Global Instabilities in Nematic Liquid Crystals

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11.28.2017 at BIRS

Partial Order in Materials: at the Triple Point of Mathematics, Physics and Applications

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Ericksen-Leslie System

General EL system is given by :

$$\begin{cases} \partial_t n + u \cdot \nabla n - \Omega\, n + \frac{\lambda_2}{\lambda_1} An = \frac{1}{|\lambda_1|}\left(\Delta n + |\nabla n|^2 n\right) + \frac{\lambda_2}{\lambda_1}\langle An, n\rangle\, n; \\ \partial_t u + u \cdot \nabla u + \nabla p = -\nabla \cdot (\nabla n \odot \nabla n) + \nabla \cdot \sigma^L; \\ \mathrm{div} u = 0; \end{cases} .$$

Here σ^L is the Leslie stress tensor which can be read as :

$$\alpha_1 \langle An, n\rangle\, n \otimes n + \alpha_2 N \otimes n + \alpha_3 n \otimes N + \alpha_4 A + \alpha_5 (An) \otimes n + \alpha_6 n \otimes (An).$$

Simplified EL system

To simplify and meanwhile preserve the dissipative properties of EL, the following simplified system was proposed by LIN in 1991:

$$\begin{cases} \partial_t \phi + v \cdot \nabla \phi - \Delta \phi = |\nabla \phi|^2 \phi\,, & \text{in } \mathbb{R}^n \times (0,\infty); \\ \partial_t v + v \cdot \nabla v - \Delta v = -\nabla p - \nabla \cdot \left(\nabla\phi \odot \nabla\phi \right), & \text{in } \mathbb{R}^n \times (0,\infty); \\ \nabla \cdot v = 0, & \text{in } \mathbb{R}^n \times (0,\infty). \end{cases}$$

Here v is velocity. ϕ is the orientation variable. $n = 2, 3$.

This system is a coupled system with Navier-Stokes equation and the transported heat flow of harmonic maps. We review some known results in heat flow of harmonic maps.

(1). Guan-Gustafuson-Tsai and Gustafuson-Nakanishi-Tsai studied global m-equivariant solution of the heat flow of harmonic maps in 2D. Solution constructed exist globally in time if $|m| \geq 2$. When $|m| \geq 3$, asymptotic limit exists. When $|m| = 2$, solution exists globally in time, but can osciallate or blow up at $t = \infty$. When $|m| = 1$, they find a solution blow up at finite time. The spatial domain considered by the authors is $\mathbb{R}^2$;

(2). For the bounded domain case, Ding et. al. showed the existence of blow-up at finite time. The optimal blow-up rate for the heat flow of harmonic maps on 2D bounded domain were shown by Raphaël-Schweyer (2011);

(3). Angenent-Hulshof (2005) constructed a solution which exists globally in time and blows up as $t \to \infty$;

Navier-Stokes Equation in $\mathbb{R}^2$

(1). As far as the existence result for the vorticity formulation of Navier-Stokes equation be concerned, Giga-Miyakawa-Osada proved in 1988 the global well-posedness in $\mathbb{R}^2$ with a given initial vorticity in measure space;

(2). In 1994 Ben-Artzi showed that for any given L^1 vorticity, the Navier-Stokes equation can be solved globally in 2-D;

(3). The result in (2) has been generalized to include any positive Radon measure valued initial vorticity in a book by Ben-Artzi.

(4). Giga-Kambe (1988) considered stability of Oseen vortices. They assumed the smallness of the circulation Reynolds number;

(5). Later in 2005, Galley-Wayne dropped the smallness assumption on the circulation Reynolds number. They obtained global stability result for the Oseen vortices in 2D.

Ericksen-Leslie Equation

Employing methods from harmonic maps, Ginzburg-Landau approximation and Navier-Stokes, many authors have studied the well-posedness of Ericksen-Leslie system in both 2-D and 3-D, for both strong solution and global weak solution. Some results are listed as follows.

(1). In 2-D bounded domain case, Lin-Lin-Wang in 2010 proved the global existence of weak solutions for the simplified Ericksen-Leslie system. The solution obtained by them might blow up at finite times;

(2). As for the 3-D case, in 2013, the existence theory of the Ericksen-Leslie system has been established by Wang-Zhang-Zhang. In 2014, Lin-Wang considered global existence of the simplified (EL) system in 3-D with initial director field being in $\mathbb{S}^2_+$;

(3). Recently finite-time blow-up solutions in 3D bounded domain have also been found by Huang-Lin-Liu-Wang;

(4). In the Ginzburg-Landau approximation of the general Ericksen-Leslie system, some stability results have also been established by Wu-Xu-Liu (2013).

