On Convex Bodies Generated by Borel Measures

joint work with Han Huang

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Centroid bodies

- Given a Borel probability measure μ on $\mathbb{R}^n$, and $p \geq 1$, the L_p-centroid body $Z_p(\mu)$ is defined by its support function:

$$\forall \theta \in \mathbb{S}^{n-1}, \quad h_{Z_p(\mu)}(\theta) = \left(\int_{\mathbb{R}^n} |\langle \theta, x \rangle|^p \, \mathrm{d}\mu(x) \right)^{1/p}.$$

- For μ log-concave, $Z_1(\mu) \approx Z_2(\mu)$, and by putting μ in isotropic position, it follows that

$$\int_{\mathbb{R}^n} \|x\|_{Z_1(\mu)} \, \mathrm{d}\mu(x) \approx \left(\int_{\mathbb{R}^n} \|x\|^2_{Z_2(\mu)} \, \mathrm{d}\mu(x) \right)^{1/2} = \sqrt{n}.$$

- Question: is it true that for any non-degenerate probability measure μ:

$$\int_{\mathbb{R}^n} \|x\|_{Z_1(\mu)} \, \mathrm{d}\mu(x) \geq c\sqrt{n}\,?$$

Answer: yes.

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Generating convex sets by measures

Definition

Given a Borel measure μ on $\mathbb{R}^n$, we define the convex set:

$$\mathrm{M}(\mu) = \left\{ \int_{\mathbb{R}^n} y f(y) \mathrm{d}\mu(y) : 0 \le f \le 1, \int_{\mathbb{R}^n} f \mathrm{d}\mu = 1 \right\}.$$

- If $\mu = \sum_{i=1}^N \delta_{x_i}$, then $\mathrm{M}(\mu) = \mathrm{conv}(x_1, \ldots, x_N)$.
- If $\mu(\mathbb{R}^n) < 1$, then $\mathrm{M}(\mu) = \emptyset$.
- For $\mu(\mathbb{R}^n) = 1$, then $\mathrm{M}(\mu) = \{\int_{\mathbb{R}^n} x \mathrm{d}\mu(x)\}$ is a singleton (the center of mass of μ).

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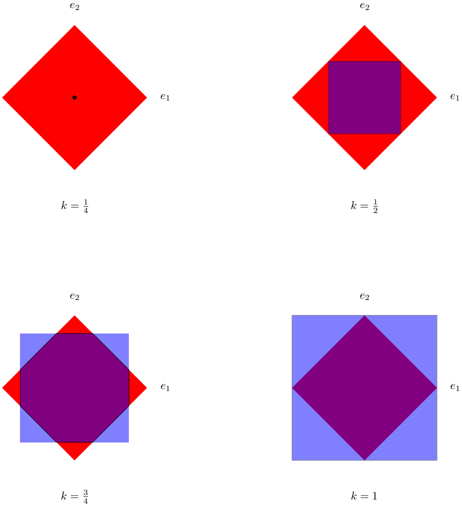
Examples

Discrete generating measures

1) If $\mu = \frac{1}{k}\sum_{i=1}^{2} \delta_{\pm e_i}$ then:



Examples

Discrete generating measures

- In general, if $\mu = \sum_{i=1}^{m} w_i \delta_{x_i}$ then $\mathrm{M}(\mu)$ is a polytope. More precisely, it is the linear image of

$$P = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^m : 0 \leq \lambda_i \leq 1, \sum_{i=1}^{m} \lambda_i w_i = 1 \right\} \subseteq \mathbb{R}^m$$

under the map $F(\boldsymbol{\lambda}) = \sum_{i=1}^{m} \lambda_i w_i x_i$.

- Also satisfied:

$$\mathrm{M}(\mu) \subseteq \mathrm{conv}(x_1, \cdots, x_m) \cap Z(w_1 x_1, \ldots, w_m x_m),$$

where $Z(w_1 x_1, \ldots, w_m x_m)$ is the Minkowski sum of $[0, w_i x_i]$.

