Global Existence and Regularity for the Active Liquid Crystal System

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Outline

- Active Hydrodynamics
- The Models
- Incompressible Active Liquid Crystals
- Inhomogeneous Active Liquid Crystals
- Compressible Active Liquid Crystals
- Open Problems

Active Hydrodynamics

- with active constituent particles that have collective motion
- maintained out of equilibrium by internal energy sources

$\neq$ Passive ones that maintained out of equilibrium by the external force applied at the boundary of the system

Applications

In biophysics, synthetic chemistry, colloidal physics, and materials science:

- microtubule bundles, cytoskeletal filaments, actin filaments
- dense suspensions of microswimmers, bacteria, catalytic motors
- nonliving analogues, such as monolayers of vibrated granular rods
- ......

Liquid-crystalline order in a myxobacterial flock

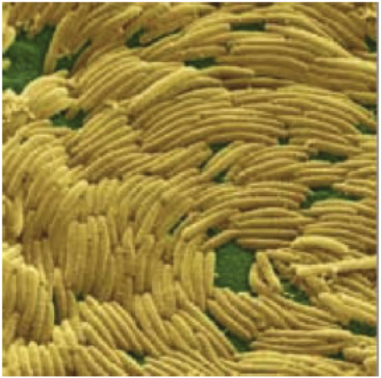


Image courtesy of Gregory Velicer (Indiana University Bloomington) and Juergen Bergen (Max-Planck Institute for Developmental Biology).

Sardine school

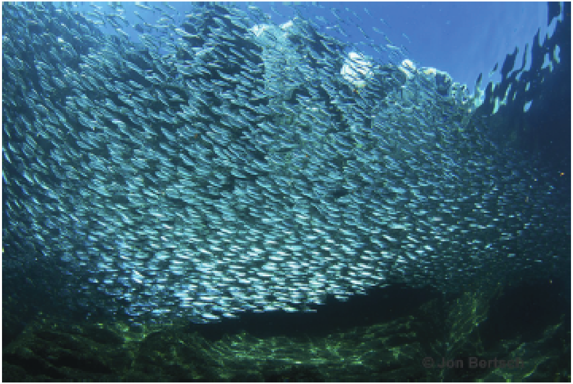
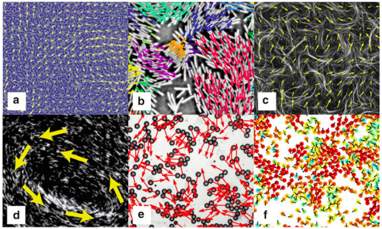


Image courtesy of Jon Bertsch, from underwater images form the Sea of Cortez:

http://www.thalassagraphics.com/blog/?p=167

Soft active systems:



(Image courtesy of David Saintillan (University of California) and Michael J. Shelley (New York University))

(a) collective motion in a suspension of swimming Bacillus subtilis, where arrows show the velocity field; (b) dynamic clusters in swarms of bacteria, where arrows show the direction of motion of the particles; (c) spontaneous motion in a suspension of microtubules and kinesin motors confined at a two-dimensional interface; (d) large-scale swirling motion in a suspension of actin filaments transported by wall-tethered myosin molecular motors; (e) swarming of self-propelling liquid droplets in a Hele-Shaw cell; (f) long-range order of vibrated polar disks on a two-dimensional substrate.

- Active Hydrodynamics + Liquid Crystal model =Active Liquid Crystals
- Nematic Liquid Crystal model theories: Doi-Onsager theory (86'), Oseen-Frank theory (33', 58'), Ericksen-Leslie theory (61', 68'), Landau - de Gennes theory (95').

Properties of the Q-tensor

- 3×3 symmetric and traceless matrix
- the Q-tensor is $\begin{cases} \text{isotropic, if } Q = 0 \\ \text{uniaxial, if it has two equal eigenvalues} \\ \text{biaxial, otherwise} \end{cases}$

Modeling and Numerical Simulation

- Giomi-Mahadevan-Chakraborty-Hagan, 2011, show that active nematic suspensions behave as excitable media, and relaxation oscillations is similar to biological pumps.
- Ravnik-Yeomans, 2012, use numerical modelling to study the flow patterns of an active nematic confined in a cylindrical capillary.
- Giomi-Bowick-Ma-Marchetti, 2013, defects in active liquid crystals.
- Saintillan-Shelley, 2013, use continuum kinetic theories to model active suspension systems.
- Blow-Thampi-Yeomans, 2014, dynamical simulations of a 2D active nematic fluid in coexistence with an isotropic fluid.
- Marchetti-Joanny-Ramaswamy-Liverpool-Prost-Rao-Simha, Ahmadi, Lau-Lubensky, Calderer$\cdots$

Active Liquid Crystal System

$$
\begin{cases}
c_t + (u\cdot\nabla)c = \nabla\cdot f(c,\nabla c, Q, \nabla\cdot Q),\\
\rho_t + \nabla\cdot(\rho u) = 0,\\
(\rho u)_t + \nabla\cdot(\rho u\otimes u) + \nabla P - \mathcal{L}u = \nabla\cdot\tau + \nabla\cdot\sigma,\\
Q_t + (u\cdot\nabla)Q + S(\nabla u, Q) - \lambda|Q|D = \Gamma H[Q,c],
\end{cases}
$$

c: concentration, ρ: density, u: velocity, Q: nematic tensor order parameter, $P=\rho^\gamma$: pressure

$\mathcal{L}u = \mu\Delta u + (\nu+\mu)\nabla \mathrm{div} u$: Lamé operator

$H[Q,c]$: molecular tensor

Active Liquid Crystal System

$$\begin{cases} c_t + (u\cdot\nabla)c = \nabla\cdot f(c, \nabla c, Q, \nabla\cdot Q), \\ \rho_t + \nabla\cdot(\rho u) = 0, \\ (\rho u)_t + \nabla\cdot(\rho u\otimes u) + \nabla P - \mathcal{L}u = \nabla\cdot\tau + \nabla\cdot\sigma, \\ Q_t + (u\cdot\nabla)Q + S(\nabla u, Q) - \lambda|Q|D = \Gamma H[Q, c], \end{cases}$$

stress tensor

$$S(\nabla u, Q) = \Omega Q - Q\Omega$$

$$\tau = -\nabla Q \odot \nabla Q$$

$$\sigma = \underbrace{-\lambda|Q|H[Q,c] + QH[Q,c] - H[Q,c]Q}_{\text{elastic stress tensor}} + \underbrace{\sigma_* c^2 Q}_{\text{active contribution}}$$

Active Liquid Crystal System

$$
\begin{cases}
c_t + (u \cdot \nabla) c = \nabla \cdot f(c, \nabla c, Q, \nabla \cdot Q), \\
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(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P - \mathcal{L} u = \nabla \cdot \tau + \nabla \cdot \sigma, \\
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$\sigma_* c^2 Q$ describes the contractile or extensile stress exerted by the active particles. $\sigma_* > 0$: contractile case; $\sigma_* < 0$: extensile case

