Monotonicity and Symmetry results for nonlocal elliptic problem in the half space

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Mostly Maximum Principle
Banff

Joint work B. Barrios, L. del Pezzo, J. García-Melián

Problem

We consider the problem:

$$(*) \quad \begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathbb{R}^N_+, \\ \quad u = 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+, \end{cases}$$

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The function $f : [0, +\infty) \to \mathbb{R}$ is **Lipschitz or C^1** and we deal only with **nonnegative solutions**.

Case of s=1

Questions: for a **nonnegative** solution u of

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- u is **symmetric**:

$$u(x) = v(x_N)?$$

Some answers

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So, the case $f(0) \geq 0$ is **essentially closed**.

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  - Either $u > 0$ and u **is monotone**;
  - Or $u(x) = u_0(x_N)$, where u_0 is the unique one-dimensional solution of the problem verifying $u_0(0) = u_0'(0) = 0$.

STILL OPEN

$$f(0) < 0 \text{ and } N \geq 3.$$

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Therefore for symmetry it is usually assumed that

$$\boxed{M = \sup_{\mathbb{R}^N_+} u < +\infty.}$$

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- A. Farina, E. Valdinoci (2010): $\boxed{u \text{ positive, } N = 3 \text{ and } f \text{ locally Lipschitz}}$ or $\boxed{N = 4, 5 \text{ and special } f\text{'s}} \Longrightarrow u$ is symmetric.

STILL OPEN

$$N \geq 3 \text{ and } f(M) > 0.$$

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Conjecture A (BCN, ASNP 1997)

If u is positive, then $f(M) = 0$ and therefore u is one-dimensional.

An interesting example related to this conjecture:

$$\begin{cases} -\Delta u = u - 1 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

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For $N \geq 4$. C. Cortázar, M. Elgueta, J. García-Melián. (2016).

Some extension are know for p-Laplacian, fully nonlinear elliptic operator, etc...

$$\begin{cases} (-\Delta)^s u = f(u) & \text{in } \mathbb{R}^N_+, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \mathbb{R}^N_+, \end{cases} \tag{1}$$

here

$$(-\Delta)^s u(x) = \frac{1}{2} \int_{\mathbb{R}^N} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{N+2s}} dy,$$

Previous cases for positive monotone nonlinearity f:

- M. M. Fall, T. Weth (2016): (Green function in the half space)

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- W. Chen, Y. Fang, R. Yang (2015) for $f(t) = t^p$ (Green function in the half space).

Main Results Monotonicity

Theorem

Assume $f \in C^1(\mathbb{R})$ and let u be a bounded, nonnegative, nontrivial classical solution of (1). Then u is positive and

$$\frac{\partial u}{\partial x_N} > 0 \quad \text{in } \mathbb{R}^N_+.$$

Results one dimensional case

When $s = 1$, however, the corresponding problem

$$\begin{cases} -u'' = f(u) & \text{in } \mathbb{R}_+, \\ u(0) = 0 \end{cases} \tag{2}$$

Easy to see that there exists a bounded solution of (2) if and only if $\rho = \|u\|_{L^\infty(\mathbb{R}_+)}$ verifies $f(\rho) = 0$ and

$$F(t) < F(\rho) \text{ for all } t \in [0, \rho), \tag{F}$$

where F is the primitive of f vanishing at zero, $F(t) = \int_0^t f(\tau) d\tau$.

Results one dimensional case

Theorem

Assume f is locally Lipschitz and $\rho > 0$ is such that $f(\rho) = 0$ and condition (F) *is verified. Then there exists a unique positive solution u of $(*)(N = 1)$ with the property*

$$\|u\|_{L^\infty(\mathbb{R})} = \rho.$$

Moreover, u is strictly increasing and defining $\ell_0 := \lim_{x\to 0^+} \frac{u(x)}{x^s}$ then

$$\ell_0 = \frac{(2F(\rho))^{\frac{1}{2}}}{\Gamma(1+s)}. \tag{3}$$

Finally, all bounded positive solutions of $(*)(N = 1)$ are of the above form.

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Remark:If $f(\rho) \leq 0$ then $f(\rho) = 0$

Ideas of the Proof.

Key points

$$F(\rho) - F(u(a)) = \frac{c(1,s)}{2} \int_{-\infty}^{+\infty} \frac{(u(a)-u(y))^2}{|a-y|^{1+2s}} dy$$

$$+(1+2s) \int_a^{+\infty} \int_{-\infty}^{a} \frac{(u(x)-u(y))^2}{|x-y|^{2+2s}} dy dx$$

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Ideas of the Proof.

Define

$$I_{\delta,M} := \iint_{A_{\delta,M}} u'(x) \frac{u(x)-u(y)}{|x-y|^{1+2s}} dydx,$$

$$\begin{aligned} I_{\delta,M} &= \frac{1}{2} \iint_{A_{\delta,M}} \frac{(u(x)-u(y))_x^2}{|x-y|^{1+2s}} dydx \\ &= \frac{1}{2} \iint_{A_{\delta,M}} \left(\frac{(u(x)-u(y))^2}{|x-y|^{1+2s}} \right)_x dydx + \frac{1+2s}{2} \iint_{A_{\delta,M}} \frac{(x-y)(u(x)-u(y))^2}{|x-y|^{3+2s}} dydx. \end{aligned}$$

Ideas of the Proof.

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$$= \frac{1}{2}\oint_{\partial A_{\delta,M}} \frac{(u(x)-u(y))^2}{|x-y|^{1+2s}}dy + \frac{1+2s}{2}\iint_{A^3_{\delta,M}} \frac{(u(x)-u(y))^2}{(x-y)^{2+2s}}dydx.$$

Ideas of the Proof.

$$F(\rho) = \mathcal{K}(s)\ell_0^2, \tag{4}$$

where

$$\begin{aligned}\mathcal{K}(s) \; &= \frac{c(1,s)}{2}\left(-\frac{1}{2s} - \int_{-1}^{1} \frac{((t+1)^s - 1)^2}{|t|^{1+2s}}\,dt + \int_{1}^{+\infty} \frac{t^{2s} - ((t+1)^s - 1)^2}{t^{1+2s}}\,dt\right. \\ &\left. +(1+2s)\int_{1}^{+\infty}\int_{0}^{1} \frac{(t^s - \tau^s)^2}{(t-\tau)^{2+2s}}\,d\tau\,dt\right).\end{aligned} \tag{5}$$

Thanks!

Thank you for your attention!