ETH zürich

α-Harmonicity in Sub-Riemannian Geometry

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α-Harmonic Maps

They are critical points of the nonlocal energy

$$\mathcal{L}^{\alpha}(u) = \int_{\mathbb{R}^k} |(-\Delta)^{\frac{\alpha}{2}} u(x)|^2 dx^k \,, \tag{1}$$

where $u \in \dot{H}^{\alpha}(\mathbb{R}^k, \mathcal{N})$, $\mathcal{N} \subset \mathbb{R}^m$ is an at least C^2 closed (compact without boundary) n-dimensional manifold, and

$$\widehat{(-\Delta)^{\frac{\alpha}{2}} u} = |\xi|^{\alpha} \hat{u}.$$

Case of Harmonic maps into a sub-manifold $\mathcal{N}$ of $\mathbb{R}^m$

Critical points $u \in W^{1,2}(B^k, \mathcal{N})$ of

$$\mathcal{L}(u) = \int_{B^k} |\nabla u|^2 dx^k \tag{2}$$

satisfy under the pointwise constraint $u(x) \in \mathcal{N}$

$$-\Delta u = A(u)(\nabla u, \nabla u) \quad \Longleftrightarrow \quad P_T(u)\, \Delta u = 0,$$

$A(z)(X, Y)$, 2^{nd} fundamental form of $\mathcal{N}$ at $z \in \mathcal{N}$ along $(X, Y) \in (T_z\mathcal{N})^2$, $P_T(z)$ is the orthogonal matrix projection onto $T_z\mathcal{N}$.

Horizontal Harmonic Maps

A Polarization of $\mathbb{R}^m$ is a C^1 field of orthogonal projections

$$\begin{cases} P_T \in C^1(\mathbb{R}^m, M_m(\mathbb{R})) \quad \text{s.t.} \quad P_T \circ P_T = P_T \\ P_N := Id - P_T \quad \text{and} \quad \forall z,\ X,\ Y \quad \langle P_T(z)X, P_N(z)Y\rangle = 0 \\ \|\partial_z P_T\|_{L^\infty} < +\infty, \ \ \text{rank}(P_T) = n \end{cases}$$

Horizontal Harmonic Maps in $W^{1,2}(B^k, \mathbb{R}^m)$

$$\left.\begin{array}{l} P_T(u)\,\Delta u = 0 \\ P_N(u)\,\nabla u = 0 \end{array}\right. \quad \text{in} \quad \mathcal{D}'(B^k)$$

If P_T is integrable, namely $P_N[P_T X, P_T Y] = 0 \;\Rightarrow\;$ back to harmonic maps.

Horizontal Harmonic Maps satisfy:

$$\begin{aligned} 0 = P_T(u)\Delta u &= \operatorname{div}(P_T(u)\nabla u) - \nabla P_T(u)\nabla u \\ &= \operatorname{div}(\nabla u) - \nabla P_T(u)\nabla u \\ &= \Delta u + [\nabla P_N P_T - \underbrace{(\nabla P_N P_T)^t]\nabla u}_{=0} \end{aligned}$$

Therefore:

$$-\Delta u = \Omega\,\nabla u$$

with $\Omega := (\nabla P_N\, P_T - P_T\, \nabla P_N)$ and $\Omega^t = -\Omega$.

If $k = 2 \quad \Rightarrow \quad$ Regularity for Systems with Antisymmetric Potentials in 2-D [Rivière, 2005]

- Locally there exists $A \in W^{1,2} \cap L^\infty$ invertible such that

$$\mathrm{div}\,(\nabla A - A\,\Omega) = 0.$$

- Poincaré Lemma $\Rightarrow$ there is $B \in W^{1,2}$ such that

$$\nabla^\perp B := \nabla A - A\,\Omega$$

$$\Delta u + \Omega \cdot \nabla u = 0 \qquad \Longleftrightarrow \qquad \mathrm{div}\left(A\,\nabla u + B\,\nabla^\perp u\right) = 0$$

Thus [Coifman, Lions, Meyer, Semmes, 1989]

$$\operatorname{div}(A\nabla u) = \nabla^{\perp}B\cdot\nabla u \quad \in \mathcal{H}^1 \quad \text{Hardy space}$$

But

$$\operatorname{curl}(A\nabla u) = \nabla^{\perp}A\cdot\nabla u \quad \in \mathcal{H}^1 \quad \text{Hardy space}$$