$\Gamma > 0$, $\mu > 0$, $\nu + \frac{2}{3}\mu \geq 0$, b, $\lambda \in \mathbb{R}$,

- μ, ν: viscosity constants
- $\gamma > \frac{3}{2}$: adiabatic constant
- Γ^{-1}: rotational viscosity
- λ: nematic alignment parameter
- b : material-dependent elastic constants

Main Difficulties

- c-equation: Too complicated
- $\lambda\nabla\cdot(|Q|H[Q,c])$, $\sigma_*\nabla\cdot(c^2Q)$ involves interaction of Q and c
- Q-equation: $\lambda|Q|D \Rightarrow \mathrm{tr}\,Q = 0$? ($\mathrm{tr}D = \mathrm{div}\,u = 0$, if $\mathrm{div}\,u = 0$)

Incompressible Active Liquid Crystal system

$$
\begin{cases}
\partial_t Q + (u\cdot\nabla)Q + Q\Omega - \Omega Q - \lambda|Q|D = \Gamma H[Q], \\
\partial_t u + (u\cdot\nabla)u + \nabla P - \mu\Delta u = -\nabla\cdot(\nabla Q \odot \nabla Q) \\
\qquad -\lambda\nabla\cdot(|Q|H[Q]) + \nabla\cdot(Q\Delta Q - \Delta Q Q) + \kappa\nabla\cdot Q, \\
\nabla\cdot u = 0.
\end{cases}
$$

$$
H[Q] = \Delta Q - aQ + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{d}\mathrm{I}_d] - cQ\mathrm{tr}(Q^2).
$$

Inhomogeneous Active Liquid Crystal system

$$
\begin{cases}
\rho_t + \nabla\cdot(\rho u) = 0,\\
(\rho u)_t + \nabla\cdot(\rho u\otimes u) + \nabla P - \mu\Delta u = -\nabla\cdot(\nabla Q\odot\nabla Q)\\
\qquad -\lambda\nabla\cdot(|Q|H[Q]) + \nabla\cdot(Q\Delta Q - \Delta Q Q) + \kappa\nabla\cdot Q,\\
\partial_t Q + (u\cdot\nabla)Q + Q\Omega - \Omega Q - \lambda|Q|D = \Gamma H[Q],\\
\nabla\cdot u = 0.
\end{cases}
$$

$$
H[Q] = \Delta Q - aQ + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{d}\mathrm{I}_d] - cQ\mathrm{tr}(Q^2).
$$

$$\begin{cases} c_t + u \cdot \nabla c = D_0 \Delta c, \\ \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla \rho^\gamma - \mathcal{L} u = \nabla \cdot (F(Q) \mathrm{I}_3 - \nabla Q \odot \nabla Q) \\ \qquad + \nabla \cdot (Q \Delta Q - \Delta Q Q) + \sigma_* \nabla \cdot (c^2 Q), \\ Q_t + (u \cdot \nabla) Q + Q\Omega - \Omega Q = \Gamma H[Q, c]. \end{cases}$$

$$H[Q,c] = \Delta Q - \frac{c - c_*}{2} Q + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{3} \mathrm{I}_3] - c_* Q \mathrm{tr}(Q^2)$$

$$F(Q) = \frac{1}{2} |\nabla Q|^2 + \frac{1}{2} \mathrm{tr}(Q^2) + \frac{c_*}{4} \mathrm{tr}^2(Q^2)$$

$$\mathcal{L} u = \mu \Delta u + (\nu + \mu) \nabla \mathrm{div} u$$

Previous Results

- Ball-Majumdar, 2009, define Ball-Majumdar bulk potential.
- Paicu-Zarnescu, 2011, 2012, existence of global weak solutions in 2D, 3D, strong solutions in 2D.
- Wilkinson, 2012, existence and regularity of weak solutions on the $d-$dimensional torus over a certain singular potential.
- Feireisl-Rocca-Schimperna-Zarnescu, 2012, global-in-time weak solutions in case of Ball-Majumdar's singular free energy bulk potential.
- Wang-Xu-Yu, 2015, existence and long time dynamics of globally defined weak solutions, compressible case.
- Guilln-Gonzlez-Rodrguez-Bellido, 2015, existence of weak solutions and a uniqueness criterion for an initial-boundary problem in a bounded domain.
- De Anna-Zarnescu, 2016, the uniqueness of weak solutions in 2D.
- Abels-Dolzmann-Liu, Wang-Zhang-Zhang, $\cdots$

Incompressible Active Liquid Crystal System

$$\begin{cases} \partial_t Q + (u\cdot\nabla)Q + Q\Omega - \Omega Q - \lambda|Q|D = \Gamma H, \\ \partial_t u + (u\cdot\nabla)u + \nabla P - \mu\Delta u = -\nabla\cdot(\nabla Q\odot\nabla Q) \\ \qquad -\lambda\nabla\cdot(|Q|H) + \nabla\cdot(Q\Delta Q - \Delta Q Q) + \kappa\nabla\cdot Q, \\ \nabla\cdot u = 0. \end{cases}$$

$$H = \Delta Q - aQ + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{d}\mathrm{I}_d] - cQ\mathrm{tr}(Q^2).$$

A priori Estimates

$$E(t) := \underbrace{\int_{\mathbb{R}^d} \frac{1}{2}|u|^2 \mathrm{d}x}_{\text{kinetic energy}} + \underbrace{\int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla Q|^2 + \frac{a}{2}|Q|^2 - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|Q|^4 \right) dx}_{\text{free energy of the molecules}}$$

$$\frac{d}{dt}E(t) + \mu \int_{\mathbb{R}^d} |\nabla u|^2 dx + \Gamma \int_{\mathbb{R}^d} \mathrm{tr}(H^2) dx = \sum_{i=1}^{4} \mathcal{I}_i$$

$$\mathcal{I}_1 = (u \cdot \nabla Q, \Delta Q) - (\nabla \cdot (\nabla Q \odot \nabla Q), u)$$

$$\mathcal{I}_2 = (\Omega Q - Q\Omega, \Delta Q) - (\nabla \cdot (Q\Delta Q - \Delta Q Q), u)$$

$$\mathcal{I}_3 = \lambda(|Q|H, \nabla u) - \lambda(|Q|D, H)$$

$$\mathcal{I}_4 = \kappa(\nabla \cdot Q, u)$$

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$$\mathcal{I}_1 = (u \cdot \nabla Q, \Delta Q) - (\nabla \cdot (\nabla Q \odot \nabla Q), u) = 0$$

$$\mathcal{I}_2 = (\Omega Q - Q\Omega, \Delta Q) - (\nabla \cdot (Q\Delta Q - \Delta Q Q), u) = 0$$

$$\mathcal{I}_3 = \lambda(|Q|H, \nabla u) - \lambda(|Q|D, H) = 0$$

$$\mathcal{I}_4 = \kappa(\nabla \cdot Q, u) \leq C(\|\nabla Q\|_{L^2}^2 + \|u\|_{L^2}^2)$$

