Polytopal Combinatorics

and.

Generalizations.

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Thanks to the Simons Foundation.

P n-dim'l polytope

The $\underline{f\text{-vector}}$ $(f_0, \text{---}, f_{n-1})$

$$f_i = \#\ i\text{-dim'l faces}.$$

[Steinitz 1906]   Characterized f-vectors
                    of 3-dim'l polytopes.

[Stanley 1978; Billera-Lee 1980].   Characterized
                                      f-vectors of
                                      simplicial polytopes.

Open 2: Characterize f-vectors
         of n-dim'l polytopes, $n \geq 4$.

flag vectors

P n-dim'l polytope

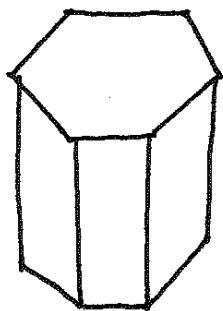For $S = \{s_1 < \cdots < s_k\} \subseteq \{0, 1, \ldots, n-1\}$ let

$$f_S = \# \quad F_1 \subsetneq \cdots \subsetneq F_k$$

where $\dim(F_i) = s_i$.

The **flag f-vector**

$$(f_S)_{S \subseteq \{0, 1, \ldots, n-1\}}$$

ex.

The **flag h-vector**

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} f_T$$

[Stanley 1979]

$$h_S = h_{\bar{S}}.$$

S	f_S	h_S	u_S
∅	1	1	aaa
0	12	11	baa
1	18	17	aba
2	8	7	aab
01	36	7	bba
02	36	17	bab
12	36	11	abb
012	72	1	bbb.

The **ab-index**

$$\sum_S h_S \cdot u_S$$

$$\overline{\Psi}\left(\text{⬡}\right) = 1\ aaa + 11\ baa + 17\ aba + 7\ aab$$
$$+\ 7\ bba + 17\ bab + 11\ abb + 1\ bbb$$

$$= (a+b)^3 + 10\ baa + 16\ aba + 6\ aab$$
$$+\ 6\ bba + 16\ bab + 10\ abb$$

$$= (a+b)^3 + 6\ (a+b)(ab+ba) + 10\ (ab+ba)(a+b)$$

Let
$$c = a+b$$
$$d = ab + ba$$

Then
$$\overline{\Psi}\left(\text{⬡}\right) = c^3 + 6\,cd + 10\,dc.$$

The <u>cd-index</u>.

Theorem: [Bayer - Klapper 1991]

P polytope then $\Phi(P) \in \mathbb{Z}\langle c,d\rangle$.

P Eulerian poset then Φ

Eulerian: $\mu(\upsilon x, y) = (-1)^{\rho(\upsilon x, y)}$ for every interval $[\upsilon x, y]$ in a graded poset P.

Equivalently, in every non-trivial interval $[\upsilon x, y]$:

$$\# \text{ elts of even rank} = \# \text{ elts of odd rank}.$$

Some cd-history.

1980's:

    [Bayer - Billera]   Generalized Dehn-Sommerville relations.

1990's:

    [Bayer - Klapper]   Existence of Φ; "cd-index is a basis for GDDS"

    [Purtill]. Φ (n-simplex) $\Longleftrightarrow$ André and signed André perms.
                  Φ (n-cube)

    [Stanley]. $\Phi \geq 0$ for $\mathcal{L}$ (polytope), more generally, for S-shellable face poset of a regular CW-complex

    [Ehrenborg - Readdy]   Coalgebraic techniques; Geometric operations.

1990's. (cont'd)

[Billera - Ehrenborg - Readdy]. Zonotopes span;
Φ (oriented matroids),

[Billera - Ehrenborg]. Φ (n-polytope) $\geq \Phi$ (n-simplex),

2000's.

[Karu]. Φ (Gorenstein* posets) ≥ 0.

[Karu - Ehrenborg]. Φ (Gorenstein* lattices) $\geq \Phi(B_n)$.

[Ehrenborg]. New inequalities for flag vectors of polytopes.

