Rook placements and Jordan forms

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Motivation: Higman's Conjecture

Let $U_n(\mathbb{F}_q)$ be the group of $n \times n$ unipotent upper-triangular matrices over $\mathbb{F}_q$.

Let $k_n(q)$ be the number of conjugacy classes of $U_n(\mathbb{F}_q)$.

Higman's Conjecture: $k_n(q)$ is a polynomial in q.

We'll work on an easier related problem.

Counting nilpotent matrices

A Jordan block is

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Question: How many $n \times n$ upper-triangular nilpotent matrices over $\mathbb{F}_q$ are conjugate to J_λ, for $\lambda \vdash n$?

Counting nilpotent matrices

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Over $\mathbb{F}_2$,

$$X = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 \\ & & 0 & 1 & 1 \\ & & & 0 & 0 \\ & & & & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix} = J_{(3,2)}.$$

Question: How many $n \times n$ upper-triangular nilpotent matrices over $\mathbb{F}_q$ are conjugate to J_λ, for $\lambda \vdash n$?

A first example

Let $F_\lambda(q)$ be the number of $\uparrow\Delta 0$ matrices conjugate to J_λ.

Let $n = 3$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- $F_{(1,1,1)}(q) = 1$.
- $F_{(3)}(q) = (q-1)^2 q$.
- $F_{(2,1)}(q) = q^3 - (q-1)^2 q - 1 = (q-1)(2q+1)$.

How can we compute this in general?

A recursive formula

Theorem. [Borodin]

$$F_\lambda(q) = \sum_{\mu\colon \lambda \gtrdot \mu} c_{\lambda\mu}(q) F_\mu(q),$$

where if the added box is in the jth column, then

$$c_{\lambda\mu}(q) = \begin{cases} q^{|\mu|-\ell(\mu)}, & \text{if } j = 1, \\ \left(q^{\mu'_{j-1}-\mu'_j} - 1\right) q^{|\mu|-\mu'_{j-1}}, & \text{otherwise.} \end{cases}$$

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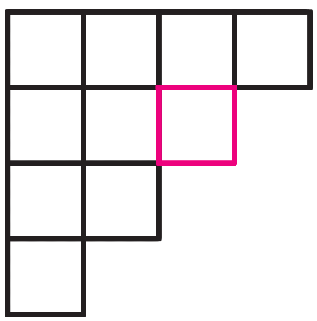
Example. $\mu = (4,2,2,1)$, $\lambda = (4,3,2,1)$.



$$c_{\lambda\mu}(q) = (q^2 - 1)q^6$$

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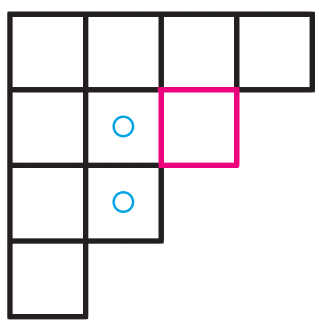

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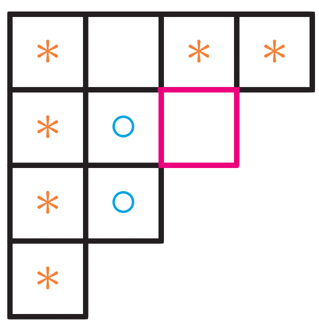