- If $\mu(\mathbb{R}^n) \leq 2$ and $\mu(\{0\}) \geq 1$, then $\mathrm{M}(\mu) = Z(w_1 x_1, \ldots, w_m x_m)$

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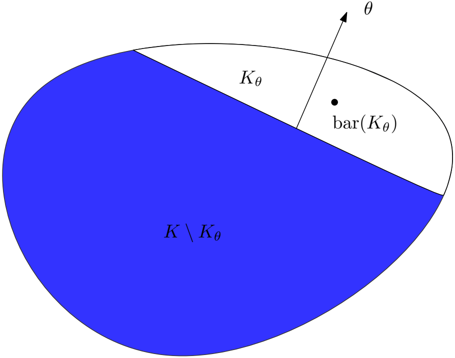
uniform measures on convex bodies

2) If μ is uniform on a convex body $K \subseteq \mathbb{R}^n$ with $\mathrm{vol}(K) > 1$, then $\mathrm{M}(\mu)$ is related to the floating body K_1 of K:



$$K_\theta = \{x \in K \,:\, \langle x, \theta \rangle \geq r\} \qquad \mathrm{vol}(K_\theta) = 1$$

$$K_1 = \bigcap_{\theta \in S^{n-1}} (K \setminus K_\theta) \qquad \mathrm{bar}(K_\theta) \in \mathrm{M}(\mu)$$

Approximation of convex bodies by polytopes

- Let $K \subseteq \mathbb{R}^n$ be a centered convex body. For $R > 1$, consider:
$$d_R(K) = \inf\left\{N \in \mathbb{N} : \exists P = \mathrm{conv}(x_1, \ldots, x_N) \subseteq \mathbb{R}^n, \frac{1}{R}P \subseteq K \subseteq P\right\}.$$
- For $R = \infty$, we trivially have $d_\infty(K) = n+1$ (take a big simplex). However, consider:
$$D_R(K) = \inf\left\{\sum_{i=1}^{N} \|x_i\|_K : \exists P = \mathrm{conv}(x_1, \ldots, x_N) \subseteq \mathbb{R}^n, \frac{1}{R}P \subseteq K \subseteq P\right\}.$$
Note that for any $R < \infty$, $d_R(K) \le D_R(K) \le R d_R(K)$.
- For $R = \infty$, $D_\infty(K)$ coincides with the vertex index of K, which was introduced by Bezdek and Litvak:
$$\mathrm{vein}(K) = \inf\left\{\sum_{i=1}^{N} \|x_i\|_K : K \subseteq P = \mathrm{conv}(x_1, \ldots, x_N)\right\}.$$

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Approximation of convex bodies by measure-generated sets

- Our definition of

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leads to the following new quantities:

$$d_R^*(K) = \inf\left\{ \mu(\mathbb{R}^n) : \frac{1}{R}\mathrm{M}(\mu) \subseteq K \subseteq \mathrm{M}(\mu) \right\},$$

$$D_R^*(K) = \inf\left\{ \int_{\mathbb{R}^n} \|x\|_K \, \mathrm{d}\mu(x) : \frac{1}{R}\mathrm{M}(\mu) \subseteq K \subseteq \mathrm{M}(\mu) \right\},$$

$$\mathrm{vein}^*(K) = \inf\left\{ \int_{\mathbb{R}^n} \|x\|_K \, \mathrm{d}\mu(x) : K \subseteq \mathrm{M}(\mu) \right\}.$$

Upper bounds

Theorem

Let K be a centered convex body in $\mathbb{R}^n$. Then for $1 < R \le n$ one has

$$d_R^*(K) \le \exp\left(1 + \frac{n-1}{R-1}\right), \text{ and } D_R^*(K) \le R\exp\left(1 + \frac{n-1}{R-1}\right).$$

In particular, $\mathrm{vein}^*(K) \le D_n^*(K) = e^2 n$

Theorem

Let $K = -K$ be a convex body in $\mathbb{R}^n$. Then

$$d_{\sqrt{n}}^*(K) \le C, \text{ and } D_{\sqrt{n}}^*(K) \le Cn.$$

- The results follow by taking appropriate uniform measures + John's position / Brunn Minkowski .