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$$\mathrm{tr}(Q^3) \leq \frac{\varepsilon}{4}|Q|^4 + \frac{1}{\varepsilon}|Q|^2$$

A priori Estimates

$$E(t) := \underbrace{\int_{\mathbb{R}^d} \frac{1}{2}|u|^2 \mathrm{d}x}_{\text{kinetic energy}} + \underbrace{\int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla Q|^2 + \frac{a}{2}|Q|^2 - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|Q|^4 \right) dx}_{\text{free energy of the molecules}}$$

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$$\mathrm{tr}(Q^3) \leq \frac{\varepsilon}{4}|Q|^4 + \frac{1}{\varepsilon}|Q|^2$$

$$(M + \frac{a}{2})|Q|^2 - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|Q|^4 \geq \frac{M}{2}|Q|^2 + \frac{c}{8}|Q|^4 \geq 0$$

A priori Estimates

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$$E^M(t) := E(t) + M\|Q\|_{L^2}^2 \geq 0$$

A priori Estimates

$$E(t) := \underbrace{\int_{\mathbb{R}^d} \frac{1}{2}|u|^2 \mathrm{dx}}_{\text{kinetic energy}} + \underbrace{\int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla Q|^2 + \frac{a}{2}|Q|^2 - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|Q|^4 \right) dx}_{\text{free energy of the molecules}}$$

$$\frac{d}{dt}E(t) + \mu \int_{\mathbb{R}^d} |\nabla u|^2 dx + \Gamma \int_{\mathbb{R}^d} \mathrm{tr}(H^2) dx \le C(\|\nabla Q\|_{L^2}^2 + \|u\|_{L^2}^2)$$

$$\mathrm{tr}(Q^3) \le \frac{\varepsilon}{4}|Q|^4 + \frac{1}{\varepsilon}|Q|^2$$

$$(M + \frac{a}{2})|Q|^2 - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|Q|^4 \ge \frac{M}{2}|Q|^2 + \frac{c}{8}|Q|^4 \ge 0$$

$$E^M(t) := E(t) + M\|Q\|_{L^2}^2 \ge 0$$

$$\frac{dE^M(t)}{dt} + \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2 + \frac{c^2\Gamma}{2}\|Q\|_{L^6}^6 \le C\left(\|u\|_{L^2}^2 + \|Q\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2 + \|Q\|_{L^4}^4\right)$$

Existence of Weak Solutions for $d = 2, 3$

Theorem (Chen-Majumdar-Wang-Z., '17)

Suppose $d = 2, 3$. There exists a weak solution (Q, u) to the incompressible active LCs, subject to the initial conditions

$$Q|_{t=0} = Q_0(x) \in H^1(\mathbb{R}^d), u|_{t=0} = u_0(x) \in L^2(\mathbb{R}^d), \nabla \cdot u_0 = 0,$$

in $\mathcal{D}'(\mathbb{R}^d)$.
Moreover, the solution satisfies,

$$Q \in L^\infty_{loc}(\mathbb{R}_+; H^1) \cap L^2_{loc}(\mathbb{R}_+; H^2), \quad u \in L^\infty_{loc}(\mathbb{R}_+; L^2) \cap L^2_{loc}(\mathbb{R}_+; H^1).$$

Regularized System

$$\begin{cases} \partial_t Q + (R_\varepsilon u)\cdot\nabla Q + Q(R_\varepsilon \Omega) - (R_\varepsilon\Omega)Q - \lambda|Q|(R_\varepsilon D) = \Gamma H, \\ \partial_t u + \mathcal{P}(R_\varepsilon u\cdot\nabla u) - \mu\Delta u = \varepsilon\mathcal{P}\nabla\cdot R_\varepsilon(\nabla R_\varepsilon u|\nabla R_\varepsilon u|^2) \\ \qquad -\varepsilon\mathcal{P}R_\varepsilon(\partial_\alpha Q_{\beta\gamma}(R_\varepsilon u\cdot\nabla Q_{\beta\gamma})|R_\varepsilon u\cdot\nabla Q|) \\ \qquad -\mathcal{P}\nabla\cdot R_\varepsilon(\nabla Q\odot\nabla Q) - \lambda\nabla\cdot\mathcal{P}R_\varepsilon(|Q|H) \\ \qquad +\mathcal{P}\nabla\cdot R_\varepsilon(Q\Delta Q - \Delta Q Q) + \kappa\mathcal{P}\nabla\cdot R_\varepsilon Q, \\ (Q,u)|_{t=0} = (R_\varepsilon Q_0, R_\varepsilon u_0). \end{cases}$$

- R_ε: convolution operator with kernel $\varepsilon^{-d}\chi(\varepsilon^{-1}\cdot)$. $\chi\in C_0^\infty$ is a radial positive function, with $\int_{\mathbb{R}^d}\chi(y)dy = 1$.
- $\mathcal{P} : L^2(\Omega)\to\{\mathbf{w}\in(L^2(\Omega))^d : \nabla\cdot\mathbf{w} = 0\}$.

Friedrichs' Scheme

$$\begin{cases} \partial_t Q^{(n)} + J_n((\mathcal{P}J_nR_\varepsilon u^n\cdot\nabla)J_nQ^{(n)}) - J_n(\mathcal{P}J_nR_\varepsilon\Omega^nJ_nQ^{(n)}) \\ \quad +J_n(J_nQ^{(n)}\mathcal{P}J_nR_\varepsilon\Omega^n) - \lambda J_n(|J_nQ^{(n)}|\mathcal{P}J_nR_\varepsilon D^n) = \Gamma H^{(n)}, \\ \partial_t u^n + \mathcal{P}J_n((\mathcal{P}J_nR_\varepsilon u^n\cdot\nabla)\mathcal{P}J_nu^n) - \mu\Delta\mathcal{P}J_nu^n \\ \quad = -\varepsilon\mathcal{P}J_nR_\varepsilon(\partial_\alpha J_nQ^{(n)}_{\beta\gamma}(R_\varepsilon J_nu^n\cdot\nabla J_nQ^{(n)}_{\beta\gamma})|R_\varepsilon J_nu^n\cdot\nabla J_nQ^{(n)}|) \\ \quad +\varepsilon\mathcal{P}\nabla\cdot J_nR_\varepsilon(\nabla J_nR_\varepsilon u^n|\nabla J_nR_\varepsilon u^n|^2) + \kappa\mathcal{P}\nabla\cdot J_nR_\varepsilon Q^{(n)}, \\ \quad -\mathcal{P}\nabla\cdot J_nR_\varepsilon(\nabla J_nQ^{(n)}\odot\nabla J_nQ^{(n)}) - \lambda\mathcal{P}\nabla\cdot J_nR_\varepsilon(|J_nQ^{(n)}|H^{(n)}) \\ \quad +\mathcal{P}\nabla\cdot J_nR_\varepsilon(J_nQ^{(n)}\Delta J_nQ^{(n)} - \Delta J_nQ^{(n)}J_nQ^{(n)}) \\ (Q^{(n)},u^n)|_{t=0} = (J_nR_\varepsilon Q_0, J_nR_\varepsilon u_0). \end{cases}$$