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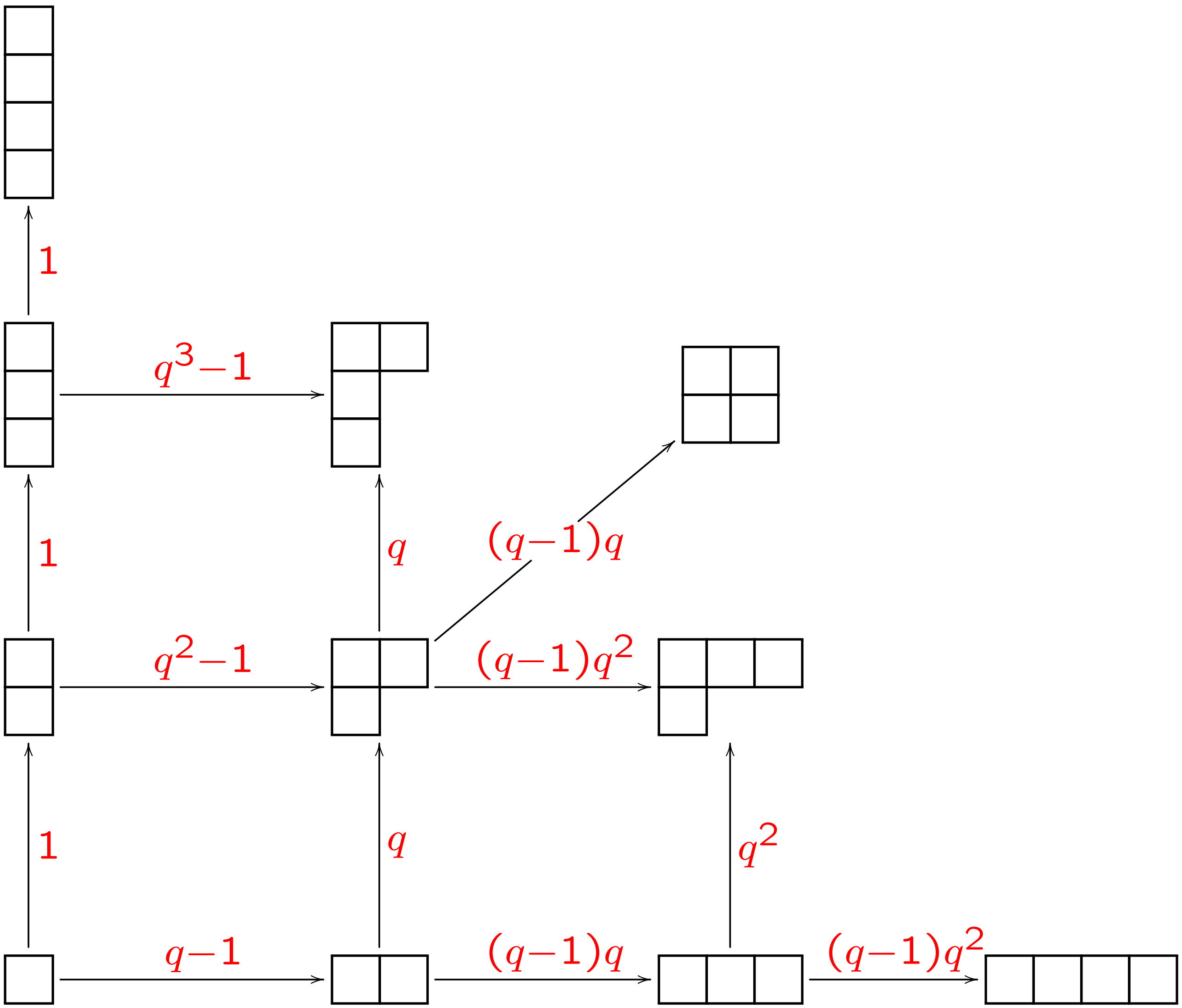
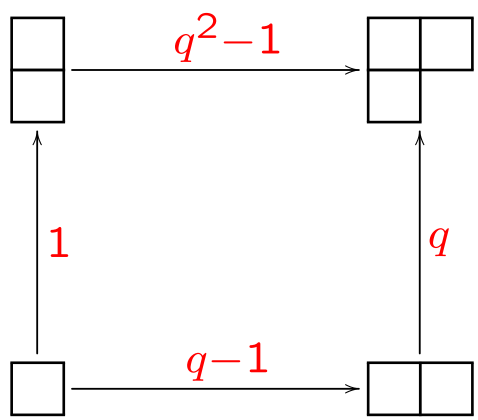
Visualize $c_{\lambda\mu}(q)$ on Young's lattice



Rephrase the formula

$$F_\lambda(q) = \sum_{\substack{\text{paths} \\ \text{to } \lambda}} \prod_e \text{edge weights} = \sum_{T \in SYT(\lambda)} F_T(q).$$

Example.



$$F_{(2,1)}(q) \;=\; 1 \cdot (q^2 - 1) \;+\; (q-1) \cdot q \;= (q-1)(2q+1)$$

1	3
2	

1	2
3	

Some consequences

1. $F_\lambda(q) \in \mathbb{Z}[q]$.