(6). Uniqueness of the Ericksen-Leslie system is also considered by Lin-Wang and Li-Titi-Xin;

(5). Compressible Ericksen-Leslie system is also considered by Hu-Wu (2013), Huang-Wang-Wen (2012), Jiang-Jiang-Wang (2014) and Li-Xu-Zhang.

What happens when vorticity meets orientation variable ?

Vorticity Formulation of Simplified EL System

Taking curl on both sides of the second equation yields:

$$\begin{cases} \partial_t \phi + v \cdot \nabla \phi - \Delta \phi = |\nabla \phi|^2 \phi \,, & \text{in } \mathbb{R}^2 \times (0, \infty); \\ \partial_t \omega + v \cdot \nabla \omega - \Delta \omega = -\nabla \times \nabla \cdot \left(\nabla \phi \odot \nabla \phi \right) , & \text{in } \mathbb{R}^2 \times (0, \infty). \end{cases}$$

Here the velocity v can be recovered by the Biot-Savart law:

$$v = \mathrm{K} * \omega, \qquad \text{where } \mathrm{K}(x) = \frac{1}{2\pi} \frac{x^{\perp}}{|x|^2}.$$

Theorem (Chen-Y.)

Suppose that (ϕ_0, ω_0) *is an initial data in* $\mathrm{H}^1_e(\mathbb{R}^2; \mathbb{S}^2) \times \mathrm{L}^1(\mathbb{R}^2)$. *Then there exists a* $T_* > 0$ *and a smooth solution, denoted by* (ϕ, ω), *of the vorticity formulation of EL on* $(0, T_*)$ *so that*

(i). *As* $t \downarrow 0$, *we have*

$$\big(\phi(\cdot, t) - e, \omega(\cdot, t)\big) \longrightarrow \big(\phi_0 - e, \omega_0\big), \quad \textit{strongly in } \mathrm{H}^1 \times \mathrm{L}^1.$$

The velocity $v = \mathrm{K} * \omega$ *satisfies*

$$v(\cdot, t) \longrightarrow v_0 = \mathrm{K} * \omega_0, \ \textit{strongly in } \big(\mathrm{L}^1 \cap \mathrm{L}^p\big)^*, \textit{ for all } p > 2.$$

Moreover we also have

$$\big(\phi - e, \omega\big) \in \mathrm{L}^\infty\big([0, T_*]; \mathrm{H}^1\big(\mathbb{R}^2\big)\big) \times \mathrm{L}^\infty\big([0, T_*]; \mathrm{L}^1\big(\mathbb{R}^2\big)\big).$$

(ii). Fixing a $\tau \in (0, T_*)$ and denoting by $\bar{\omega}$ the unique mild solution of the following initial value problem:

$$\begin{cases} \partial_t \bar{\omega} - \Delta \bar{\omega} + \bar{v} \cdot \nabla \bar{\omega} = 0, & \text{on } \mathbb{R}^2 \times (\tau, \infty); \\ \bar{\omega}(\cdot, \tau) = \omega(\cdot, \tau); \qquad \bar{v} = \mathrm{K} * \bar{\omega}, & \end{cases}$$

then we can decompose the velocity field v into the sum

$$v = \bar{v} + v^*, \qquad \text{on } \mathbb{R}^2 \times [\tau, T_*]. \tag{1}$$

The velocity field v^* lies in the space

$$\mathrm{L}^\infty\big([\tau, T_*]; \mathrm{L}^2(\mathbb{R}^2)\big) \cap \mathrm{L}^2\big([\tau, T_*]; \mathrm{H}^1(\mathbb{R}^2)\big)$$

and satisfies the global energy inequality given below:

$$\int_{\mathbb{R}^2 \times \{t_2\}} \big|v^*\big|^2 + \big|\nabla\phi\big|^2 + \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \big|\nabla v^*\big|^2 + \big|\Delta\phi + |\nabla\phi|^2\phi\big|^2$$

$$\leq \exp\left\{\mathrm{c}\int_{t_1}^{t_2} \big\|\nabla\bar{v}\big\|_\infty\right\} \int_{\mathbb{R}^2\times\{t_1\}} \big|v^*\big|^2 + \big|\nabla\phi\big|^2.$$

Moreover as $t \downarrow \tau$, $v^*(\cdot, t)$ converges to 0 strongly in L^2.