- $H^{(n)} = \Delta J_nQ^{(n)} - aJ_nQ^{(n)} + bJ_n[(J_nQ^{(n)})^2 - \frac{\mathrm{tr}(J_nQ^{(n)})^2}{d}\mathrm{I}_d] - cJ_n(J_nQ^{(n)}\mathrm{tr}(J_nQ^{(n)})^2)$.
- J_n: the mollifying operator, $\mathcal{F}(J_nf)(\xi) := 1_{[2^{-n},2^n]}(|\xi|)\mathcal{F}(f)(\xi)$.

Bernstein-Type Inequalities

For any $1 \leq p \leq q \leq \infty$, and $w \in L^p(\mathbb{R}^d)$,
$||D^k u||_{L^p} = \sup_{|\alpha|=k} ||\partial^\alpha u||_{L^p}$

- $C2^{nk}||J_n w||_{L^p} \leq ||D^k J_n w||_{L^p} \leq C2^{nk}||J_n w||_{L^p}$.
- $||D^k J_n w||_{L^q} \leq C2^{(k+d(\frac{1}{p}-\frac{1}{q}))n}||J_n w||_{L^p}$.

- Cauchy-Lipschitz theorem
  $\Rightarrow$ local existence of unique solution
  $(Q^{(n)}, u^n) \in C^1([0, T_n); L^2(\mathbb{R}^d; \mathbb{R}^{d\times d}) \times L^2(\mathbb{R}^d, \mathbb{R}^d))$.
- $(\mathcal{P}J_n)^2 = \mathcal{P}J_n$, $J_n^2 = J_n$
  $\Rightarrow$ $(J_n Q^{(n)}, \mathcal{P}J_n u^n)$ is also a solution.
- Uniqueness $\Rightarrow$ $(J_n Q^{(n)}, \mathcal{P}J_n u^n) = (Q^{(n)}, u^n)$.

Simplified Approximation System

$$\begin{cases} \partial_t Q^{(n)} + J_n((R_\varepsilon u^n \cdot \nabla)Q^{(n)}) - J_n(R_\varepsilon \Omega^n Q^{(n)} - Q^{(n)}(R_\varepsilon \Omega^n)) \\ \quad -\lambda J_n(|Q^{(n)}| R_\varepsilon D^n) = \Gamma \bar{H}^{(n)}, \\ \partial_t u^n + \mathcal{P} J_n((R_\varepsilon u^n \cdot \nabla) u^n) - \mu \Delta u^n \\ \quad = -\varepsilon \mathcal{P} J_n R_\varepsilon (\partial_\alpha Q^{(n)}_{\beta\gamma} (R_\varepsilon u^n \cdot \nabla Q^{(n)}_{\beta\gamma}) | R_\varepsilon u^n \cdot \nabla Q^{(n)}|) \\ \qquad + \varepsilon \mathcal{P} \nabla \cdot J_n R_\varepsilon (\nabla R_\varepsilon u^n |\nabla R_\varepsilon u^n|^2) \\ \qquad - \mathcal{P} \nabla \cdot J_n R_\varepsilon (\nabla Q^{(n)} \odot \nabla Q^{(n)}) - \lambda \mathcal{P} \nabla \cdot J_n R_\varepsilon (|Q^{(n)}| \bar{H}^{(n)}) \\ \qquad + \mathcal{P} \nabla \cdot J_n R_\varepsilon (Q^{(n)} \Delta Q^{(n)} - \Delta Q^{(n)} Q^{(n)}) + \kappa \mathcal{P} \nabla \cdot J_n R_\varepsilon Q^{(n)}, \\ (Q^{(n)}, u^n)|_{t=0} = (J_n R_\varepsilon Q_0, J_n R_\varepsilon u_0), \end{cases}$$

where $(Q^{(n)}, u^n) \in C^1([0, T_n); H^\infty(\mathbb{R}^d) \times H^\infty(\mathbb{R}^d))$ and

$$\bar{H}^n = \Delta Q^{(n)} - a Q^{(n)} + b J_n[(Q^{(n)})^2 - \frac{\mathrm{tr}(J_n(Q^{(n)})^2)}{d} \mathrm{I}_d] - c J_n(Q^{(n)} \mathrm{tr}(Q^{(n)})^2).$$

$$[0, T_n) \Rightarrow [0, T^*]$$

$$\frac{d}{dt}E_n^M(t) + \frac{\mu}{2}||\nabla u^n||_{L^2}^2 + \frac{\Gamma}{2}||\Delta Q^{(n)}||_{L^2}^2 + \frac{c^2\Gamma}{2}||J_n(Q^{(n)}|Q^{(n)}|^2)||_{L^2}^2$$
$$+ \frac{\varepsilon}{2}||R_\varepsilon u^n \cdot \nabla Q^{(n)}||_{L^3}^3 + \frac{\varepsilon}{2}||\nabla R_\varepsilon u^n||_{L^4}^4$$
$$\leq C(\varepsilon)\left(||Q^{(n)}||_{L^2}^2 + ||Q^{(n)}||_{L^4}^4 + ||u^n||_{L^2}^2 + ||\nabla Q^{(n)}||_{L^2}^2\right)$$

Global Existence for $(u^n, Q^{(n)})$

$$[0, T_n) \Rightarrow [0, T^*]$$

$$\frac{d}{dt}E_n^M(t) + \frac{\mu}{2}||\nabla u^n||_{L^2}^2 + \frac{\Gamma}{2}||\Delta Q^{(n)}||_{L^2}^2 + \frac{c^2\Gamma}{2}||J_n(Q^{(n)}|Q^{(n)}|^2)||_{L^2}^2$$
$$+ \frac{\varepsilon}{2}||R_\varepsilon u^n \cdot \nabla Q^{(n)}||_{L^3}^3 + \frac{\varepsilon}{2}||\nabla R_\varepsilon u^n||_{L^4}^4$$
$$\leq C(\varepsilon)\left(||Q^{(n)}||_{L^2}^2 + ||Q^{(n)}||_{L^4}^4 + ||u^n||_{L^2}^2 + ||\nabla Q^{(n)}||_{L^2}^2\right)$$

$$[0, T^*] \Rightarrow [0, T]$$

Global Existence for $(u^n, Q^{(n)})$

$$\sup_n ||R_\varepsilon u^n \cdot \nabla Q^{(n)}||_{L^3(0,T;L^3)} \leq C(\varepsilon),$$

$$\sup_n ||\nabla R_\varepsilon u^n||_{L^4(0,T;L^4)} \leq C(\varepsilon),$$

$$\sup_n ||Q^{(n)}||_{L^2(0,T;H^2)\cap L^\infty(0,T;H^1\cap L^4)} \leq C(\varepsilon),$$

$$\sup_n ||J_n(Q^{(n)}|Q^{(n)}|^2)||_{L^2(0,T;L^2)} \leq C(\varepsilon),$$

$$\sup_n |||\nabla Q^{(n)}||Q^{(n)}|||_{L^2(0,T;L^2)} \leq C(\varepsilon),$$

$$\sup_n ||u^n||_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)} \leq C(\varepsilon),$$