2. For every SYT T,

$$\deg F_T(q) = \binom{n}{2} - n(\lambda).$$

3. Coefficient of highest term of $F_\lambda(q)$ is $f^\lambda = \#\mathrm{SYT}(\lambda)$.

4. $(q-1)^{n-\ell(\lambda)}$ divides $F_\lambda(q)$. In fact,

$$G_\lambda(q) = F_\lambda(q)/(q-1)^{n-\ell(\lambda)} \in \mathbb{N}[q].$$

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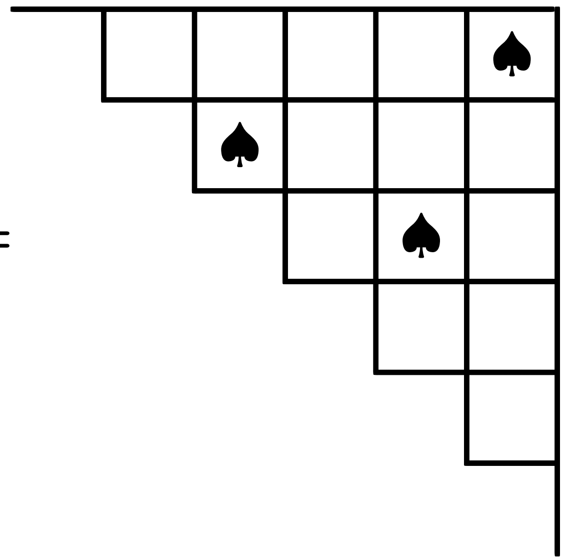
Question: Is there a combinatorial interpretation of the non-negative coefficients in $G_\lambda(q)$?

Rook placements

Example. Staircase board $B(0, 1, 2, 3, 4, 5)$.

$C =$



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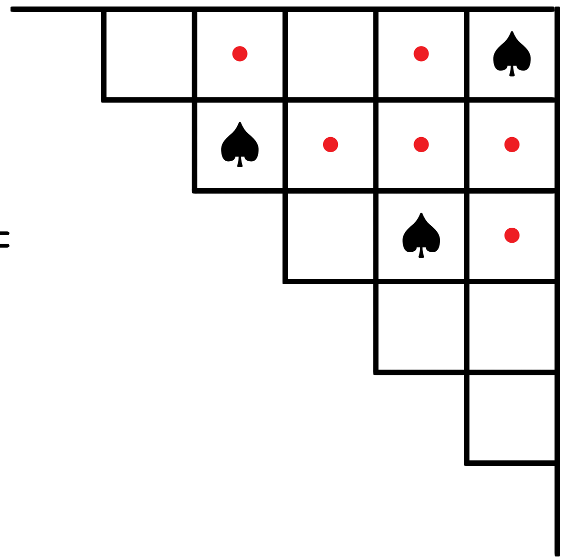
$C =$ 

$\mathrm{rk}(C) = 3$

$\mathrm{ne}(C) = 6$

$F_C(q) = (q-1)^3 q^6$

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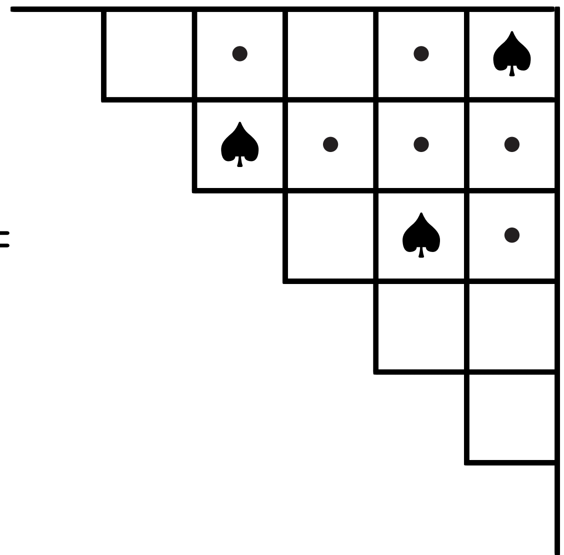
$C =$ 

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Theorem. [Haglund] The number of $n \times n$ matrices over $\mathbb{F}_q$ with rank k and with support on the board B is

$$(q-1)^{\mathsf{rk}(C)} \sum_{C \in \mathcal{C}(B,k)} q^{\mathsf{ne}(C)}.$$

Idea: Is it possible to refine this formula with respect to Jordan forms?

