the previous assumptions, If

(i) $\mathcal{F}(x, [u^\lambda]) = \mathrm{tr}(A(x)D^2u^\lambda(x))$ with classical ellipticity condition,

or

(ii) $\mathcal{F}(x, [u^\lambda])$ is integro-differential with $\beta \in (1, 2)$,

then (Lip) holds with C, ϕ_μ independent of λ

Comments

- extension to the parabolic case

$$\begin{cases} \frac{\partial u}{\partial t} - \mathcal{F}(x,[u]) + \langle b(x), Du\rangle + H(x,Du) = f(x), \ (x,t) \in \mathbb{R}^N \times (0,+\infty) \\ u(\cdot,0) = u_0 \in \mathcal{E}_\mu \text{ satisfying (Lip)} \end{cases}$$

- Results we were interested in :

– nondegenerate equations since one needs a strong maximum principle for large time behavior

– in the unbounded setting, it generalizes the fact that ellipticity must control first-order nonlinearities at least with linear growth [Ishii-Lions 1990], [Barles-Souganidis 2001]

- Lipschitz estimates are independent of λ in the stationary case and independent of t in the parabolic case.

Results : degenerate equations

Theorem. For any $\mu > 0$.
Let $u^\lambda \in C(\mathbb{R}^N) \cap \mathcal{E}_\mu$ be a solution of (HJ).
Under the previous assumptions, assume that H satisfies

$$|\frac{\partial H}{\partial x}| \leq C(1+|p|), \qquad |\frac{\partial H}{\partial p}| \leq C(1+|x|).$$

If

(i) $\mathcal{F}(x,[u^\lambda]) = \mathrm{tr}(A(x)D^2u^\lambda(x))$ possibly degenerate

or

(ii) $\mathcal{F}(x,[u^\lambda])$ is integro-differential with $\beta \in (0,1)$,

then there exists $\alpha > 0$ such that (Lip) holds with C, ϕ_μ independent of λ.

Same result in the parabolic case.

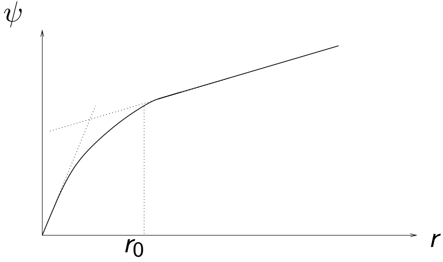
Rough idea of the proof in the nondegenerate case

Goal : Prove that

$$M := \max_{x,y\in\mathbb{R}^N} \{u^\lambda(x) - u^\lambda(y) - \varphi(x,y)\},$$

$$\text{with } \varphi(x,y) = \psi(|x-y|)(\phi_\mu(x) + \phi_\mu(y)),$$

is nonpositive for some suitable concave function ψ



This implies easily (Lip).

By contradiction, assume $M > 0$ and achieved at $(\bar{x}, \bar{y})$.

Proof in the nondegenerate case $r := |\bar{x} - \bar{y}|$, $\varphi(\bar{x}, \bar{y}) = \psi(r)\Phi$, $\Phi = \phi_\mu(\bar{x}) + \phi_\mu(\bar{y})$

Using the equation

$$\begin{aligned}
& \lambda(u^\lambda(\bar{x}) - u^\lambda(\bar{y})) - \Big(\mathcal{F}(\bar{x}, [u^\lambda]) - \mathcal{F}(\bar{y}, [u^\lambda])\Big) \\
& + (\langle b(\bar{x}), D_x\varphi\rangle - \langle b(\bar{y}), -D_y\varphi\rangle) \\
\leq \quad & -H(\bar{x}, D_x\varphi) + H(\bar{y}, -D_y\varphi) + f(\bar{x}) - f(\bar{y}) \\
\leq \quad & C(1 + \psi'(r) + r)\Phi,
\end{aligned}$$

Estimate of Ornstein-Uhlenbeck term :
$\langle b(\bar{x}), D_x\varphi\rangle - \langle b(\bar{y}), -D_y\varphi\rangle \geq$
$\quad \alpha r\psi'(r)\Phi + \psi(r)\left(\langle b(\bar{x}), D\phi_\mu(\bar{x})\rangle + \langle b(\bar{y}), D\phi_\mu(\bar{y})\rangle\right)$

Proof in the nondegenerate case : local case

Estimate of $-\mathcal{F}(\bar{x},[u^\lambda])+\mathcal{F}(\bar{y},[u^\lambda])$, local case

Ishii-Lions' method :

$$-\mathrm{tr}(A(\bar{x})X)+\mathrm{tr}(A(\bar{y})Y)\geq$$
$$-\rho\psi''(r)\Phi-\mathrm{tr}(A(\bar{x})D^2\phi_\mu(\bar{x}))-\mathrm{tr}(A(\bar{y})D^2\phi_\mu(\bar{y}))$$

Global estimate :

$$(\alpha r\psi'(r)-\rho\psi''(r))\Phi+\mathcal{L}[\phi_\mu](\bar{x})+\mathcal{L}[\phi_\mu](\bar{y})\leq C(1+\psi'(r)+r)\Phi$$

with $\mathcal{L}[\phi_\mu]=-\mathrm{tr}(AD^2\phi_\mu)+\langle b,D\phi_\mu\rangle-C|D\phi_\mu|\geq\phi_\mu-K$ by fundamental property of Ornstein-Uhlenbeck operator.

Contradiction :

- when $0<r<r_0$: $-\psi''(r)\geq C\psi'(r)$ by a suitable choice of $\psi(r)=1-e^{-Lr}$. "ellipticity range", strict concavity of ψ
- when $r\geq r_0$ big enough : $\alpha r\psi'(r)\geq Cr, C\psi'(r)$, action of Ornstein-Uhlenbeck operator where ψ is linear.

Proof in the nondegenerate case : nonlocal case

Same idea in the **nonlocal case** : use of the strict ellipticity when r small and the Ornstein-Uhlenbeck operator when r is big.

Estimate of $-\mathcal{F}(\bar{x}, [u^\lambda]) + \mathcal{F}(\bar{y}, [u^\lambda])$ more complicated : relies on the "ellipticity" of the integro-differential operator.
Adaptation of ideas of Barles-Chasseigne-Ciomaga-Imbert to generalize Ishii-Lions' method in the nonlocal framework.
Use of different concave functions ψ.
Need of a first step to prove Hölder continuity and to improve Hölder contnuity to Lipschitz continuity.