(iii). If $\omega_0 \in \mathrm{L}^1 \cap \mathrm{L}^p$ for some $p > 1$, then τ in part (ii) can take value 0. The decomposition of the velocity field v in (1) holds on $\mathbb{R}^2 \times [0, T_*]$.

The solution obtained in the last theorem can be extended globally.

Theorem (Chen-Y.)

The solution obtained in the last theorem exists on $\mathbb{R}^2 \times (0, \infty)$ *if*

$$\left\|\nabla \phi_0\right\|_2 \le \epsilon.$$

Here $\epsilon > 0$ *is a number suitably small. Moreover on* $(0, \infty)$*, the extended solution is smooth.*

Using the energy inequality in (ii) of the first theorem, we have

Theorem (Chen-Y.)

Given $(\phi_0, \omega_0) \in \mathrm{H}_e^1\big(\mathbb{R}^2; \mathbb{S}^2\big) \times \mathrm{L}^1\big(\mathbb{R}^2\big)$*, there exists a global weak solution of the vorticity formulation of the simplified EL equation.*

How about explicit long-time descriptions of solutions ?

I. Oscillation Instability

Suppose the following m-equivariant ansatz for solutions:

$$v = \frac{x^{\perp}}{|x|^2} w(r,t) \qquad \text{and} \qquad \phi = e^{m\theta \mathrm{R}} \psi(r,t).$$

Simple calculations imply that

$$\left(0, e^{m\theta\mathrm{R}+\alpha\mathrm{R}} \, \mathbf{h}\left(\frac{r}{\sigma}\right)\right)$$

is an equilibrium solution of the simplified EL system. Here $\alpha \in \mathbb{R}$ and $\sigma > 0$ are two constants. R is the generator of the horizontal rotations. The vector field $\mathbf{h}$ is a 3-vector with

$$\mathbf{h}_1(r) = \frac{2}{r^{|m|} + r^{-|m|}}, \quad \mathbf{h}_2(r) \equiv 0, \quad \mathbf{h}_3(r) = \frac{r^{|m|} - r^{-|m|}}{r^{|m|} + r^{-|m|}}.$$

Theorem (Chen-Y.)

Suppose the following perturbation on the equilibrium solutions:

$$W_{in}(r) = \underbrace{\omega\left(1 - e^{-r^2/4}\right)}_{\text{OS Vortices with Reynolds Number } \omega} + W_{in}^*(r),$$

$$\psi_{in}(r) = \mathbf{h}(r) + \gamma(r)\mathbf{h}(r) + z_1(r)\mathbf{e}_2 + z_2(r)\mathbf{h}(r) \times \mathbf{e}_2.$$

Then

(i). The simplified EL admits a classical global solution;

(ii). For all $t \geq 0$, the solution W can be decomposed by

$$W = W^{os} + W_1^* + W_2^*,$$

where

$$W^{os} = \underbrace{\omega\left(1 - e^{-r^2/4(t+1)}\right)}_{\text{Main Flow with } \infty\text{-Kinetic Energy}},$$

$$W_1^* = \underbrace{\beta\left(\frac{r}{t+1}\right)^2 e^{-r^2/4(t+1)}}_{\text{Secondary Flow with } 1/t^2 \text{ - decay Kinetic Energy}}, \text{ with } \beta = \frac{1}{8}\int_0^\infty W_{in}^*.$$

W_1^*/r^2 - heat kernel in $\mathbb{R}^4$. Moreover for all $t \geq 0$, it holds

$$\left\| \frac{W_2^*}{r} \right\|_{L^2}^2 \lesssim (1+t)^{-(p-1)}, \qquad \text{for some } p > 3;$$

(iii). There exist $(\alpha(t), \sigma(t))$ with $(\alpha_{in}, \sigma_{in}) = (0, 1)$ and

$$z(\rho, t) = z_1(\rho, t) + i z_2(\rho, t) \qquad \text{with } \rho = r \big/ \sigma(t),$$

by which ψ can be expressed as follows:

$$\psi(r, t) = e^{\alpha(t)\mathrm{R}} \Big\{ \mathbf{h}(\rho) + \gamma(\rho, t)\mathbf{h}(\rho) + z_1(\rho, t)\mathbf{e}_2 + z_2(\rho, t)\mathbf{h}(\rho) \times \mathbf{e}_2 \Big\}.$$