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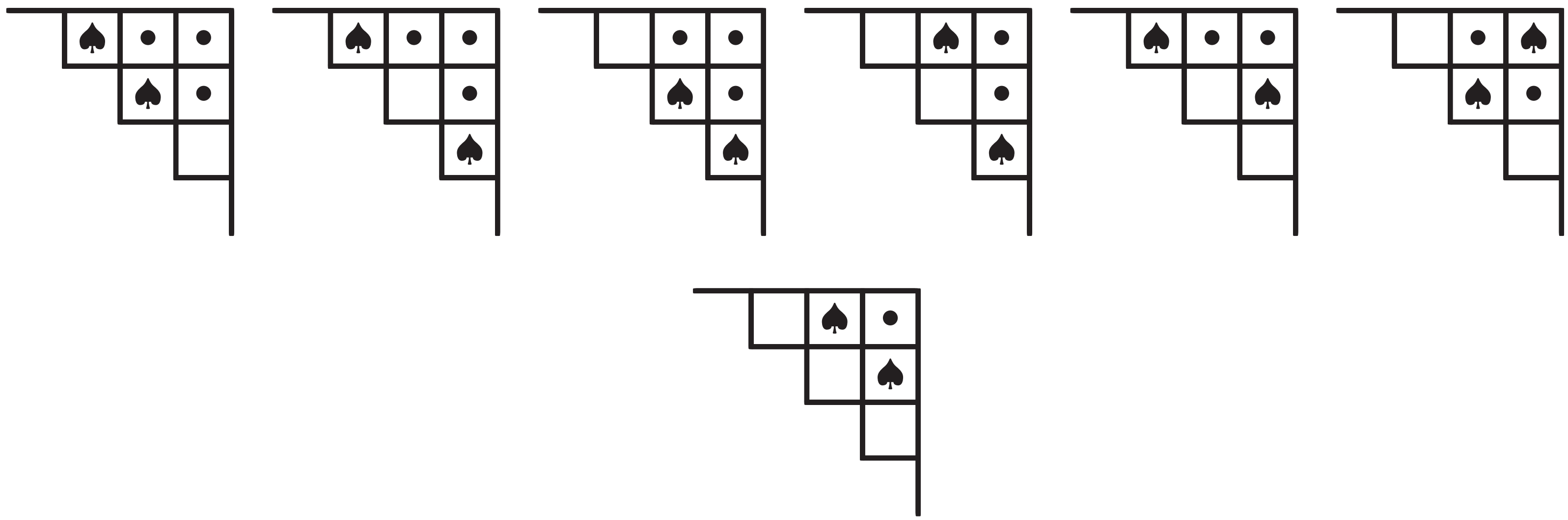

