Space-time characterization of coherence

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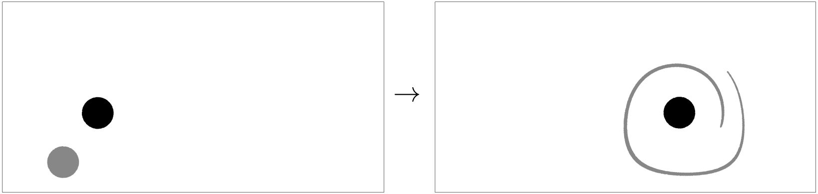
Joint work with Gary Froyland (UNSW)

Motivation

Coherent are sets with a particular property in state space with a specific behavior in time

- Bundles of trajectories (Talk by Padberg-Gehle)
- Sets that don't disperse over time under the combined effect of dynamics and diffusion (Talks by Froyland, Junge, and Karrasch)



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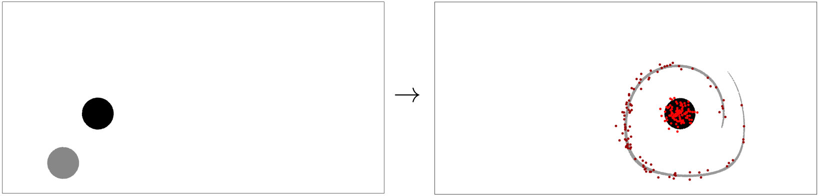

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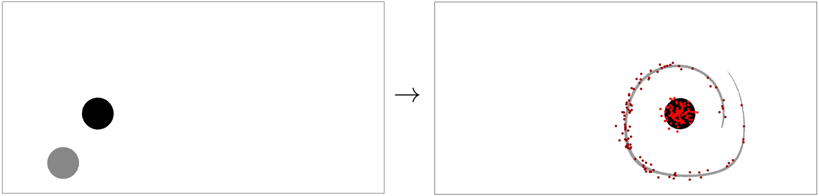


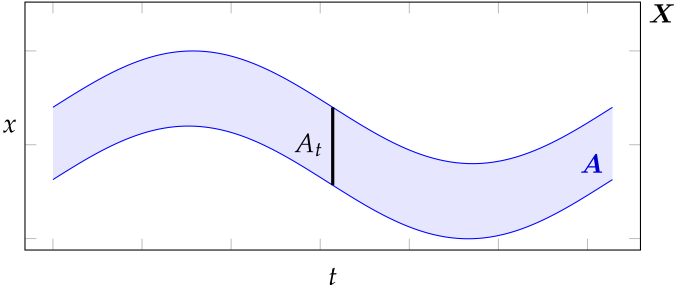
Aim: Try to understand coherent sets in their spatio-temporal entirety.

Augmented system

- With $\boldsymbol{x} = (\theta, x) \in \boldsymbol{X}$ we have the augmented system

$$\begin{aligned} \dot{\theta}_t &= 1 \\ dx_t &= v(\theta_t, x_t)dt + \varepsilon\, dw_t \end{aligned} \quad \Longleftrightarrow \quad d\boldsymbol{x}_t = \boldsymbol{v}(\boldsymbol{x}_t)dt + \boldsymbol{\Sigma}\, d\boldsymbol{w}_t$$

- Generates autonomous (homogeneous) augmented system $\boldsymbol{x}_t$
- For family of sets $\{A_r\}_{r\in(s,t)}$: augmented set $\boldsymbol{A} = \bigcup_{r\in(s,t)} \{r\} \times A_r$



Transfer operator on augmented space

- Ensemble of states $\boldsymbol{x}_0$ with density $\boldsymbol{f} \in L^1(\boldsymbol{X})$
- Transfer operator $\boldsymbol{\mathcal{P}}_t : L^1(\boldsymbol{X}) \to L^1(\boldsymbol{X})$:

$$\boldsymbol{x}_0 \sim \boldsymbol{f} \quad \Longrightarrow \quad \boldsymbol{x}_t \sim \boldsymbol{\mathcal{P}}_t \boldsymbol{f}$$

- For $s, t \geq 0$, $\boldsymbol{\mathcal{P}}_{s+t} = \boldsymbol{\mathcal{P}}_s \boldsymbol{\mathcal{P}}_t$: one-parameter semigroup
  Compare with $\exp(s+t) = \exp(s)\exp(t)$
- (Infinitesimal) generator $\boldsymbol{\mathcal{G}}$, with $\frac{d}{dt}\boldsymbol{\mathcal{P}}_t = \boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{P}}_t$. Intuitively

$$\text{“}\boldsymbol{\mathcal{P}}_t = \exp(t\boldsymbol{\mathcal{G}})\text{”}$$

- Spatio-temporal Fokker–Planck operator

$$\boldsymbol{\mathcal{G}}\boldsymbol{f} = -\mathrm{div}(\boldsymbol{f}\boldsymbol{v}) + \frac{\varepsilon^2}{2}\Delta_x \boldsymbol{f}$$