For all $t \geq 0$, z satisfy the time-decay estimate given as below:

$$\| z \|_X \;\lesssim\; (1 + t)^{-(p-1)\big/2}, \quad \text{for some } p > 3;$$

(iv). As $t \to \infty$, we have

$$\sigma(t) \longrightarrow \sigma_\infty > 0 \quad \text{and} \quad \alpha(t) + \frac{m\omega}{4} \ln t \longrightarrow \alpha_\infty.$$

It is the second limit above generate the oscillation instability of the simplified EL system. The orientation variable rotates counterclockwisely or clockwisely depends on the sign of $m\omega$.

II. Concentration Instability

Consider the equation of simplified EL on $\mathbb{R}^3$ and use the following twisted *m*-equivariant ansatz:

$$v = \frac{W(r,t)}{r^2}\begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ V(r,t) \end{pmatrix}, \ \ \psi = e^{\mu x_3 \mathrm{R} + m\theta \mathrm{R}}\varphi(r,t).$$

Here $\mu > 0$ is the twisted rate. Our solution constructed is twisted and periodic along the x_3 - axis.

Theorem (Chen-Kim-Y.)

Under similar perturbation assmptions as the last theorem, we have

(i). Solution exists classically and globally.

(ii). The functions W and V can be decomposed into

$$W = W^{os} + W_1^* + W_2^* \qquad \textit{and} \qquad V = V_1 + V_2,$$

respectively. Moreover V_1, V_2, W_1^*, W_2^* satisfy

$$\| V_1 \|_{\mathrm{L}^\infty}^2 + \left\| \frac{W_1^*}{r^2} \right\|_{\mathrm{L}^\infty} \lesssim t^{-1}, \; \| V_2 \|_{\mathrm{L}^2} + \left\| \frac{W_2^*}{r} \right\|_{\mathrm{L}^2} \lesssim (1+t)^{-1};$$

(iii). The variable z can be estimated by

$$\int_0^\infty \exp\left\{\frac{2\mu^2}{m^2}s\right\} \|z(\cdot,s)\|_X^2 \,\mathrm{d}s \lesssim 1;$$

(iv). The scaling function σ decays exponentially:

$$\sigma(t) \sim \exp\left\{-\frac{\mu^2}{m^2}t\right\}, \quad \text{as } t \to \infty.$$

Remark 1 It is this exponential decay that makes the orientation variable blow up at $t = \infty$.

Remark 2 One can also observe that as $t \to \infty$, the orientation variable will converges to $\mathbf{e}_3$ for all $r > 0$. At $r = 0$, the orientation variable equals to $-\mathbf{e}_3$. In other words the solution will blow up to a constant map exponentially. The bubble is m-equivariant harmonic map.

Remark 3 For the work of Gustafuson et al. they found a global solution of heat flow of harmonic maps in 2-D, which do not blow up at $t = \infty$. The reason for our blow-up is also different from the work Angenent-Hulshof. Ours is due to non-zero twist rate.

Remark 4 Compare our two results, when the twist rate $\mu = 0$, the problem without twist is reduced to the oscillation problem. But we find that the two system behave quite different when $t \to \infty$. In fact

$$\begin{array}{ccc} \text{Twist Model} & \xrightarrow{\mu \longrightarrow 0} & \text{Vorticity formulation of EL in 2-D} \\ \Big\downarrow t \to \infty & & \Big\downarrow t \to \infty \\ \text{Blow up} & & \text{Slow oscillation} \end{array}$$

Therefore even for $\mu > 0$ suitably small, the twist model should not be regarded as a perturbation system of the un-twist case.

In the following, we take a brief look at the proofs of the above consequences.

