PERSISTENCE-BASED SUMMARIES FOR METRIC GRAPHS

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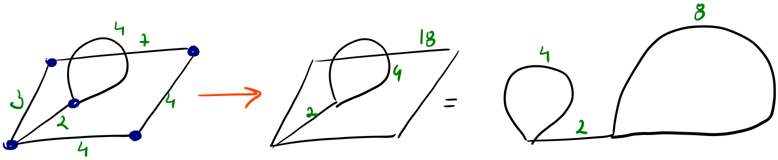
Metric graphs

- Input: Weighted graph $G = (V, E, L)$ with a weight function $L : E \rightarrow R_{\geq 0}$
- Output: metric graph (G, d_G)
  - homeomorphic to a 1-dim stratified space
  - Every point of every edge is a point in this space-it is infinite!
  - Embedding does not matter.



Motivation

Why metric graphs

- Data often comes from a hidden space that is graph-like: cosmic networks, road maps,...
- Graphs are often simplest meaningful way to represent non-linear structure of the data: internet, social networks, structural and functional connectome, etc.

(Metric) Graph comparison

- Graph isomorphism: conjecturally NP complete (for metric graphs there is a problem of noise and small deformations)
- Distances: discrete or not-computable (Gromov-Hausdorff compares metric graphs as metric spaces and it is NP-hard even to approximate within a constant factor!)

TDA TO THE RESCUE

QUALITATIVE SUMMARIES OF METRIC GRAPHS

- Relate properties of a metric graph G and the homology of an associated complex.
- Identify which topological properties of a graph are contained in the persistence diagram
- Give complete characterization of the 1-dimensional persistence diagrams for metric graphs with the Čech complex construction in terms of graph properties.
- Goal: provide powerful insights to understanding underlying data.

TDA to the rescue

Persistance-based distances

- Construct a continuous distance on the space of metric graphs, stable under metric perturbations (noise and small deformations) using persistence of an associated simplicial complex
- Compare discriminative powers of distances based on the bottleneck distance between persistent diagrams obtained from different constructions:
  - Čech and
  - persistence distortion (PD) distance [Dey, Shi, Wang]
- Polynomial-time computable

PRIOR RESULTS

CYCLE GRAPHS

Nerve complex for a finite collection of arcs on a circle has the homotopy type of a

- point
- odd-dimensional sphere
- wedge sum of spheres with the same even dimension

[Adamaszek, Adams, Florian, Peterson, Previte-Johnson]

COROLLARY

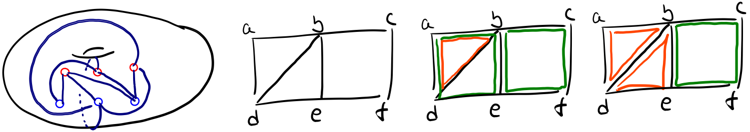
The 1-dimensional persistence diagram of a circle split into arcs of total length equal to ℓ consist of at most one bar

- $[0, \ell/4)$ *for the Čech complex, and*
- $[0, \ell/6)$ *for the Vietoris-Rips complex.*

TOPOLOGICAL GRAPH THEORY

THE GENUS OF A GRAPH

- the minimal integer n such that the graph can be drawn without crossing itself on an oriented surface of genus n.
- β_1: number of cycles in a basis for the first homology



SHORTEST SYSTEM OF LOOPS

- cycles that are shortest non-trivial paths from a vertex to itself
- shortest representatives of homology classes in lexicographical order

INTRINSIC ČECH FILTRATION

Let (G, d_G) be a metric graph.

- For any point $x \in G$ define $B(x, \epsilon) := \{y \in |G| : d_G(x, y) \leq \epsilon\}$
- Covering $U_\epsilon := \{B(x, \epsilon) : x \in |G|\}$ and let
- C_ϵ denote the nerve of U_ϵ
- *Intrinsic Čech filtration* is the set of inclusions

$$\{\mu_\epsilon^c : C_\epsilon \hookrightarrow C_{\epsilon'}\}_{\forall 0 \leq \epsilon \leq \epsilon'}.$$

- *Intrinsic Čech persistence diagram $Dg_* IC_G$*, is obtained from the induced persistence module

$$\{\mu_\epsilon^h : H_*(C_\epsilon) \to H_*(C_{\epsilon'})\}_{\forall 0 \leq \epsilon \leq \epsilon'}$$

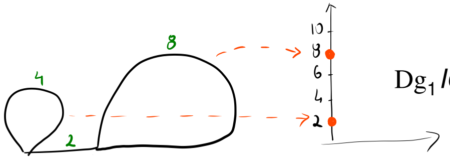
Characterization of the 1-dim persistence diagrams

Theorem (GGPSWWZ '17)

Let G be a metric graph of genus g with

- *a shortest cycle basis $\beta = \{\gamma_1, \ldots, \gamma_g\}$*
- *with cycles γ_i of length ℓ_i for $1 \le i \le g$ such that $\ell_1 \le \ldots \le \ell_g$*

Then the 1-dimensional intrinsic Čech persistence diagram of G, $\mathrm{Dg}_1 IC(G)$, consists of the following collection of points on the y-axis:



$$\mathrm{Dg}_1 IC(G) = \left\{ \left[0, \frac{\ell_i}{4}\right) : 1 \le i \le g \right\}$$

Proof of the main theorem

- For small $\delta > 0$, C_δ has the same homotopy type as G.
- C_δ^0 inherits metric from G
- All γ_i's are born at δ (consider it to be 0)
- Each γ_i must die at $\frac{\ell_i}{4}$ or earlier since $\forall x, y, z \in \gamma_i$, $B\left(x, \frac{\ell_i}{4}\right) \bigcap B\left(y, \frac{\ell_i}{4}\right) \bigcap B\left(z, \frac{\ell_i}{4}\right) \neq \emptyset$.