Coherent families

- Escape rates

$$\boldsymbol{E}(\boldsymbol{A}) = -\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big(\boldsymbol{x}_r \in \boldsymbol{A},\, 0 \le r \le t\big)$$

$$\Updownarrow$$

$$E(\{A_r\}) = -\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}\big(x_r \in A_r,\, s \le r \le t\big)$$

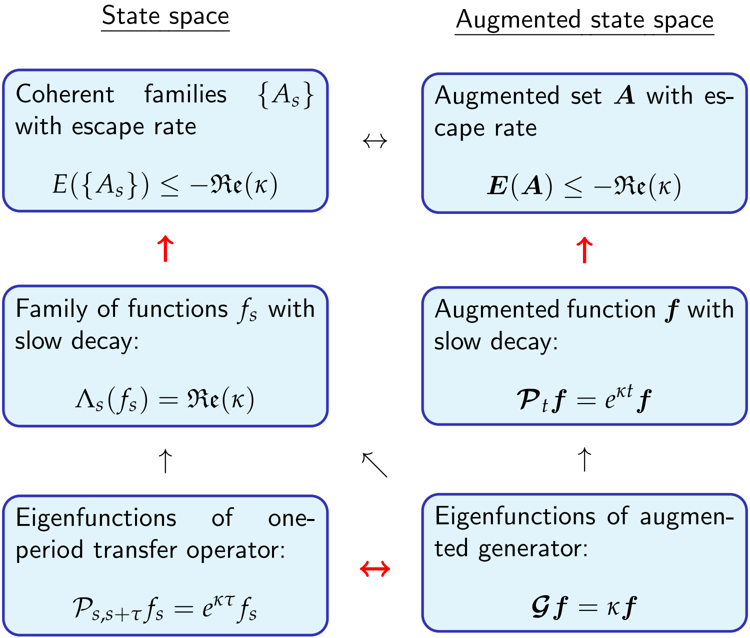
- Periodic forcing: $v(t,\cdot) = v(t+\tau,\cdot)$

• Let $\boldsymbol{\mathcal{G}} \boldsymbol{f} = \kappa \boldsymbol{f}$
• Set $\boldsymbol{A}^{\pm} = \{\pm \boldsymbol{f} \ge 0\}$
Then

$$\boldsymbol{E}(\boldsymbol{A}^{\pm}) \le -\mathfrak{Re}(\kappa)$$

[Froyland, K., preprint]

- Cf. *evolution semigroups* [Howland '74], [Chicone, Latushkin '00] & others; and also the talk by Gonzalez Tokman



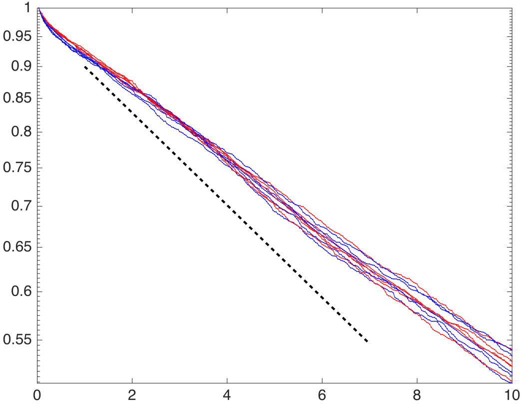
State space
Augmented state space
Coherent families $\{A_s\}$ with escape rate
$E(\{A_s\}) \leq -\mathfrak{Re}(\kappa)$
Augmented set $\boldsymbol{A}$ with escape rate
$\boldsymbol{E}(\boldsymbol{A}) \leq -\mathfrak{Re}(\kappa)$
Family of functions f_s with slow decay:
$\Lambda_s(f_s) = \mathfrak{Re}(\kappa)$
Augmented function $\boldsymbol{f}$ with slow decay:
$\boldsymbol{\mathcal{P}}_t \boldsymbol{f} = e^{\kappa t} \boldsymbol{f}$
Eigenfunctions of one-period transfer operator:
$\mathcal{P}_{s,s+\tau} f_s = e^{\kappa\tau} f_s$
Eigenfunctions of augmented generator:
$\boldsymbol{\mathcal{G}} \boldsymbol{f} = \kappa \boldsymbol{f}$

Double gyre: real eigenvalues

Double gyre: escape rates

Double gyre: escape rates

Escape rates by numerical simulation: from $\{A_t^+\}$ (blue) and from $\{A_t^-\}$ (red). Dashed line: eigenvalue bound.