(1). We take initial data so that

$$\varphi_{in}(r) = e^{\Theta_{in}\mathrm{R}}\Big\{\, \mathbf{h}(\rho) + \gamma_{in}(\rho)\mathbf{h}(\rho) + z_{1,in}(\rho)\mathbf{e}_2 + z_{2,in}(\rho)\mathbf{h}(\rho) \times \mathbf{e}_2 \,\Big\},$$

(2). We assume the solution satisfy the ansatz:

$$\varphi(r,t) = e^{\Theta(t)\,\mathrm{R}}\Big\{\, \mathbf{h}(\rho) + \gamma(\rho,t)\mathbf{h}(\rho) + z_1(\rho,t)\mathbf{e}_2 + z_2(\rho,t)\mathbf{h}(\rho)\times\mathbf{e}_2 \,\Big\},$$

Then the z variable satisfies the equation

$$\partial_t z + \frac{1}{\sigma^2} N z = \mathrm{Mod} + \mathrm{HT}, \qquad \text{where}$$

$$\mathrm{Mod} := -\Big\{\big(1+\gamma\big)\mathbf{h}_1 + i\,\mathbf{h}_3 z\Big\}\left(\Theta' + \mu V + \frac{mW}{r^2}\right)$$

$$+\frac{\sigma'}{\sigma}\Big\{i\big(1+\gamma\big)m\mathbf{h}_1 + \rho\,\partial_\rho z\Big\} + \mu^2\Big\{i\big(1+\gamma\big)\mathbf{h}_1\mathbf{h}_3 + i\mathbf{h}_1^2 z_2 - \mathbf{h}_3^2 z\Big\};$$

$$\mathrm{HT} := \frac{i}{\sigma^2}\,\frac{2m\mathbf{h}_1}{\rho}\,\partial_\rho\gamma$$

$$+\left(\frac{m^2}{\rho^2\sigma^2} + \mu^2\right) z\Big\{z_1^2 + \big(\gamma\mathbf{h}_1 - z_2\mathbf{h}_3\big)^2 + 2\mathbf{h}_1\big(\gamma\mathbf{h}_1 - z_2\mathbf{h}_3\big)\Big\}$$

(3). The operator N is given by

$$-N := -\mathrm{L}_{\mathbf{h}}^{*}\mathrm{L}_{\mathbf{h}} = \partial_{\rho\rho}+\frac{1}{\rho}\partial_{\rho}+\frac{m^2}{\rho^2}\Big(2\mathbf{h}_1^2-1\Big), \text{ where } \mathrm{L}_{\mathbf{h}} = \partial_{\rho}+\frac{m}{\rho}\mathbf{h}_3(\rho).$$

The kernel space of operator N is non-trvial. It is spanned by $\mathbf{h}_1$. If we force z to be orthogonal to $\mathbf{h}_1$, then we take inner product with $\mathbf{h}_1$ on both sides of the equation for z. The equation satisfied by $\sigma(t)$ and $\Theta(t)$ can be estimated as follows:

$$\|z\|_X \left|\Theta'\right| \lesssim \|z\|_X t^{-1/2} + \|z\|_X^2 + \epsilon_* \sigma^{-2} \|z\|_X^2$$

$$+ \|z\|_X^2 \left| \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right| + \epsilon_*^{-1} \int_0^\infty V_2^2 + \frac{\left(\partial_r W^*\right)^2}{r^2},$$

and

$$\left| \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right| \lesssim \|z\|_X + \|z\|_X t^{-1/2} + \sigma^{-2} \|z\|_X^2 + \epsilon_*^{-1} \int_0^\infty V_2^2 + \frac{\left(\partial_r W^*\right)^2}{r^2}.$$

Here we have used the following nonlinear cancellation

$$\int_0^\infty \mathbf{h}_1^2 \mathbf{h}_3 \, \rho \, \mathrm{d}\rho = m^{-1} \int_0^\infty \mathbf{h}_1^2 \, \rho \, \mathrm{d}\rho.$$

(4). We also need the following weighted energy estimate for z

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sigma^2\int_0^\infty |z|^2\rho\,\mathrm{d}\rho\right]+\mu^2\sigma^2\int_0^\infty |z|^2\rho\,\mathrm{d}\rho+c_*\|z\|_X^2$$
$$\lesssim\ \sigma^2\|z\|_X+\sigma^2\|z\|_X\,t^{-1/2}+\sigma^2\epsilon_*^{-1}\int_0^\infty V_2^2+\frac{\left(\partial_r W^*\right)^2}{r^2}.$$

Now we make our bootstrap assumption:

$$\text{(A.1).} \qquad \big(1-\varepsilon/2\big)\, e^{-\frac{\mu^2}{m^2}\,t}\sigma_{in} \;\le\; \sigma \;\le\; \big(1+\varepsilon/2\big)\, e^{-\frac{\mu^2}{m^2}\,t}\sigma_{in};$$

$$\text{(A.2).} \qquad \int_0^\infty V_2^2 \;\le\; \epsilon_*^{3/2}\,\frac{1}{(1+t)^2}.$$

Here $V_2 = V - V_1$, where V_1 is the solution of the standard heat equation with $V_1(0) = V_{in}$.