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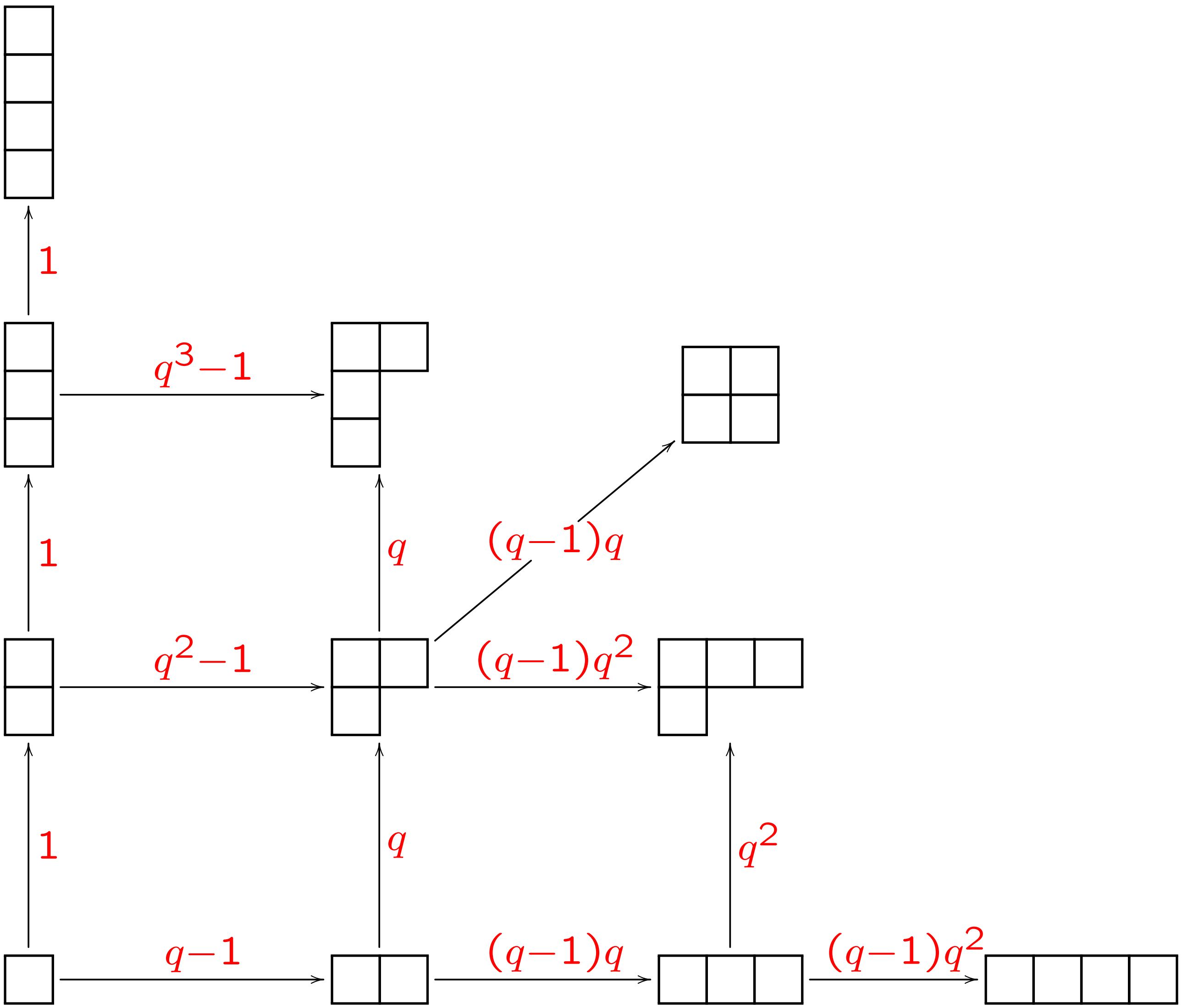