Convergence Result $n \to \infty$

- $\partial_t(u^n, Q^{(n)})$ is bounded in $L^1(0, T; H^{-N}(\mathbb{R}^d))$
- by the classical Aubin-Lions compactness Lemma,

$$
\begin{aligned}
&Q^{(n)} \rightharpoonup Q_\varepsilon \quad \text{in} \quad L^2(0, T; H^2),\\
&Q^{(n)} \to Q_\varepsilon \quad \text{in} \quad L^2(0, T; H^{2-\delta}_{loc}), \quad \forall \delta \in (0, 2+N)\\
&Q^{(n)}(t) \rightharpoonup Q_\varepsilon(t) \quad \text{in} \quad H^1 \quad \forall t \in \mathbb{R}_+,\\
&Q^{(n)}(t) \rightharpoonup Q_\varepsilon \quad \text{in } L^p(0, T; H^1),\\
&Q^{(n)} \to Q_\varepsilon \quad \text{in } L^p(0, T; H^{1-\delta}_{loc}) \quad \forall \delta \in (0, 1+N), p \in [2, \infty],\\
&u^n \rightharpoonup u_\varepsilon \quad \text{in} \quad L^2(0, T; H^1),\\
&u^n \to u_\varepsilon \quad \text{in} \quad L^2(0, T; H^{1-\delta}_{loc}) \quad \forall \delta \in (0, 1+N),\\
&u^n(t) \rightharpoonup u_\varepsilon(t) \quad \text{in} \quad L^2 \quad \forall t \in \mathbb{R}_+.
\end{aligned}
$$

$$Q_\varepsilon \in L^\infty_{loc}(\mathbb{R}_+; H^1) \cap L^2_{loc}(\mathbb{R}_+; H^2), u_\varepsilon \in L^\infty_{loc}(\mathbb{R}_+; L^2) \cap L^2_{loc}(\mathbb{R}_+; H^1).$$

For any compactly supported $\psi \in C^\infty([0,\infty)\times\mathbb{R}^d;\mathbb{R}^d)$, with $\nabla\cdot\psi = 0$,

$$
\begin{aligned}
&\lambda\int_0^\infty\int_{\mathbb{R}^d}\mathcal{P}J_nR_\varepsilon(|Q^{(n)}|\bar{H}^n)\cdot\nabla\psi\mathrm{d}x\mathrm{d}t\\
=&\lambda\int_0^\infty\int_{\mathbb{R}^d}|Q^{(n)}|\bar{H}^n\cdot J_nR_\varepsilon\nabla\psi\mathrm{d}x\mathrm{d}t\\
=&\lambda\int_0^\infty\int_{\mathbb{R}^d}(\Delta Q^{(n)} - aQ^{(n)} + b[(Q^{(n)})^2 + \frac{\mathrm{tr}(Q^{(n)})^2}{d}\mathrm{I}_d] - cQ^{(n)}\mathrm{tr}(Q^{(n)})^2)\\
&\cdot|Q^{(n)}|J_nR_\varepsilon\nabla\psi\mathrm{d}x\mathrm{d}t\\
\to&\lambda\int_0^\infty\int_{\mathbb{R}^d}|Q_\varepsilon|H_\varepsilon\cdot\nabla R_\varepsilon\psi\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Let $n \to \infty$, obtain a weak solution $(u_\varepsilon, Q_\varepsilon)$ to the regularized system

$$
\begin{cases}
\partial_t Q + (R_\varepsilon u)\cdot\nabla Q + Q(R_\varepsilon\Omega) - (R_\varepsilon\Omega)Q - \lambda|Q|(R_\varepsilon D) = \Gamma H,\\
\partial_t u + \mathcal{P}(R_\varepsilon u\cdot\nabla u) - \mu\Delta u = \varepsilon\mathcal{P}\nabla\cdot R_\varepsilon(\nabla R_\varepsilon u|\nabla R_\varepsilon u|^2)\\
\qquad -\varepsilon\mathcal{P}R_\varepsilon(\partial_\alpha Q_{\beta\gamma}(R_\varepsilon u\cdot\nabla Q_{\beta\gamma})|R_\varepsilon u\cdot\nabla Q|)\\
\qquad -\mathcal{P}\nabla\cdot R_\varepsilon(\nabla Q\odot\nabla Q) - \lambda\nabla\cdot\mathcal{P}R_\varepsilon(|Q|H)\\
\qquad +\mathcal{P}\nabla\cdot R_\varepsilon(Q\Delta Q - \Delta Q Q) + \kappa\mathcal{P}\nabla\cdot R_\varepsilon Q,\\
(Q,u)|_{t=0} = (R_\varepsilon Q_0, R_\varepsilon u_0).
\end{cases}
$$

$Q_\varepsilon \in L^\infty_{loc}(\mathbb{R}_+; H^1)\cap L^2_{loc}(\mathbb{R}_+; H^2)$, $u_\varepsilon \in L^\infty_{loc}(\mathbb{R}_+; L^2)\cap L^2_{loc}(\mathbb{R}_+; H^1)$.

Uniform Bounds for Regularized System

- $(Q_\varepsilon, u_\varepsilon)$ smooth.
- Uniform bounds w.r.t. ε

$$\sup_\varepsilon ||Q_\varepsilon||_{L^2(0,T;H^2)\cap L^\infty(0,T;H^1\cap L^4)\cap L^6(0,T;L^6)} \leq C,$$

$$\sup_\varepsilon ||u_\varepsilon||_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)} \leq C,$$

$$\varepsilon ||R_\varepsilon u \cdot \nabla Q||^3_{L^3(0,T;L^3)} < C,$$

$$\varepsilon ||\nabla R_\varepsilon u||^4_{L^4(0,T;L^4)} < C.$$

- $\partial_t(Q_\varepsilon, u_\varepsilon)$ is bounded in $L^1(0, T; H^{-2})$.