(5). Employing the energy estimate for z twice and using (A.1)-(A.2), we get the weighted X-norm estimate of z :

$$\int_0^t \sigma^{-2} \| z \|_X^2 \, \mathrm{d}s \lesssim \epsilon_*^{1/2}$$

$$+ \int_0^\infty \left| z_{in}(\rho) \right|^2 \rho \, \mathrm{d}\rho + \left(\sigma_{in}^2 \epsilon_*^{1/2} + \sigma_{in}^2 \int_0^\infty \left| z_{in}(\rho) \right|^2 \rho \, \mathrm{d}\rho \right)^{1/2}.$$

Here X is the space endowed with the following norm

$$\| z \|_X^2 = \int_0^\infty \left(\left| \partial_\rho z \right|^2 + \frac{| z |^2}{\rho^2} \right) \rho \mathrm{d}\rho.$$

(6). This estimate together with the estimates for σ imply

$$\int_0^t \left| \frac{\sigma'}{\sigma} + \frac{\mu^2}{m^2} \right| \ll 1,$$

which improves the assumption (A.1).

(7). To improve (A.2), we perform a Schonbek type estimate and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty V_2^2 + \mathrm{R}_*^2\int_0^\infty V_2^2 \lesssim \|z\|_X^2 + \mathrm{R}_*^6\epsilon_*^2\sigma_{in}^2.$$

Taking $\mathrm{R}_* = 3(1+t)^{-1}$ and solving the resulting inequality, we can improve the assumption (A.2). In fact we have

$$\int_0^\infty V_2^2 \le \frac{\epsilon_*^2}{(1+t)^2}.$$

This decay is better than the one in (A.2).

To prove oscillation instability, more will be involved. The main observation is the following linearized equation of fluid. That is

$$\partial_t W^* = \partial_{rr} W^* - \frac{\partial_r W^*}{r} + \dots$$

Oseen vortex solution is automatically a solution. But there is another one. One can easily calculate that W^*/r^2 satisfies

$$\partial_t \left(\frac{W^*}{r^2}\right) = \Delta_4 \left(\frac{W^*}{r^2}\right) + \dots.$$

Therefore we can split W^* into two parts $W^* = W_1^* + W_2^*$, where

$$W_1^* = \beta\, r^2\, \Gamma_4, \qquad \text{with } \Gamma_4 \text{ the heat kernel in } \mathbb{R}^4.$$

The coefficient β is chosen so that

$$\int_{\mathbb{R}^4} \frac{W_2^*}{r^2} = 0,$$

which provides better time decay for W_2^*/r^2 than W_1^*/r^2. The rigorous proof needs to apply Schonbek type estimate and bootstrap it twice.

III. Fréedericksz transition

There is a third instability called Fréedericksz transition. It is generated in terms of the external magnetic field. Assuming the one-constant approximation in the Oseen-Frank theory, the free energy functional for a nematic liquid crystal with a magnetic field potential is given by

$$\frac{1}{2}\int_{\Omega} |\nabla \mathbf{n}|^2 + \left\{ |\mathbf{H}|^2 - (\mathbf{n}\cdot\mathbf{H})^2 \right\}.$$

Here $\mathbf{H}$ is the applied magnetic field.

On the space time $\Omega \times \mathbb{R}_+$, the simplified Ericksen-Leslie equation with applied magnetic field can be read as follows:

$$\begin{cases} \partial_t \mathbf{n} + u \cdot \nabla \mathbf{n} - \Delta \mathbf{n} = |\nabla \mathbf{n}|^2 \mathbf{n} + \Big[\big(\mathbf{n} \cdot \mathbf{H}\big)\mathbf{H} - \big(\mathbf{n} \cdot \mathbf{H}\big)^2 \mathbf{n}\Big]; \\ \partial_t u + u \cdot \nabla u - \Delta u = -\nabla p - \nabla \cdot \big(\nabla \mathbf{n} \odot \nabla \mathbf{n}\big); \\ \mathrm{div} u = 0. \end{cases}$$

Here Ω is a smooth, bounded domain in $\mathbb{R}^2$.