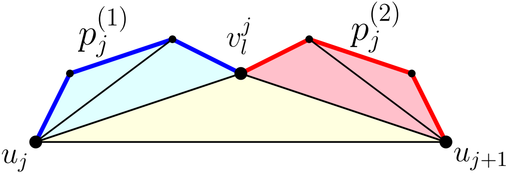
Need to show

A No other cycles are created in C_ϵ, $\epsilon > \delta$ due to interference from other cycles: β spans 1-dim persistence

B For $i = 1, \ldots, g$, $[\gamma_i]$ does not die before $\epsilon = \dfrac{\ell_i}{4}$: β is linearly independent

Proof of Part A

The map $\mu^h_\epsilon : H^{(1)}_\delta \to H^{(1)}_\epsilon$ is surjective, has a right inverse up to homotopy.



In other words, there exists a combinatorially defined map $\rho : C^{(1)}_\delta \to C^{(1)}_\epsilon$ such that for every $[\eta] \in H^{(1)}_\epsilon$

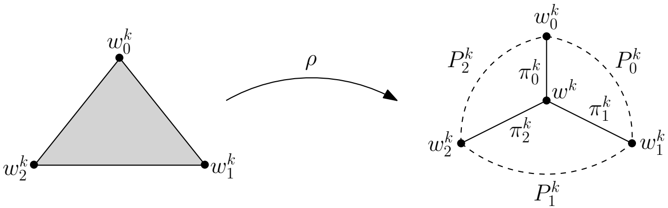
$$\mu^h_\epsilon([\rho(\eta)]) = [(\mu^c_\epsilon \circ \rho)(\eta)] = [\eta] \in H^{(1)}_\epsilon$$

Proof of Part B

For any $i = 1, \ldots, g$, the set

$$\{[\mu_\epsilon^c(\gamma_i)], [\mu_\epsilon^c(\gamma_{i+1})], \ldots, [\mu_\epsilon^c(\gamma_g)]\}$$

is a basis for $H_\epsilon^{(1)}$ where $\frac{\ell_{i-1}}{4} \leq \epsilon < \frac{\ell_i}{4}$ and $\ell_0 = 0$.



$\rho(t_k) = \sum_{n=0}^{n=2} \pi_n^k + P_n^k - \pi_{(n+1 \mod 3)}^k$ is the sum of three cycles, each of length at most $\epsilon + \epsilon + 2\epsilon = 4\epsilon < \ell_i$.

SUMMARY

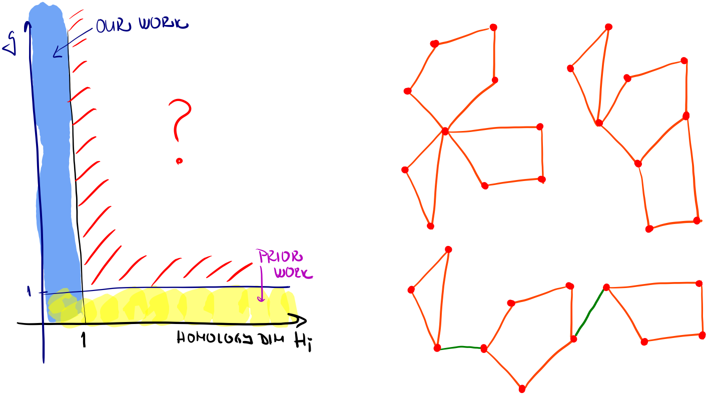


FIGURE: What is missing? Graphs that we can not distinguish...

Intrinsic Čech distance

Intrinsic Čech distance $d_{IC}(G_1, G_2)$

Let d_B denore the bottleneck distance between the two intrinsic Čech persistence diagrams in dimension 1. Then

$$d_{IC}(G_1, G_2) := d_B(Dg_1 IC_{G_1}, Dg_1 IC_{G_2}),$$

[Chazal Cohen, Steiner, Guibas, Memoli, Oudout 2009]

Modified bottleneck distance $\delta(D_1, D_2)$

Given persistence diagrams D_1 and D_2 let the distance between $(x, y) \in D_1$ and $(x', y') \in D_2$ be $|x - x'| + |y - y'|$.

