Geometry of triple tensor products

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We want to study the volume ratio and cotype 2 of triple tensor products.

Definition of volume ratio

Let X be a n-dimensional, normed space with unit ball B_X. Then

$$\mathrm{vr}(X) = \inf_{\mathcal{E} \subseteq B_X} \left(\frac{\mathrm{vol}_n(B_X)}{\mathrm{vol}_n(\mathcal{E})} \right)^{\frac{1}{n}}$$

where $\mathcal{E}$ is an ellipsoid contained in B_X.

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For all finite-dimensional normed spaces X and Y

$$\mathrm{vr}(X) \leq d(X, Y)\, \mathrm{vr}(Y).$$

Example

For all $n \in \mathbb{N}$

$$\mathrm{vr}(\ell_p^n) \sim \begin{cases} 1 & 1 \leq p \leq 2 \\ n^{\frac{1}{2}-\frac{1}{p}} & 2 \leq p \leq \infty \end{cases}$$

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Khintchine inequality

There are positive constants c_1 and c_2 such that for all $n \in \mathbb{N}$ and all sequences of real numbers $a_1, \dots, a_n$

$$c_1 \left(\sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}} \leq \frac{1}{2^n} \sum_{\epsilon} \left| \sum_{i=1}^{n} \epsilon_i a_i \right| \leq c_2 \left(\sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}}$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ denote all possible signs ± 1.

This can be reformulated as follows: There are positive constants c_1 and c_2 such that for all $n \in \mathbb{N}$ and all sequences of real numbers $a_1, \dots, a_n$

$$c_1 \left(\sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}} \leq \int_0^1 \left| \sum_{i=1}^{n} r_i(t) a_i \right| dt \leq c_2 \left(\sum_{i=1}^{n} |a_i|^2 \right)^{\frac{1}{2}}$$

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For all x_1 and x_2 in $\mathbb{R}^2$ we have

$$2\|x_1\|_2^2 + 2\|x_2\|_2^2 = \|x_1 - x_2\|_2^2 + \|x_1 + x_2\|_2^2.$$

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The parallelogram equality can be generalized to higher dimensional parallelotopes

$$\sum_{i=1}^{n} \|x_i\|_2^2 = \frac{1}{2^n} \sum_{\epsilon} \left\| \sum_{i=1}^{n} \epsilon_i x_i \right\|_2^2$$

where $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ denotes all sequences of signs ± 1.

Definition

Let $1 \leq p \leq 2$. A Banach space X is of type p if there is a constant T such that for all $n \in \mathbb{N}$ and all vectors $x_1, \ldots, x_n$ in X

$$\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\| dt \leq T \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}}. \tag{1}$$

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For all Banach spaces X and Y

$$T_p(X) \leq d(X, Y) T_p(Y).$$

Definition

Let $2 \leq q < \infty$. A Banach space X has cotype q if there is a constant C such that for all $n \in \mathbb{N}$ and all vectors $x_1, \ldots, x_n$ in X

$$C \int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\| dt \geq \left(\sum_{i=1}^n \|x_i\|^q \right)^{\frac{1}{q}}. \tag{2}$$

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His result can also be phrased like this: A Banach space is isomorphic to a Hilbert space if and only if it has type 2 and cotype 2.

Example

(i) Let $1 \leq p \leq 2$. Then the cotype-2 constant of L_p is less than $\sqrt{2}$.
(ii) (Tomczak-Jaegermann) The Schatten classes C_p have cotype 2 for $1 \leq p \leq 2$.

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The norm on C_1 is the same as

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Theorem (Briet+Naor+Regev)

For $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$

$$\ell_p \otimes_\pi \ell_q \otimes_\pi \ell_r$$

does not have non-trivial cotype .

Theorem (Bourgain-Milman)

Let X be a finite dimensional, normed space. Then

$$\mathrm{vr}(X) \leq C_2(X) \ln(2C_2(X)).$$

As an application of the invariant volume ratio: Euclidean decomposition of normed spaces.

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Theorem (Kashin, Szarek)

There is a positive constant c such that for all $n \in \mathbb{N}$ and all $2n$-dimensional, normed spaces X there are two n-dimensional subspaces E and F with $E \cap F = \{0\}$ and

$$d(E, \ell_2^n) \leq c\,\mathrm{vr}(X) \quad \text{and} \quad d(F, \ell_2^n) \leq c\,\mathrm{vr}(X).$$

Theorem (S., Tomczak-Jaegermann)

$1 \le p \le q \le \infty$, then

$$\mathrm{vr}\left(\ell_p^n \otimes_\pi \ell_q^n\right) \asymp_{p,q} \begin{cases} 1, & q \le 2 \\ n^{\frac{1}{2}-\frac{1}{q}}, & p \le 2 \le q,\ \frac{1}{p}+\frac{1}{q} \ge 1 \\ n^{\frac{1}{p}-\frac{1}{2}}, & p \le 2 \le q,\ \frac{1}{p}+\frac{1}{q} \le 1 \\ n^{\max\left(\frac{1}{2}-\frac{1}{p}-\frac{1}{q},0\right)}, & p \ge 2 \end{cases}$$

Theorem

Let $n \in \mathbb{N}$ and $1 \leq p \leq q \leq r \leq \infty$. Then

$$\mathrm{vr}\left(\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n\right) \asymp_{p,q,r} \begin{cases} 1, & r \leq 2 \\ n^{\max\left(\frac{1}{2}-\frac{1}{q}-\frac{1}{r},0\right)} & p \leq 2 \leq q\,, \frac{1}{p}+\frac{1}{q}+\frac{1}{r} \geq 1 \\ n^{\frac{1}{p}-\frac{1}{2}} & p \leq 2 \leq q\,, \frac{1}{p}+\frac{1}{q}+\frac{1}{r} \leq 1 \\ n^{\max\left(\frac{1}{2}-\frac{1}{p}-\frac{1}{q}-\frac{1}{r},0\right)} & p \geq 2 \end{cases}$$

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- Identify the ellipsoid of maximal volume contained in the unit ball.