If we assume bend geometry as an ansatz for the system (book of Stewart), that is

$$\mathbf{n} = \begin{pmatrix} \sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \qquad \text{and} \quad \mathbf{H} = \mathrm{H}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

then the above equation can be reduced to

$$\begin{cases} \partial_t \phi + u \cdot \nabla\phi - \Delta\phi = \dfrac{\mathrm{H}^2}{2} \sin 2\phi; \\ \partial_t u + u \cdot \nabla u - \Delta u = -\nabla \mathrm{q} - \left(\Delta\phi + \dfrac{\mathrm{H}^2}{2} \sin 2\phi \right) \nabla\phi; \\ \mathrm{div} u = 0. \end{cases}$$

Here q is given by $\mathrm{p} + \dfrac{1}{2}|\nabla\phi|^2 + \dfrac{\mathrm{H}^2}{4} \cos 2\phi$.

Supplied the above equation with $u = 0$ and $\phi = 0$ on boundary, it is clear that $(0, 0)$ is a trivial equilibrium state of the system. If we let H_c^2 be the first Dirichlet eigenvalue on Ω, then we can easily have
(i). when $\mathrm{H} < \mathrm{H}_c$, this equilibrium state is locally stable;
(ii). if $\mathrm{H} > \mathrm{H}_c$, then this equilibrium is linearly unstable. Indeed by W. M. Ni et al, when $\mathrm{H} > \mathrm{H}_c$, the elliptic sine-gordon equation admits a positive solution ϕ_*.

The Fréedericksz transition studied in [Chen, Kim, Y., 2017] is about the nonlinear transition from $(0, 0)$ state to $(0, \phi_*)$ state. Here ϕ_* is the unique positive solution of the sine-Gordon equation with zero Dirichlet data.

General results on long time behavior of (u, ϕ) system.

Let (u, ϕ) be a global solution to the (u, ϕ) system with the initial data $(\phi_{in}, u_{in}) \in \left\{ H_0^1(\Omega) \cap H^2(\Omega) \right\} \times H_0^1(\Omega)$. Then there exist a smooth solution ϕ_∞ to the elliptic sine-Gordon problem and a constant $\theta \in (0, 1/2)$ such that

$$\left\| u(t) \right\|_{H^1(\Omega)} + \left\| \phi(t) - \phi_\infty \right\|_{H^2(\Omega)} \leq C \, (1+t)^{-\frac{\theta}{1-2\theta}} \qquad \forall t > 0. \tag{3}$$

Here constants C and θ depend on Ω, H and the initial data (ϕ_{in}, u_{in}).

This result is quite similar to the one studied by Lin-Liu and Wu et al for the Ginzburg-Landau relaxation of the Ericksen-Leslie system. The dacay rate can be obtain via Lojasiewicz-Simon inequality. In our current situation, we can say more about the long time behavior of the solution when H is in different regime.

(1). when H is switched on but less than H_c, we have $\phi_\infty = 0$. Moreover it holds

$$\left\|u(t)\right\|_{H^1(\Omega)} + \left\|\phi(t)\right\|_{H^2(\Omega)} \le C(1+t)^{-\frac{\theta}{1-2\theta}} \qquad \forall t > 0.$$

(2). When $\mathrm{H} > \mathrm{H}_c$ and the initial orientation angle ϕ_{in} of the liquid crystal satisfies $\phi_{in} \not\equiv 0$, and $0 \leq \phi_{in} \leq \frac{\pi}{2}$. Then the decay estimate holds with $\phi_\infty = \phi_*$. Moreover, the following estimate holds with an exponential convergence rate:

$$\left\|u(t)\right\|_{H^1(\Omega)} + \left\|\phi(t) - \phi_*\right\|_{H^2(\Omega)} \leq Ce^{-\kappa t} \qquad \forall t > 0.$$

Remark 1. This result implies that when external magnetic field is below the threshold, then it keeps orthogonal to the applied magnetic field as $t \to \infty$. But when the external magnetic field is beyond the threshold, then the transition happens;

Remark 2. This transition is a nonlinear transition with exponential convergence rate from a neighborhood of $(0, 0)$ to $(0, \phi_*)$;

Remark 3. The critical threshold of magnetic field strength is given by the first Dirichlet eigenvalue. If the total area of the liquid crystal material is given, then disk will be the optimal shape to minimize the critical applied magnetic field.

Thanks a lot for your attention !

All the above works

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partially supported by

RGC grants, Hong Kong, P.R.C.