Hence [Fefferman,Stein, 1972]

$$A\nabla u \in W^{1,1} \quad \Rightarrow \quad A\nabla u \in L^{2,1} \quad \Rightarrow \quad \nabla u \in L^{2,1} \quad \Rightarrow \quad u \in C^0$$

Concentration Compactness for H-H Maps in 2-D

Let u_k be horizontal harmonic maps with $\mathcal{L}(u_k) < C$.

$$u_k \to u_\infty \quad \text{strong in} \quad C^{1,\alpha}_{loc}\left(B^2 \setminus \{a_1 \cdots a_Q\}\right)$$

u_∞ is a horizontal harmonic map. Moreover

$$|\nabla u_k|^2\, dx^2 \quad \rightharpoonup |\nabla u_\infty|^2\, dx^2 + \sum_{j=1}^{Q} \lambda_j\, \delta_{a_j} \quad \text{in Radon meas.}$$

Question:

How much energy is dissipating ? λ_j =? Is λ_j equal to the sum of the Bubbles' Energy concentrating at a_j?

$$\lambda_j = \sum_{i=1}^{N_j} \mathcal{L}(u^{i,j})$$

where $u^{i,j} : \mathbb{C} \to \mathbb{R}^m$ is a horizontal harmonic map w.r.t P_T?

$$\Downarrow$$

No Neck Energy Problem

Neck Region

Let u_k be horiz. harm. with $\mathcal{L}(u_k) < C$.

A neck region for u_k is a union of degenerating annuli :

$A_k := B_{R_k} \setminus B_{r_k}$ s.t.

$$\lim_{k\to+\infty} \frac{R_k}{r_k} = +\infty \quad \text{and} \quad |\nabla u_k|(x) \le \frac{o_k(1)}{|x|}$$

Main Problem: Do we have

$$\int_{B_{R_k}\setminus B_{r_k}} |\nabla u_k(x)|^2 \, dx \longrightarrow 0?$$

Duality between the Lorentz spaces $L^{2,1}$ and $L^{2,\infty}$

What holds:

$$\|\nabla u_k\|_{L^{2,\infty}(B_{R_k}\setminus B_{r_k})} \longrightarrow 0.$$

Question:

$$\limsup_{k\to+\infty} \|\nabla u_k\|_{L^{2,1}(B_{R_k}\setminus B_{r_k})} < +\infty?$$

Asymptotic Expansions in Neck Region, [Laurain, Rivière (2011)]

There exist $\overrightarrow{c_k} \in \mathbb{R}^m$, $A_k \in W^{1,2} \cap L^\infty$, A_k radial, $f_k \in L^{2,1}$

$$\nabla u_k(x) = \frac{A_k(|x|)\overrightarrow{c_k}}{|x|} + f_k$$

with $\limsup_{k\to+\infty} \|f_k\|_{L^{2,1}(B_{R_k}\setminus B_{r_k})} < +\infty, \quad |\overrightarrow{c_k}| = O\left(\frac{1}{\sqrt{\log R_k/r_k}}\right)$.

Hence

$$\limsup_{k\to+\infty} \left\| \frac{1}{r}\frac{\partial u_k}{\partial \theta} \right\|_{L^{2,1}(B_{R_k}\setminus B_{r_k})} < +\infty \ \Rightarrow \ \lim_{k\to+\infty} \left\| \frac{1}{r}\frac{\partial u_k}{\partial \theta} \right\|_{L^2(B_{R_k}\setminus B_{r_k})} = 0.$$

Pohozaev $\Rightarrow$ No Neck Energy

Pohozaev Identity says

$$\forall\, r > 0 \qquad \int_{\partial B_r} \left| \frac{1}{r} \frac{\partial u_k}{\partial \theta} \right|^2 \, d\ell = \int_{\partial B_r} \left| \frac{\partial u_k}{\partial r} \right|^2 \, d\ell$$

Hence

$$\lim_{k \to +\infty} \left\| \frac{1}{r} \frac{\partial u_k}{\partial \theta} \right\|_{L^2(B_{R_k} \setminus B_{r_k})} = 0 \quad \Rightarrow \quad \lim_{k \to +\infty} \left\| \frac{\partial u_k}{\partial r} \right\|_{L^2(B_{R_k} \setminus B_{r_k})} = 0$$

Case $\alpha = \frac{1}{2}$: Half Harmonic Maps

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Critical points $u \in H^{1/2}(\mathbb{R}^k\mathcal{N})$, $\mathcal{N} \hookrightarrow \mathbb{R}^m$

$$\mathcal{L}^{1/2}(u) := \int_{\mathbb{R}^k} |(-\Delta)^{1/4} u|^2 \, dx < +\infty$$

Variations under pointwise constraint of the map : $u(x) \in \mathcal{N}$ a.e.

$$P_T(u)\,(-\Delta)^{1/2} u = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^k)$$

where $P_T(z)$ is the orthogonal matrix projection onto $T_z\mathcal{N}$.