Convergence Result $\varepsilon \to 0$

Again by Aubin-Lions Lemma, on a subsequence,

$$
\begin{aligned}
&Q_\varepsilon \rightharpoonup Q \quad \text{in} \quad L^2(0,T;H^2),\\
&Q_\varepsilon \to Q \quad \text{in} \quad L^2(0,T;H^{2-\delta}_{loc}) \quad \forall \delta \in (0,4),\\
&Q_\varepsilon(t) \rightharpoonup Q(t) \quad \text{in} \quad H^1 \quad \forall t \in \mathbb{R}_+,\\
&Q_\varepsilon \rightharpoonup Q \quad \text{in } L^p(0,T;H^1),\\
&Q_\varepsilon \to Q \quad \text{in } L^p(0,T;H^{1-\delta}_{loc}) \quad \forall \delta \in (0,3), p \in [2,\infty],\\
&u_\varepsilon \rightharpoonup u \quad \text{in} \quad L^2(0,T;H^1),\\
&u_\varepsilon \to u \quad \text{in} \quad L^2(0,T;H^{1-\delta}_{loc}) \quad \forall \delta \in (0,3),\\
&u_\varepsilon(t) \rightharpoonup u(t) \quad \text{in} \quad L^2 \quad \forall t \in \mathbb{R}_+.
\end{aligned}
$$

$$Q \in L^\infty_{loc}(\mathbb{R}_+;H^1) \cap L^2_{loc}(\mathbb{R}_+;H^2), u \in L^\infty_{loc}(\mathbb{R}_+;L^2) \cap L^2_{loc}(\mathbb{R}_+;H^1)$$

Higher Regularity of Solutions for $d = 2$

Theorem (Chen-Majumdar-Wang-Z., '17)

Let $s > 0$ and $(Q_0, u_0) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$. There exists a pair of global solution $(Q(t,x), u(t,x))$ of the incompressible active LCs, subject to initial conditions, $Q(0,x) = Q_0(x)$, $u(0,x) = u_0(x)$, and

$$Q \in L^2_{loc}(\mathbb{R}_+; H^{s+2}(\mathbb{R}^2)) \cap L^\infty_{loc}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2)),$$
$$u \in L^2_{loc}(\mathbb{R}_+; H^{s+1}(\mathbb{R}^2)) \cap L^\infty_{loc}(\mathbb{R}_+; H^{s}(\mathbb{R}^2)).$$

Moreover, we have:

$$||\nabla Q(t,\cdot)||^2_{H^s} + ||u(t,\cdot)||^2_{H^s} \leq Ce^{e^{Ct}},$$

where the constant C depends on $Q_0, u_0, a, b, c, \Gamma$.

Sobolev Space and Bony's Paraproduct

- Dyadic Partition of Unity

$$\forall \xi \in \mathbb{R}^d, \chi(\xi) + \sum_{j \geq 1} \varphi(2^{-j}\xi) = 1,$$

where $\chi \in \mathcal{D}(B(0, \frac{4}{3}))$ and $\varphi \in \mathcal{D}(\mathcal{C})$ are radial, valued in the interval $[0, 1]$, with $\mathcal{C} := \{\xi \in \mathbb{R}^d | \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$

- Homogeneous dyadic blocks Δ_j and low-frequency cut-off operators S_j

$$\Delta_j u = \mathcal{F}^{-1}(\varphi(2^{-j}\xi)\mathcal{F}u) = 2^{jd} \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\varphi)(2^j y)u(x-y)dy$$

$$S_j u = \mathcal{F}^{-1}(\chi(2^{-j}\xi)\mathcal{F}u) = 2^{jd} \int_{\mathbb{R}^d} (\mathcal{F}^{-1}\chi)(2^j y)u(x-y)dy$$

Strategy

- Sobolev norm of the space H^s:

$$||u||_{H^s} := (||S_0 u||_{L^2}^2 + \sum_{q\in\mathbb{N}} 2^{2qs}||\Delta_q u||_{L^2}^2)^{\frac{1}{2}}.$$

- Bony's paraproduct decomposition

$$\begin{aligned}\Delta_j(uv) = & \; S_{j-1}u\Delta_j v \\ & + \sum_{|j-j'|\leq 5} [\Delta_j, S_{j'-1}u]\Delta_{j'} v \\ & + \sum_{|j-j'|\leq 5} (S_{j'-1}u - S_{j-1}u)\Delta_j\Delta_{j'} v \\ & + \sum_{j'>j-5} \Delta_j(S_{j'+2}v\Delta_{j'} u)\end{aligned}$$

Strategy

- Sobolev norm of the space H^s:

$$||u||_{H^s} := (||S_0 u||_{L^2}^2 + \sum_{q\in\mathbb{N}} 2^{2qs}||\Delta_q u||_{L^2}^2)^{\frac{1}{2}}.$$

- Bony's paraproduct decomposition

$$\begin{aligned}\Delta_j(uv) = {} & S_{j-1}u\Delta_j v \\ & + \sum_{|j-j'|\le 5} [\Delta_j, S_{j'-1}u]\Delta_{j'} v \quad \text{commutator inequality} \\ & + \sum_{|j-j'|\le 5} (S_{j'-1}u - S_{j-1}u)\Delta_j\Delta_{j'} v \quad \text{Bernstein inequality} \\ & + \sum_{j'>j-5} \Delta_j(S_{j'+2}v\Delta_{j'} u) \quad \text{Young inequality}\end{aligned}$$

Strategy

- estimates of high frequencies, apply Δ_q, $q \in \mathbb{N}$
- estimates of low frequencies, apply S_0
- high regularity of the solution (Gronwall's inequality)

Weak-Strong Uniqueness for $d = 2$

Theorem (Chen-Majumdar-Wang-Z., '17)

Let $(Q_0, u_0) \in H^{s+1}(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$ with $s > 0$ be the initial data. By the first theorem, there exists a weak solution (Q_1, u_1) of the incompressible active LCs, such that

$$Q_1 \in L^\infty_{loc}(\mathbb{R}_+; H^1) \cap L^2_{loc}(\mathbb{R}_+; H^2), \quad u_1 \in L^\infty_{loc}(\mathbb{R}_+; L^2) \cap L^2_{loc}(\mathbb{R}_+; H^1).$$

The second theorem gives the existence of a strong solution (Q_2, u_2) such that

$$Q_2 \in L^\infty_{loc}(\mathbb{R}_+; H^{s+1}) \cap L^2_{loc}(\mathbb{R}_+; H^{s+2}), u_2 \in L^\infty_{loc}(\mathbb{R}_+; H^s) \cap L^2_{loc}(\mathbb{R}_+; H^{s+1}).$$

Then $(Q_1, u_1) = (Q_2, u_2)$.