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Estimating $\mathrm{vein}^*(K)$

Two precise computations

Recall: $$\mathrm{vein}(K) = \inf\left\{\sum_{i=1}^{N} \|x_i\|_K \,:\, K \subseteq P = \mathrm{conv}(x_1,\dots,x_N)\right\},$$

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$$\mathrm{vein}(B_1^n) = 2n$$

Gluskin-Litvak:

$$\sqrt{3}n^{3/2} \le \mathrm{vein}(B_2^n) \le 2n^{3/2}$$

In our case:

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$$\mathrm{vein}(B_1^n) = 2n$$

Gluskin-Litvak:

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In our case:

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Estimating $\mathrm{vein}^*(K)$

Theorem (Gluskin and Litvak, '08, '12)

Let K be a centrally symmetric convex body in $\mathbb{R}^n$. Then

$$2n = \mathrm{vein}(B_1^n) \le \mathrm{vein}(K) \le C_1 \,\mathrm{vein}(B_2^n) \le C_2 n^{3/2}.$$

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Let K be a centrally symmetric convex body in $\mathbb{R}^n$. Then

$$c\sqrt{n} \le c\,\mathrm{vein}^*(B_2^n) \le \mathrm{vein}^*(K) \le C_1 \,\mathrm{vein}^*(B_1^n) \le C_2 n.$$

- Recall: upper bound is a consequence of our upper bound on $D_n^*(K)$.

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Lower bound on $\mathrm{vein}^*(K)$: Sketch of the proof

- Fact 1: $\exists\ T \in \mathrm{GL}_n(\mathbb{R})$ and a subspace $E \subseteq \mathbb{R}^n$ with $\dim E \geq n/2$ s.t.

$$B_1^E \subseteq P_E(TK) \subseteq C\sqrt{n}B_1^E.$$

Since $\mathrm{vein}^*(K) = \mathrm{vein}^*(TK)$, we may assume that $T = Id$.

- Fact 2: enough to consider finite discrete measures.
- Fact 3: $\mathrm{vein}^*(K) \leq \mathrm{vein}^*(L)\, d_{BM}(K, L)$.

Proof: Suppse $\mu = \sum_{i=1}^m w_i \delta_{x_i}$ with $K \subseteq \mathrm{M}(\mu)$. Define $\nu = \sum_{i=1}^m w_i \delta_{P_E x_i}$. Then $P_E K \subseteq \mathrm{M}(\nu)$.Moreover, $\|x\|_K \geq \|P_E x\|_{P_E K}$ implies

$$\int_{\mathbb{R}^n} \|x\|_K \, \mathrm{d}\mu(x) \geq \int_E \|y\|_{P_E K} \, \mathrm{d}\nu(y) \geq \mathrm{vein}^*(P_E K),$$

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Relation to centroid bodies

Proposition

We have $\inf_{K=-K} \mathrm{vein}^*(K) = 2 \inf_{\mu} \int_{\mathbb{R}^n} \|x\|_{Z_1(\mu)} \, d\mu(x)$.

Corollary

We have $\inf_{\mu} \int_{\mathbb{R}^n} \|x\|_{Z_1(\mu)} \, d\mu(x) \geq C\sqrt{n}$.

Sketch of the proof:

- Suppose $K \subseteq \mathrm{M}(\mu)$. By scaling the measure and adding an atom at the origin, we may assume that $\mu(\mathbb{R}^n) = 2$, $\mu(\{0\}) = 1$. In other words, $\mu = \nu + \delta_0$ where ν is a probability measure and $K \subseteq \mathrm{M}(\nu + \delta_0)$.
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