Some Motivations

- Free Boundary Minimal Discs

A map $u \in H^{1/2}(S^1, \mathcal{N})$ is 1/2-harmonic iff its harmonic extension $\tilde{u}$ in B^2 is conformal and "cuts" $\mathcal{N}$ orthogonally.

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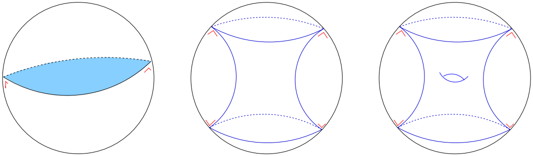

- Asymptotic Analysis of Ginzburg Landau Equation [Millot & Sire (2015)]

$$(-\Delta)^{1/2}u_\varepsilon = \frac{1}{\varepsilon}(1-|u_\varepsilon|^2)u_\varepsilon, \quad \text{in } \Omega \subseteq \mathbb{R}^k$$

Horizontal $1/2$-Harmonic Maps in $H^{1/2}(\mathbb{R}^k, \mathbb{R}^m)$:

$$\left.\begin{aligned} &P_T(u)\,(-\Delta)^{1/2}u = 0 \\ &P_N(u)\,\nabla u = 0 \end{aligned}\right\} \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^k)$$

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If P_T is integrable : back to 1/2-harmonic maps.

For $k = 1$ Horizontal 1/2-Harmonic Maps correspond to Minimal Discs with Horizontal Boundaries and Vertical Exterior Vector

Example: Hopf Distribution

In $\mathbb{C}^2 \setminus \{0\}$ let P_T be given by

$$P_T(z)\, Z := Z - |z|^{-2}\ [Z \cdot (z_1, z_2)\ (z_1, z_2) + Z \cdot (iz_1, iz_2)\ (iz_1, iz_2)]\,.$$

Horizontal $1/2$-harmonic maps u are given by solutions to the system

$$\left\{\begin{array}{ll} \frac{\partial \tilde{u}}{\partial r} \in \ \text{Span}\,\{u, i\,u\} & \\ u \cdot \frac{\partial u}{\partial \theta} = 0 & \qquad\qquad \text{in } \mathcal{D}'(S^1) \\ i\,u \cdot \frac{\partial u}{\partial \theta} = 0 & \end{array}\right.$$

where $\tilde{u}$ denotes the harmonic extension of u which defines a minimal disc.

An example of such a map is

$$u(\theta) := \frac{1}{\sqrt{2}}(e^{i\theta}, e^{-i\theta}) \quad \text{where} \quad \tilde{u}(z, \overline{z}) = \frac{1}{\sqrt{2}}(z, \overline{z}).$$

Observe that u is also an $1/2$-harmonic into S^3.

Conservation Laws for Horizontal $1/2$-Harmonic maps

Let P_T be a $H^{1/2}$-map into orthogonal projections of $\mathbb{R}^m$ and u s.t.

$$\left\{ \begin{array}{l} P_T\,(-\Delta)^{1/2} u = 0 \\ \\ P_N\,\nabla u = 0 \end{array} \right. \quad \text{in} \quad \mathcal{D}'(\mathbb{R})$$

We set

$$v = \left(\begin{array}{c} P_T\,(-\Delta)^{1/4} u \\ P_N\,(-\Delta)^{1/4} u \end{array} \right)$$

Then locally $\exists\, A \in H^{1/2} \cap L^\infty$, invertible, $B \in H^{1/2}$

$$(-\Delta)^{1/4} (A\, v) = \mathcal{J}\,(B, v)$$

with

$$\|\mathcal{J}(B, v)\|_{\mathcal{H}^1(\mathbb{R})} \leq C\, \|(-\Delta)^{1/4} B\|_{L^2(\mathbb{R})}\, \|v\|_{L^2(\mathbb{R})}$$

Blow-up Analysis of Sequence of Horizontal $1/2$-Harmonic Maps, [DL, Laurain, Rivière 2016]