Weak-Strong Uniqueness for $d = 2$

By denoting $\delta Q = Q_1 - Q_2$ and $\delta u = u_1 - u_2$, then $(\delta Q, \delta u)$ satisfies the following system:

$$\begin{cases} \partial_t \delta Q + (\delta u \cdot \nabla)\delta Q - \delta\Omega\delta Q + \delta Q\delta\Omega + \delta u \cdot \nabla Q_2 \\ \quad + u_2 \cdot \nabla \delta Q + Q_2\delta\Omega + \delta Q\Omega_2 - \delta\Omega Q_2 - \Omega_2\delta Q \\ \quad = \lambda|Q_1|\delta D + \lambda(|Q_1| - |Q_2|)D_2 \\ \quad + \Gamma\{\Delta\delta Q - a\delta Q - c(\delta Q \mathrm{tr} Q_1^2 + Q_2 \mathrm{tr}(Q_1\delta Q + \delta Q Q_2))\}, \\ \partial_t \delta u + \mathcal{P}(\delta u \cdot \nabla \delta u) = \mu\Delta\delta u - \mathcal{P}(\nabla \cdot (\nabla\delta Q \odot \nabla\delta Q)) \\ \quad + \mathcal{P}(\nabla \cdot (\delta Q\Delta\delta Q - \Delta\delta Q\delta Q)) - \mathcal{P}(u_2 \cdot \nabla\delta u + \delta u \cdot \nabla u_2) \\ \quad + \mathcal{P}(\nabla \cdot (\delta Q\Delta Q_2 + Q_2\Delta\delta Q - \Delta\delta Q Q_2 - \Delta Q_2\delta Q)) \\ \quad - \mathcal{P}(\nabla \cdot (\nabla\delta Q \odot \nabla Q_2 + \nabla Q_2 \odot \nabla\delta Q)) \\ \quad - \lambda\mathcal{P}(\nabla \cdot (|Q_1|(\Delta\delta Q - a\delta Q))) + \lambda c\mathcal{P}(\nabla \cdot (|Q_1|\delta Q \mathrm{tr} Q_1^2)) \\ \quad - \lambda\mathcal{P}(\nabla \cdot (|Q_1| - |Q_2|)(\Delta Q_2 - aQ_2)) \\ \quad + \lambda c\mathcal{P}(\nabla \cdot (|Q_1|Q_2 \mathrm{tr}(Q_1\delta Q + \delta Q Q_2))) \\ \quad + \lambda c\mathcal{P}(\nabla \cdot ((|Q_1| - |Q_2|)Q_2 \mathrm{tr} Q_2^2)) + \kappa\mathcal{P}(\nabla \cdot \delta Q). \end{cases}$$

Weak-Strong Uniqueness for $d = 2$

- multiply $\delta Q-$equation by $-\Delta \delta Q + \delta Q$
- multiply $\delta u-$equation by δu

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla P - \mu \Delta u = -\nabla \cdot (\nabla Q \odot \nabla Q) \\ \qquad -\lambda \nabla \cdot (|Q| H) + \nabla \cdot (Q \Delta Q - \Delta Q Q) + \kappa \nabla \cdot Q, \\ Q_t + (u \cdot \nabla) Q + Q\Omega - \Omega Q - \lambda |Q| D = \Gamma H, \\ \nabla \cdot u = 0, \end{cases}$$

where $\nabla Q \odot \nabla Q = \partial_i Q_{kl} \partial_j Q_{kl}$, $D = \frac{\nabla u + \nabla u^T}{2}$, $\Omega = \frac{\nabla u - \nabla u^T}{2}$,

$$H = \Delta Q - aQ + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{d} \mathrm{I}_d] - cQ\mathrm{tr}(Q^2).$$

Initial and Boundary Conditions

- Initial conditions

$$(\rho, \rho u, Q)|_{t=0} = (\rho_0(x), m_0(x), Q_0(x)), \quad x \in \mathcal{O},$$

with

$$Q_0 \in H^1(\mathcal{O}), \quad Q_0 \in S_0^3 \quad \text{a.e. in } \mathcal{O},$$

- Boundary conditions

$$u|_{\partial\mathcal{O}} = 0, \quad \frac{\partial Q}{\partial \vec{n}}|_{\partial\mathcal{O}} = 0,$$

- Compatibility conditions

$$\rho_0 \in L^\infty(\mathcal{O}), \quad \rho_0 \geq 0; \quad m_0 \in L^2(\mathcal{O}), \quad \frac{|m_0|^2}{\rho_0} \in L^1(\mathcal{O}),$$

$$m_0 = 0, \quad \text{if} \quad \rho_0 = 0.$$

Existence of Global Weak Solutions

Theorem (Wang-Z., Submitted)

The inhomogeneous active liquid crystal system admits a weak solution (ρ, u, Q) with the initial and boundary conditions.

Galerkin approximation

Main Difficulties

- $\rho_0 \geq 0$(vacuum): Hodge-de Rham type decomposition due to Lions
- higher nonlinear terms $\lambda\nabla \cdot (|Q|H)$

Global Weak Solutions

Regularity

$$\begin{cases} \rho \geq 0, \rho \in L^\infty(\mathcal{O}_T), \rho \in C([0,T]; L^p(\mathcal{O})),\ 1 \leq p < \infty, \\ u \in L^2(0,T; H_0^1(\mathcal{O})),\ \text{and } \rho|u|^2 \in L^\infty(0,T; L^1(\mathcal{O})), \\ Q \in L^\infty(0,T; H^1(\mathcal{O})) \cap L^2(0,T; H^2(\mathcal{O})),\ \text{and } Q \in S_0^3 \ a.e. \ \text{in } [0,T] \times \mathcal{O}; \end{cases}$$

Energy inequality

$$\frac{d}{\mathrm{d}t} E(t) + \frac{\mu}{2} ||\nabla u||_{L^2}^2 + \Gamma \int_{\mathcal{O}} \mathrm{tr}(H^2) \mathrm{d}x \leq C(\kappa, \mu) ||Q||_{L^2}^2.$$

$$-\int_0^T\int_{\mathcal{O}}(\rho\partial_t\zeta+\rho u\cdot\nabla\zeta)\mathrm{d}x\mathrm{d}t=\int_{\mathcal{O}}\rho_0\zeta(0,x)\mathrm{d}x;$$

$$\int_0^T\int_{\mathcal{O}}(-\rho u\cdot\partial_t\psi-(\rho u\otimes u):\nabla\psi+\mu\nabla u:\nabla\psi)\mathrm{d}x\mathrm{d}t-\int_{\mathcal{O}}m_0(x)\cdot\psi(0,x)\mathrm{d}x$$
$$=\int_0^T\int_{\mathcal{O}}\nabla\cdot(-\nabla Q\odot\nabla Q-\lambda|Q|H[Q]+Q\Delta Q-\Delta QQ+\kappa Q)\cdot\psi\mathrm{d}x\mathrm{d}t;$$