Estimating the volume of the unit ball is done by a generalization of a theorem of Chevet.

Theorem (Chevet)

Let $(\Omega,\mathbb{P})$ be a probability space and let X and Y Banach spaces. Then for all $n,m\in\mathbb{N}$, all $x_1,\dots,x_n\in X$ and $y_1,\dots,y_m\in Y$, and all independent $N(0,1)$-random variables $\alpha_i,\beta_j,g_{i,j}$, $i=1,\dots,n,\ j=1,\dots,m$,

$$\frac{\Lambda}{2}\le\int_\Omega\left\|\sum_{i,j=1}^{n,m}g_{i,j}(\omega)x_i\otimes y_j\right\|_\varepsilon d\mathbb{P}(\omega)\le 2\sqrt{2}\,\Lambda,$$

where

$$\Lambda = \sup_{\|x^*\|=1}\left(\sum_{i=1}^{n}|x^*(x_i)|^2\right)^{1/2}\int_\Omega\left\|\sum_{j=1}^{m}\beta_j y_j\right\|_Y d\mathbb{P} + \sup_{\|y^*\|=1}\left(\sum_{j=1}^{m}|y^*(y_j)|^2\right)^{1/2}\int_\Omega\left\|\sum_{i=1}^{n}\alpha_i x_i\right\|_X d\mathbb{P}.$$

Lemma (3-fold Chevet inequality)

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$, $(Z, \|\cdot\|_Z)$ be Banach spaces. Assume that $x_1, \ldots, x_m \in X$, $y_1, \ldots, y_n \in Y$ and $z_1, \ldots, z_\ell \in Z$ and $g_{i,j,k}$, ξ_i, η_j, ρ_k $i = 1, \ldots, m$, $j = 1, \ldots, n$, $k = 1, \ldots, \ell$, be independent standard Gaussians random variables. Then

$$\mathbb{E}\left\| \sum_{i,j,k=1}^{m,n,\ell} g_{i,j,k} x_i \otimes y_j \otimes z_k \right\|_{X \otimes_\epsilon Y \otimes_\epsilon Z} \leq \Lambda,$$

Lemma (3-fold Chevet inequality)

where

$$\begin{aligned}\Lambda \quad := \quad & \|(x_i)_{i=1}^m\|_{w,2} \left\|(y_j)_{j=1}^n\right\|_{w,2} \mathbb{E} \left\| \sum_{k=1}^{\ell} \rho_k z_k \right\|_Z \\ & + \|(x_i)_{i=1}^m\|_{w,2} \left\|(z_k)_{k=1}^{\ell}\right\|_{w,2} \mathbb{E} \left\| \sum_{j=1}^{n} \eta_j y_j \right\|_Y \\ & + \left\|(y_j)_{j=1}^n\right\|_{w,2} \left\|(z_k)_{k=1}^{\ell}\right\|_{w,2} \mathbb{E} \left\| \sum_{i=1}^{m} \xi_i x_i \right\|_X .\end{aligned}$$

Proposition (S.)

Let B be the unit ball of the normed space E with normalized basis $e_1, \dots, e_n$. Suppose $s_1, \dots, s_n$ are positive real numbers such that for all sequences of signs $\epsilon = (\epsilon_1, \dots, \epsilon_n)$

$$\frac{1}{2^n} \sum_{\epsilon} \left\| \sum_{i=1}^{n} \epsilon_i s_i e_i \right\| \leq 1.$$

Then

$$2^n \prod_{i=1}^{n} s_i \leq \mathrm{vol}_n(B).$$

In the case of the triple tensor product we choose as our basis

$$e_i \otimes e_j \otimes e_k \qquad\qquad i, j, k = 1, \ldots, n$$

and we have to estimate

$$\frac{1}{2^{n^3}} \sum_{\epsilon} \left\| \sum_{i,j,k=1}^{n} \epsilon_{i,j,k} e_i \otimes e_j \otimes e_k \right\|$$

where $\epsilon_{i,j,k} = \pm 1$.

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Thus we have identified the ellipsoid up to a multiple factor.

The multiple factor equals the norm of the natural identity

$$id : \ell_2^{n^3} \to \ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n.$$

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Lemma

(Hardy-Littlewood) Let $1 \leq p, q \leq \infty$ with $\frac{3}{2} \leq \frac{1}{p} + \frac{1}{q}$ and let μ be given by

$$\frac{1}{\mu} = \frac{1}{2p} + \frac{1}{2q} - \frac{1}{4}$$

Then we have for all $A \in \ell_p^n \otimes_\epsilon \ell_q^n$

$$\left(\sum_{i,j=1}^{n} |a_{i,j}|^\mu \right)^{\frac{1}{\mu}} \leq \|A\|_{\ell_p^n \otimes_\epsilon \ell_q^n}.$$