- If a point (x, y) is matched to a point on a diagonal, it contributes $y - x$ to $\delta(D_1, D_2)$
- δ is within a factor of 2 of the standard bottleneck d_B

Intrinsic Čech distance

Theorem (GGPSWWZ '17)

Let $D_1 := \{(0, a_i)\}_{i=1}^{s}$ and $D_2 := \{(0, b_j))\}_{j=1}^{t}$ be persistence diagrams with $a_1 \leq \cdots \leq a_s$, $b_1 \leq \cdots \leq b_t$ and, WLOG, $s \leq t$. Let $A = \{a'_1, a'_2, \ldots a'_t\}$ and $B = \{b_1, \ldots b_t\}$, where A consists of $t - s$ zeroes at the beginning, followed by the sequence $a_1, a_2, \ldots, a_s$.

Then $\delta(D_1, D_2) = d_B(D_1, D_2) = \max_{i=1}^{t} |a'_i - b_i|$.

Corollary (GGPSWWZ '17)

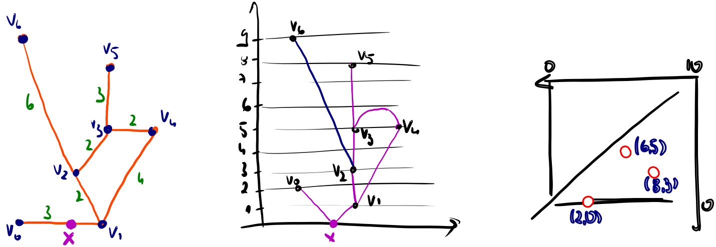
The distance between 1-dim intrinsic Čech persistence diagrams $Dg_1 IC_{G_1} = \left\{ \left(0, \frac{\ell_i}{4}\right) \right\}_{i=1}^{s}$ and $Dg_1 IC_{G_2} = \left\{ \left(0, \frac{m_j}{4}\right) \right\}_{j=1}^{t}$ associated to metric graphs G_1 and G_2 is

$$d_{IC}(G_1, G_2) = d_B(Dg_1 IC_{G_1}, Dg_1 IC_{G_2}) = \max_{i=1}^{t} \frac{|\ell'_i - m_i|}{4}.$$

Persistence with respect to intrinsic distance d_G

Persistence diagram $Dg_0(d_G, x)$

Given a metric graph G, fix a base point $x \in |G|$ and consider connected components of the superlevel sets $G \setminus B(x, \epsilon)$ with respect to the geodesic intrinsic distance.



PD distance d_{PD}

PD distance $d_{PD}(G_1, G_2)$

The persistence distortion distance between G_1 and G_2 is

$$d_{PD}(G_1, G_2) = \max \left\{ \max_{x \in G_1} \min_{y \in G_2} d_B\left(Dg_0(d_{G_1}, x), Dg_0(d_{G_2}, y)\right), \max_{y \in G_2} \min_{x \in G_1} d_B\left(Dg_0(d_{G_2}, y), Dg_0(d_{G_2}, y)\right) \right\}$$

$d_{PD}(G_1, G_2)$ is the Hausdorff distance between collections of 0-dimensional persistence diagrams with respect to all possible pairs of baspoints, as subspaces of the space of persistence diagrams equipped with the bottleneck distance.

Proposition (Dey, Shi, Wang)

$d_{PD}(G_1, G_2) \leq 6 d_{GH}(G_1, G_2)$

d_{PD} vs. d_{IC}?

Conjecture (GGPSWWZ '17)

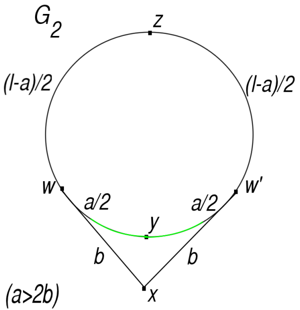
For any two metric graphs G_1 and G_2, $d_{IC}(G_1, G_2) \leq \frac{1}{2} d_{PD}(G_1, G_2)$.

Theorem (GGPSWWZ '17)

In the case of

- G_1, G_2 are metric trees; or
- G_1 is a single cycle of length ℓ and G_2 is the graph with two cycles shown in Figure with a is slightly larger than $2b$.

we have equality.



PROPOSITION

Let G_1 be a graph containing a single cycle of length ℓ and G_2 a graph containing a single cycle of length m with $\ell \geq m$. Then:

- $d_{IC}(G_1, G_2) = \dfrac{\ell - m}{4}$
- $d_{PD}(G_1, G_2) = \dfrac{\ell - m}{2}$, *and therefore* $d_{IC}(G_1, G_2) = \frac{1}{2} d_{PD}(G_1, G_2)$.

THEOREM

Let G_1 be a bouquet of circles and G_2 any metric graph. Then

$$d_{IC}(G_1, G_2) \leq \frac{1}{2} d_{PD}(G_1, G_2).$$

To be continued...

- What does the higher persistence diagram know about the underlying topology of a graph?
- Can we use these persistence summaries to distinguish common types of graph motifs?
- Characterize PD persistence?
- The geodesic persistence diagram of f_x is stable w.r.t. small perturbation of its geometric realization under the Gromov-Hausdorff distance.
- Is it possible to develop a persistence-distortion for the combinatorial graphs that would be stable with respect to some appropriate notion of perturbation?

Thank you!