$$\int_0^T\int_{\mathcal{O}}\Big(Q:\partial_t\varphi+\Gamma\Delta Q:\varphi+Q:(u\nabla_x\varphi)-(Q\Omega-\Omega Q-\lambda|Q|D):\varphi\Big)dxdt$$
$$=\Gamma\int_0^T\int_{\mathcal{O}}\Big(aQ-b\big(Q^2-\frac{\mathrm{tr}(Q^2)}{d}\mathrm{I}_d\big)+cQ\mathrm{tr}(Q^2)\Big):\varphi dxdt$$
$$-\int_{\mathcal{O}}Q_0(x):\varphi(0,x)dx;$$

$$
\begin{cases}
c_t + u \cdot \nabla c = D_0 \Delta c, \\
\rho_t + \nabla \cdot (\rho u) = 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla \rho^\gamma - \mathcal{L} u = \nabla \cdot (F(Q)\mathrm{I}_3 - \nabla Q \odot \nabla Q) \\
\qquad\qquad + \nabla \cdot (Q \Delta Q - \Delta Q Q) + \sigma_* \nabla \cdot (c^2 Q), \\
Q_t + (u \cdot \nabla) Q + Q\Omega - \Omega Q = \Gamma H[Q, c].
\end{cases}
$$

$$
H[Q,c] = \Delta Q - \frac{c - c_*}{2} Q + b[Q^2 - \frac{\mathrm{tr}(Q^2)}{3}\mathrm{I}_3] - c_* Q \mathrm{tr}(Q^2)
$$

$$
F(Q) = \frac{1}{2}|\nabla Q|^2 + \frac{1}{2}\mathrm{tr}(Q^2) + \frac{c_*}{4}\mathrm{tr}^2(Q^2)
$$

$$
\mathcal{L}u = \mu \Delta u + (\nu + \mu)\nabla \mathrm{div} u
$$

Initial and Boundary Conditions

- Initial conditions

$$(c, \rho, \rho u, Q)|_{t=0} = (c_0(x), \rho_0(x), m_0(x), Q_0(x)), \quad x \in \mathcal{O},$$
$$c_0 \in H^1(\mathcal{O}), \quad 0 < \underline{c} \le c_0 \le \bar{c} < \infty,$$
$$Q_0 \in H^1(\mathcal{O}), \quad Q_0 \in S_0^3 \quad a.e.\ in\ \mathcal{O},$$

- Boundary conditions

$$u|_{\partial\mathcal{O}} = 0, \quad \frac{\partial Q}{\partial \vec{n}}|_{\partial\mathcal{O}} = 0,$$

- Compatibility conditions

$$\rho_0 \in L^\gamma(\mathcal{O}), \quad \rho_0 \ge 0; \quad m_0 \in L^1(\mathcal{O}), \quad \frac{|m_0|^2}{\rho_0} \in L^1(\mathcal{O}),$$
$$m_0 = 0, \quad \text{if} \quad \rho_0 = 0.$$

A priori estimates

Multiplying the concentration equation by c, momentum equation by u, Q tenor equation by $-(\Delta Q - Q - c_* Q|Q|^2)$

$$\frac{dE(t)}{dt} + \frac{D_0}{2}\|\nabla c\|_{L^2}^2 + \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + (\nu+\mu)\|\mathrm{div}\, u\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2 + \frac{c_*^2\Gamma}{2}\|Q\|_{L^6}^6$$
$$\leq C(\|u\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2 + \|Q\|_{L^2}^2 + \|Q\|_{L^4}^4),$$

where

$$E(t) := \int_{\mathcal{O}} (\frac{1}{2}|c|^2 + \frac{1}{2}\rho|u|^2 + \frac{\rho^\gamma}{\gamma-1} + \frac{1}{2}|Q|^2 + \frac{1}{2}|\nabla Q|^2 + \frac{c_*}{4}|Q|^4)\mathrm{d}x.$$

Existence of Finite Energy Weak Solution

Theorem (Chen-Majumdar-Wang-Z., Submitted)

Let $\gamma > \frac{3}{2}$ *and the initial data* $(c_0(x), \rho_0(x), m_0(x), Q_0(x))$ *satisfies the compatibility conditions. Then for any* $T > 0$*, the compressible active liquid crystal system admits a finite energy weak solution* (c, ρ, u, Q) *on* $\mathcal{O}_T$*.*

Finite Energy Weak Solution

1. $c \in L^\infty(0, T; L^2(\mathcal{O})) \cap L^2(0, T; H^1(\mathcal{O}))$, $\rho \geq 0, \rho \in L^\infty(0, T; L^\gamma(\mathcal{O}))$, $u \in L^2(0, T; H_0^1(\mathcal{O}))$, $Q \in L^\infty(0, T; H^1(\mathcal{O})) \cap L^2(0, T; H^2(\mathcal{O}))$, and $Q \in S_0^3$ *a. e.* in $\mathcal{O}_T = \mathcal{O} \times [0, T]$
2. Equations are valid in $\mathcal{D}'(0, T; \mathcal{O})$
3. Energy inequality

$$\frac{dE(t)}{dt} + \frac{D_0}{2}\|\nabla c\|_{L^2}^2 + \frac{\mu}{2}\|\nabla u\|_{L^2}^2 + (\nu+\mu)\|\mathrm{div}\, u\|_{L^2}^2 + \frac{\Gamma}{2}\|\Delta Q\|_{L^2}^2 + \frac{c_*^2\Gamma}{2}\|Q\|_{L^6}^6 \leq C(\|u\|_{L^2}^2 + \|\nabla Q\|_{L^2}^2 + \|Q\|_{L^2}^2 + \|Q\|_{L^4}^4)$$

4. $\forall g \in C^1(\mathbb{R}), g'(z) \equiv 0$, for all $z \geq M$, where M is a sufficiently large constant

$$\partial_t g(\rho) + \mathrm{div}(g(\rho)u) + (g'(\rho)\rho - g(\rho))\mathrm{div}\, u = 0$$

Approximation System

$$
\begin{cases}
c_t + u \cdot \nabla c = D_0 \Delta c, \\
\rho_t + \nabla \cdot (\rho u) = \varepsilon \Delta \rho, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) + \nabla \rho^\gamma + \delta \nabla \rho^\beta + \varepsilon \nabla u \cdot \nabla \rho \\
\qquad = \mathcal{L} u + \nabla \cdot (\mathrm{F}(Q)\mathrm{I}_3 - \nabla Q \odot \nabla Q) \\
\qquad\quad + \nabla \cdot (Q \Delta Q - \Delta Q Q) + \sigma_* \nabla \cdot (c^2 Q), \\
\partial_t Q + (u \cdot \nabla) Q + Q\Omega - \Omega Q = \Gamma H[Q, c]
\end{cases}
$$

Faedo-Galerkin approximation, artificial viscosity, artificial pressure, as well as the weak convergence methods

Open Problems

- The original system coupled with complicated Concentration,
- Inviscid limit,
- Uniqueness of weak solutions for incompressible case,
- Weak-strong uniqueness for compressible case,
- $\cdots$

Thank You!