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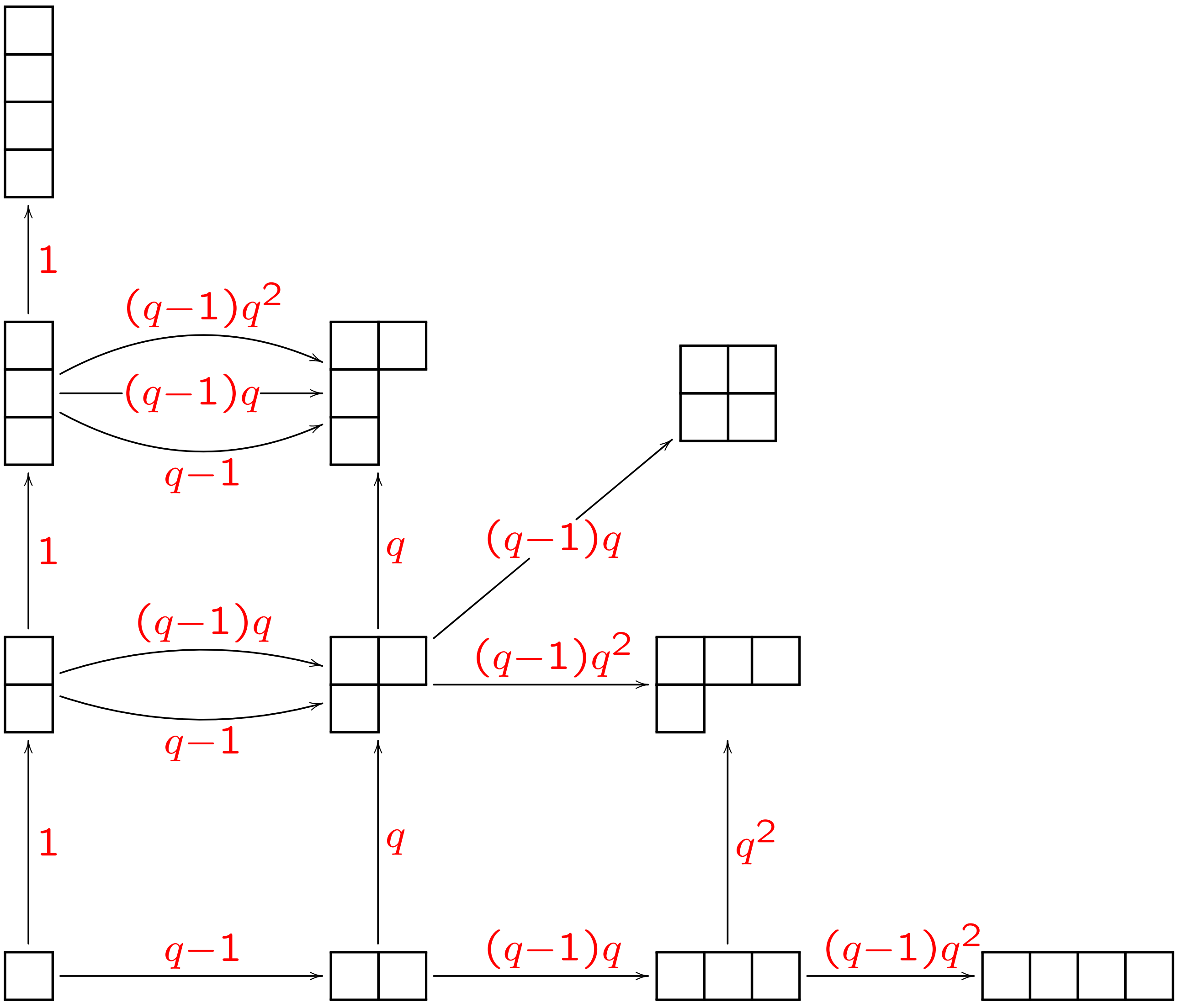
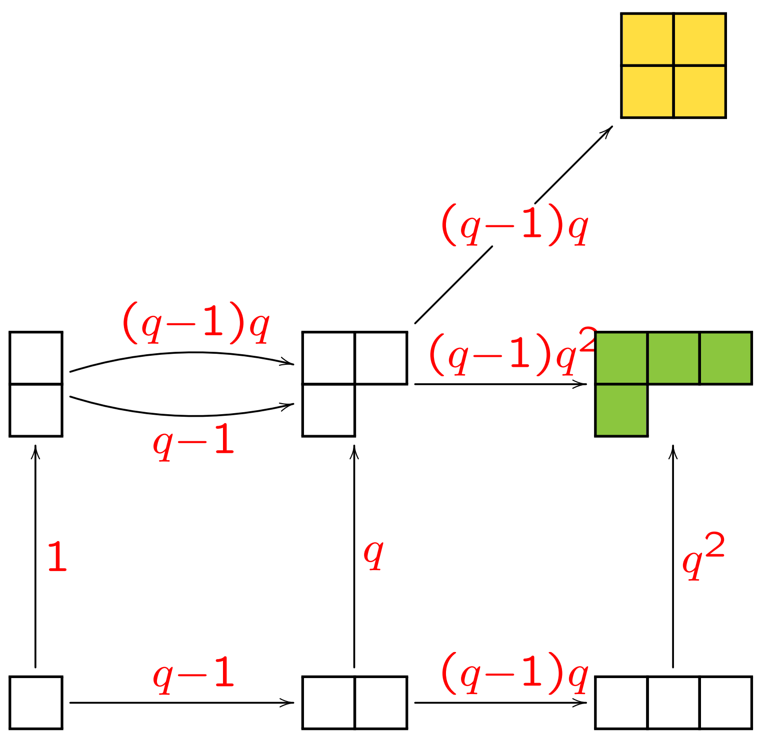


A bijection

Theorem. [Y] There exists a weight preserving bijection

$$\left\{ \begin{array}{c} \text{placements on } B \\ \text{with } k \text{ rooks} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{paths in } \mathcal{Z} \text{ to } \lambda \\ \ell(\lambda) = n - k \end{array} \right\}$$

Example. $n = 4$, $k = 2$ rooks.



