Optimal principal eigenfunction for elliptic operators with large drift

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Joint work with F. Hamel, E. Russ

Plan of the talk

- Isoperimetric problem
  - Faber-Krahn inequality
  - Adding a drift

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- Asymptotics of the optimal principal eigenfunction
  - The conjectures
  - The (partial) answers

Isoperimetric problem for the principal eigenvalue

Let λ_Ω denote the principal eigenvalue of $-\Delta$ in a bounded domain Ω, i.e.,

$$\begin{cases} -\Delta\varphi = \lambda_\Omega\varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \\ \varphi > 0 & \text{in } \Omega. \end{cases}$$

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Answer (Faber, Krahn 1920s): $\Omega =$ the ball.

Tool: Schwarz symmetrization.

Adding a drift $v \in L^\infty(\Omega)$. Let $\lambda_{\Omega,v}$ denote the principal eigenvalue:

$$\begin{cases} -\Delta\varphi - v \cdot \nabla\varphi = \lambda_{\Omega,v}\varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \\ \varphi > 0 & \text{in } \Omega. \end{cases}$$

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Answer (Hamel-Nadirashvili-Russ, Ann. of Math. 2011):

$$\Omega = \text{ the ball}, \qquad v(x) = -\tau\frac{x}{|x|}.$$

Tool: new type of symmetrization.

Optimization in a fixed domain

Let Ω be a given bounded smooth domain.
For $v \in L^\infty(\Omega)$, let λ_v be the principal eigenvalue:

$$\begin{cases} -\Delta\varphi - v \cdot \nabla\varphi = \lambda_v \varphi & \text{in } \Omega \\ \varphi = 0 & \text{on } \partial\Omega \\ \varphi > 0 & \text{in } \Omega. \end{cases}$$

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Theorem (Hamel-Nadirashvili-Russ)

The infimum is achieved by a unique $\underline{v}$. Furthermore

$$\underline{v} = \tau \frac{\nabla\varphi_\tau}{|\nabla\varphi_\tau|} \qquad \textit{whenever } \nabla\varphi_\tau \neq 0,$$

where φ_τ is the associated principal eigenfunction.

Principal eigenvalue for nonlinear operators

$\varphi_\tau \in C^2(\Omega)$ satisfies the nonlinear eigenvalue problem

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1-homogeneous $\implies$ Berestycki-Nirenberg-Varadhan approach:

$$\lambda_{gen} = \sup\{\lambda \ : \ (-\Delta - \tau|\nabla| - \lambda I) \text{ admits a positive supersolution}\}.$$

- Pucci (Felmer-Quaas)
- 1-homogeneous (Quaas-Sirakov)
- p and ∞ Laplacian (Kawohl-Lindqvist, Birindelli-Demengel, Juutinen)
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$$\lambda(\tau) = \max\{\lambda \ : \exists\varphi > 0, \ -\Delta\varphi - \tau|\nabla\varphi| \geq \lambda\varphi \ \textit{in } \Omega\}$$

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$$= \min\{\lambda \ : \exists\varphi > 0,\ -\Delta\varphi - \tau|\nabla\varphi| \leq \lambda\varphi \text{ in } \Omega,\ \varphi = 0 \text{ on } \partial\Omega\}.$$

Furthermore, the above extrema are attained only by (a multiple of) φ_τ.

Asymptotics as $\tau \to +\infty$

Theorem (Hamel-Nadirashvili-Russ)

$$e^{-R\tau} \leq \lambda(\tau) \leq e^{-r\tau}, \qquad B_r(x_1) \subset \Omega \subset B_R(x_2).$$

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Let $(x_\tau)_\tau$ be maximal points for $(\varphi_\tau)_\tau$. Then

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Conjecture 2 (—)

$$\forall x \notin \mathcal{C}, \quad \frac{\nabla\varphi_\tau(x)}{|\nabla\varphi_\tau(x)|} \to \nabla d(x) \quad \text{as} \;\; \tau \to +\infty.$$

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$$\lim_{\tau\to+\infty} \frac{\varphi_\tau(x)}{\|\varphi_\tau\|_\infty(1 - e^{-\tau d(x)})} = 1, \qquad \text{uniformly w.r.t. } x \in \Omega.$$

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$$-\Psi'' - (\tau + \varepsilon)\Psi' \geq \lambda(\tau)\Psi \qquad \text{in } (0,r)$$

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$$\psi := \Psi \circ d$$

$$-\Delta\psi - \tau|\nabla\psi| \geq \lambda(\tau)\psi + (\varepsilon\tau - \Delta d)|\nabla\psi| \qquad \text{for } d < r.$$

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Lemma

$$\sup_{\Omega} \Delta d < \infty.$$

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Blow-up around x_τ:

$$\psi_\tau(x) := \varphi_\tau\left(x_\tau + \frac{x}{\tau}\right), \qquad -\Delta\psi_\tau - |\nabla\psi_\tau| = \frac{\lambda(\tau)}{\tau^2}\psi_\tau.$$

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Lemma

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$$\forall\, 2\frac{N-1}{\tau} \le r \le r' \le R, \qquad \min_{\partial B_{r'}(x_0)} \varphi_\tau \ge \frac{1-\varepsilon r'}{1-\varepsilon r} \min_{\partial B_r(x_0)} \varphi_\tau.$$

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Covering argument.