Example 1: complex eigenvalue

- $\boldsymbol{\mathcal{G}} \boldsymbol{f} = \kappa \boldsymbol{f}$, $\kappa \in \mathbb{C}$
- $A_t^{\pm} = \{ \pm \mathfrak{Re}(e^{i \mathfrak{Im}(\kappa) t} f_t) \geq 0 \}$ is a coherent family
- Quasi-periodic family: f_t is τ-periodic, $e^{i \mathfrak{Im}(\kappa) t}$ is $\frac{2\pi}{\mathfrak{Im}(\kappa)}$-periodic

Extensions

- Finite time coherence; $t \in [t_0, t_1]$ ([Froyland, K., Plonka, In prep.])
- Ergodic base dynamics

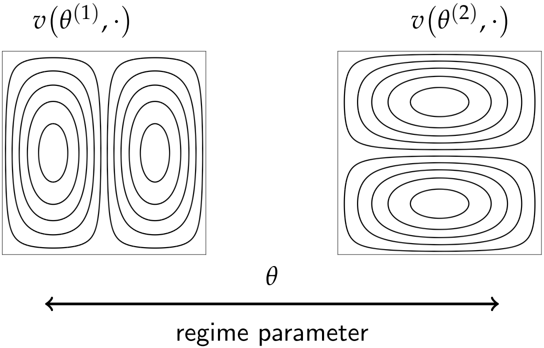
$$\begin{aligned} \dot{\theta}_t &= g(\theta_t) \\ dx_t &= v(\theta_t, x_t)dt + \varepsilon\, dw_t \end{aligned}$$

- Non-deterministic regime dynamics ([K., Plonka, In prep.])

$$\begin{aligned} \dot{\theta}_t &= g(\theta_t) + noise \\ dx_t &= v(\theta_t, x_t)dt + \varepsilon\, dw_t \end{aligned}$$

Each framework characterizable by augmented generator $\mathcal{G}$

Turbulent superstructures



Augmented-space dynamics:

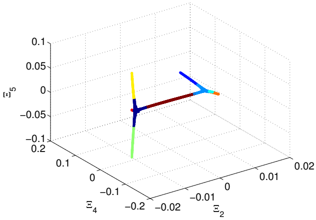
$$
\begin{aligned}
\dot{\theta}_t &= \text{noise/non-trivial dynamics} \\
\dot{x}_t &= v(\theta_t, x_t) + \text{noise}
\end{aligned}
$$

Statistically persistent coherent families = stable sets in augmented space.

Something different...

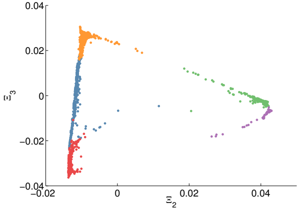
"Skeleton of transport" or "Transport coordinates"
[Banisch, K., To appear]

Embedding



Clusters / coherent sets

Ocean drifters



Conclusion

Summary:

- Spatio-temporal characterization of coherence
- Transfer operator (generator) in augmented space
- Coherent families for different types of dynamical models
- Skeleton of transport

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Turbulent Superstructures

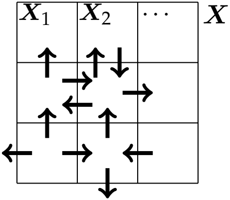
G. Froyland and P. Koltai. *Estimating long-term behavior of periodically driven flows without trajectory integration*. Preprint, arXiv:1511.07272, 2015.

R. Banisch and P. Koltai. *Understanding the geometry of transport: diffusion maps for Lagrangian trajectory data unravel coherent sets*. To appear in Chaos, 2016.

G. Froyland, O. Junge, and P. Koltai. *Estimating long-term behavior of flows without trajectory integration: The infinitesimal generator approach*. SIAM Journal on Numerical Analysis, 51(1):223–247, 2013.

Appendix

Discretization I



Discrete generator $G^{(n)}$:

$$G_{ij}^{(n)} = \begin{cases} \frac{1}{\mathrm{vol}(X_j)} \int_{\partial X_i \cap \partial X_j} \langle v, n_j \rangle^+ \, d\sigma, & i \neq j \\ -\frac{1}{\mathrm{vol}(X_i)} \int_{\partial X_i} \langle v, n_i \rangle^+ \, d\sigma, & i = j, \end{cases}$$

- $G^{(n)}$ computable *without* trajectory simulation.
- $G^{(n)}$ is a sparse matrix.
- $G^{(n)}$ is the spatial discretization of the upwind scheme.
- $G^{(n)}$ generates Markov jump process, i.e. $e^{tG^{(n)}}$ is a stoch. matrix

Double gyre with Ulam's method

- Eigenfunctions of transition matrix $P^{(n)}$ from Ulam's method for $(s,t) = (0,1)$
- Same number of vector field evaluations as for the augmented generator