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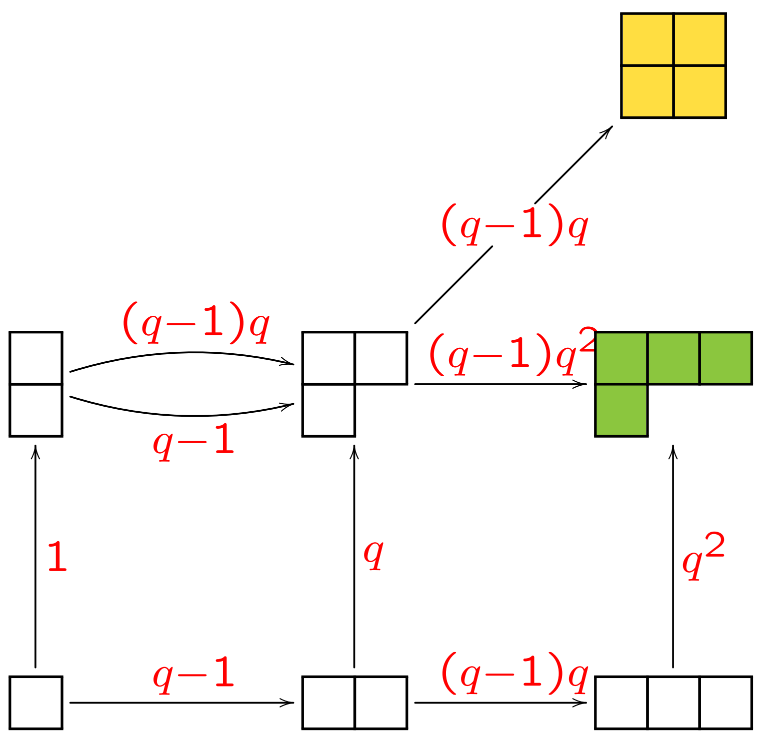
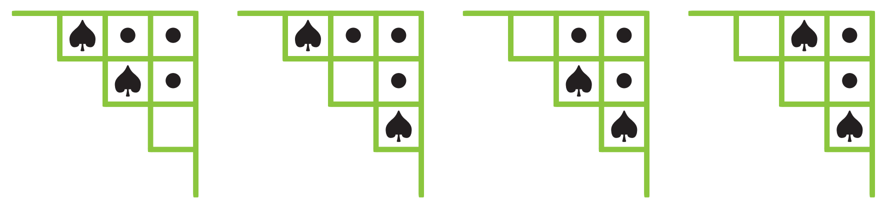
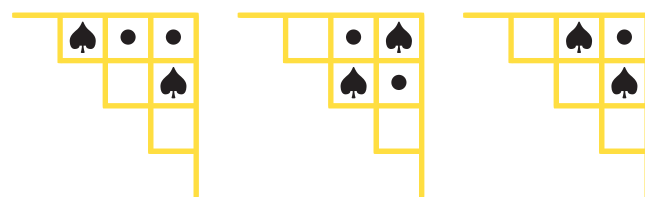




$$F_{(3,1)}(q) = (q-1)^2(3q^3 + q^2)$$



$$F_{(2,2)}(q) = (q-1)^2(2q^2 + q)$$

A second bijection

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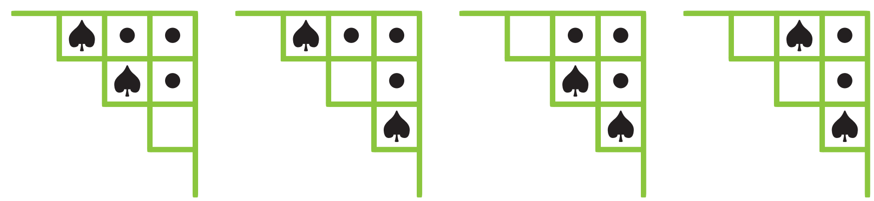

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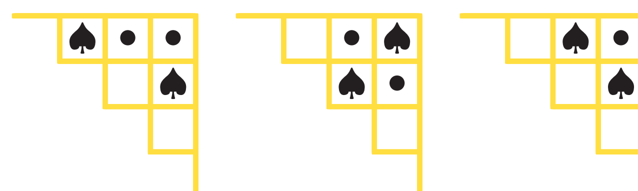