Asymptotic Expansions in Necks: Let u_k be a sequence of Horizontal $1/2$-Harmonic Maps

$$\limsup_{k\to+\infty} \|(-\Delta)^{1/4}(u_k)\|_{L^2(S^1)} < +\infty.$$

Then there are $\overrightarrow{c_k} \in \mathbb{R}^m$, $f_k \in L^{2,1}$, $A_k \in H^{1/2} \cap L^\infty(\mathbb{R}^m, GL_m)$:

$$(-\Delta)^{1/4} u_k(x) = \frac{(A_k(x) + A_k(-x))\overrightarrow{c_k}}{|x|^{1/2}} + f_k,$$

with $\lim_{k\to+\infty} \|f_k\|_{L^{2,1}(B_{R_k} \setminus B_{r_k})} < +\infty$, $|\overrightarrow{c_k}| = O\left(\frac{1}{\sqrt{\log R_k/r_k}}\right)$.

It follows:

$$\lim_{k\to+\infty}\left\|((-\Delta)^{1/4}u_k(x)-(-\Delta)^{1/4}u_k(-x))\right\|_{L^{2,1}(B_{R_k}\setminus B_{r_k})}<+\infty$$

On the other hands it always holds

$$\|(-\Delta)^{1/4}u_k\|_{L^{2,\infty}(B_{R_k}\setminus B_{r_k})}\longrightarrow 0.$$

Hence

$$\lim_{k\to+\infty}\left\|((-\Delta)^{1/4}u_k(x)-(-\Delta)^{1/4}u_k(-x))\right\|_{L^{2}(B_{R_k}\setminus B_{r_k})}=0$$

Pohozaev Identities for Horizontal $1/2$- Harmonic Maps on S^1

Let $u \in W^{1,2}(S^1, \mathbb{R}^m)$ satisfy

$$\frac{du}{d\theta} \cdot (-\Delta)^{1/2} u = 0 \;\; \text{a.e in } S^1.$$

Then

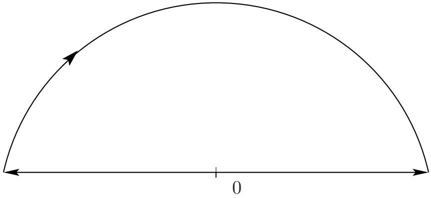
$$|u^+| = |u^-| \;\; \text{and} \;\; u^+ \cdot u^- = 0$$

where

$$u^+ := \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \cos(\theta) \; d\theta = \frac{1}{\pi} \int_0^{2\pi} (u(\theta) + u(-\theta)) \cos(\theta) \; d\theta$$

$$u^- := \frac{1}{2\pi} \int_0^{2\pi} u(\theta) \sin(\theta) \; d\theta = \frac{1}{\pi} \int_0^{2\pi} (u(\theta) - u(-\theta)) \; \sin(\theta) \; d\theta$$

$$\int_{\partial B(0,r)\cap R^2_+} \frac{\partial \tilde{u}}{\partial \theta} d\theta = (u(r) - u(0)) - (u(-r) - u(0)) = (u - u(0))^-(r)$$



$$\int_{B(0,r)\cap\{y=0\}} \frac{\partial \tilde{u}}{\partial r} dr = u(r) + u(-r) - 2u(0) = (u - u(0))^+(r)$$

Pohozaev Identities $\Rightarrow$

$$\lim_{k\to+\infty}\left\|((-\Delta)^{1/4}u_k(x)+(-\Delta)^{1/4}u_k(-x))\right\|_{L^2(B_{R_k}\setminus B_{r_k})}=0$$

$$\Downarrow$$

Let u_k be a sequence of Horizontal $1/2$-Harmonic Maps

$$\limsup_{k\to+\infty}\|(-\Delta)^{1/4}(u_k)\|_{L^2(S^1)}<+\infty$$

If in addition

$$\limsup_{k\to+\infty}\int_{S^1}|(-\Delta)^{1/2}u_k|(\theta)\,d\theta<+\infty$$

then the No Neck Property holds.

For $1/2-$harmonic maps into manifolds it always holds:

$$\|(-\Delta)^{1/2}u\|_{L^1(S^1)} \leq C\, \|u\|^2_{H^{1/2}(S^1)}$$

$$\Downarrow$$

The Energy Identity holds for $1/2-$harmonic maps into arbitrary closed manifolds:

$$\int_{S^1} |(-\Delta)^{1/4}u_k|^2\, d\theta \to \int_{S^1} |(-\Delta)^{1/4}u_\infty|^2\, d\theta + \sum_{i,j} \int_{S^1} |(-\Delta)^{1/4}\tilde{u}^{i,j}_\infty|^2\, d\theta.$$