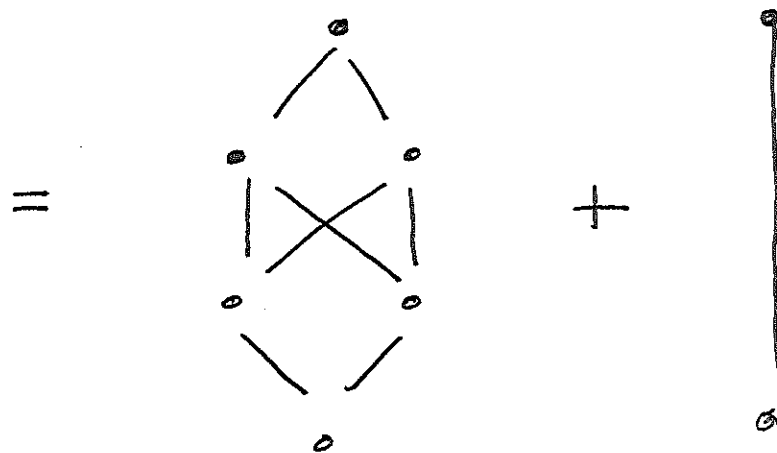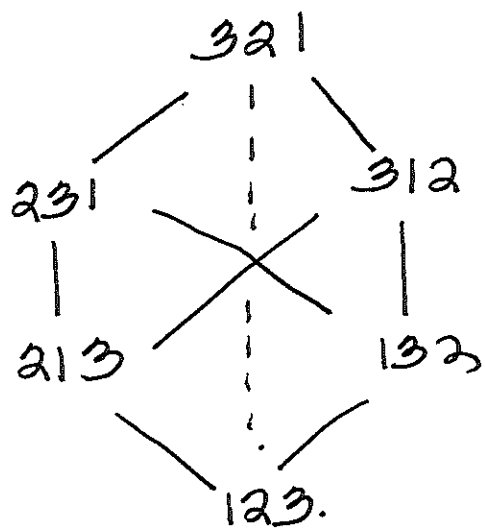[Ehrenborg - Readdy - Slone]. Combinatorics of arrangements of subspaces + subtori.

2010 and beyond:

Generalizations ...

[Billera - Brenti]. $\overline{\underline{I}}$ (Bruhat graphs) via
quasi-symmetric functions;
Kazhdan — Lusztig theory.
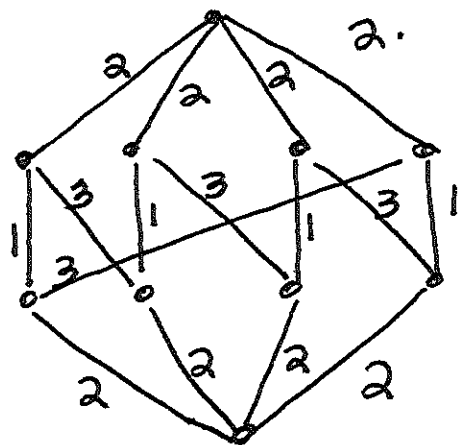
ex. (W,S) Coxeter system.
     Bruhat graph.

$$\overline{\underline{I}} \quad = \quad c^2 \;+\; 1.$$

$\overline{\underline{I}}$ nonhomogeneous!

[Ehrenborg–Readdy]. Balanced acyclic digraphs.

ex.

(direct all
edges upward).

4 max'l chains labeled   232
4 max'l _____   212.

$$\sum_c w(c) = 4ab + 4ba = 4d.$$

(usual.
$\overline{\Phi}$(n-gon) = c^2 + (n-2)d),

Generalize R-labeling
setting, + Bruhat graphs.

[Ehrenborg-Goresky-Readdy]. Whitney stratifications.*

ex. $\overline{\overline{F}}$ ( ) = $c^2 - d$.

$\overline{\overline{F}}$ ( ) = $3dc - 2cd$.

Generalize: Eulerianess.
loosen grading condition.
face poset.

* convex polytopes, regular cell complexes, real or complex
algebraic sets, analytic sets, semi-analytic sets,
quotients of smooth manifolds by compact group actions.

Open Questions / Current Work,

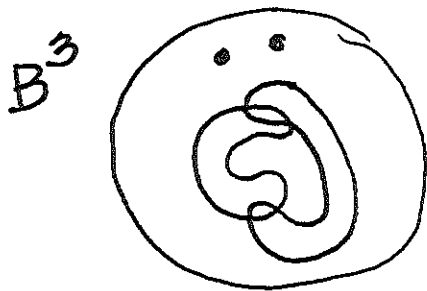①. Find a combinatorial interpretation
   for the cd-coefficients.

   [Purtill] n-simplex + n-cube
   [Karu] operators on sheaves of v.s.

②. Conj: [Billera - Brenti].
          $\Xi \geq 0$ for Bruhat graphs.

③. Inequalities for Ξ (manifold arrangements) ?

B^3

Intersection
poset

S^0 S^0 S^0
S^1 S^1