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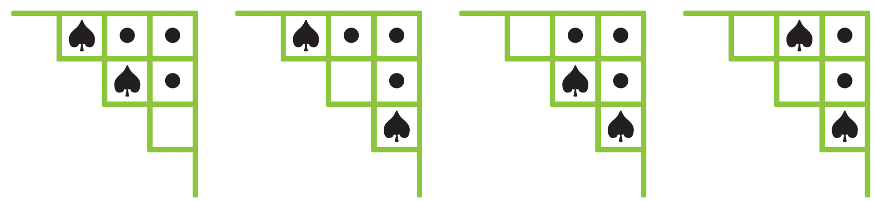

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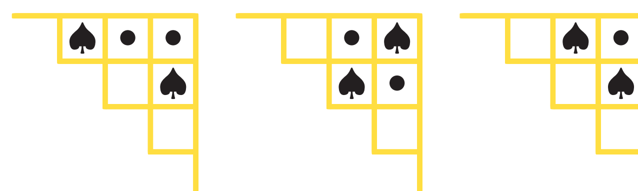

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Recap

So far, $F_\lambda(q)$ counts matrices, and it can be refined:

$$F_\lambda(q) = \sum_{\substack{\text{tableau } T \\ \text{has type } \lambda}} F_T(q) = \sum_{\substack{\text{placement } C \\ \text{has type } \lambda}} F_C(q).$$

Question: Is there an algebraic/geometric meaning for $G_C(q)$?

Example. Compare the polynomials

$$F_{(2,1)}(q) = (q-1)(2q+1) \qquad Q_{(1^3)}^{(2,1)}(q) = 2q+1$$

$$F_{(3,1)}(q) = (q-1)^2(3q^3+q^2) \qquad Q_{(1^4)}^{(3,1)}(q) = 3q+1$$

$$F_{(2,2)}(q) = (q-1)^2(2q^2+q) \qquad Q_{(1^4)}^{(2,2)}(q) = (q+1)(2q+1)$$

Flags

$Q^\lambda_\rho(q)$ is Green's polynomial.

Theorem. [Hotta, Springer] $Q^\lambda_{(1^n)}(q)$ is the number of $\mathbb{F}_q$-rational points in the variety X_λ of η-stable flags, where η is a nilpotent matrix conjugate to J_λ.

Example. Let $\eta = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Over $k \supseteq \mathbb{F}_2$, with $V = k^3$,

$$\{0\}, \langle \mathrm{e}_1 \rangle, \langle \mathrm{e}_1, \mathrm{e}_3 \rangle, V$$
$$\{0\}, \langle \mathrm{e}_3 \rangle, \langle \mathrm{e}_1, \mathrm{e}_3 \rangle, V$$
$$\{0\}, \langle \mathrm{e}_1 + \mathrm{e}_3 \rangle, \langle \mathrm{e}_1, \mathrm{e}_3 \rangle, V$$
$$\{0\}, \langle \mathrm{e}_1 \rangle, \langle \mathrm{e}_1, \mathrm{e}_2 \rangle, V$$
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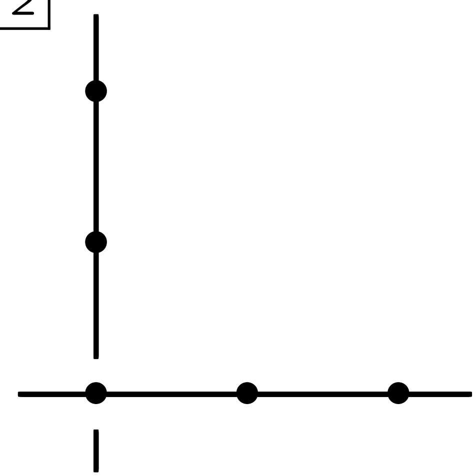
Example. The variety $X_{(2,1)}$ has two irreducible subvarieties.

$(V_2 \supset \langle \mathbf{e}_1 \rangle, V_2 \not\supset \ker \eta)$ $X_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}$



$X_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}$ $(V_2 = \ker \eta)$

$$G_{21}(q) = q + 1 + q$$