Convergence result of a non montone scheme for HJB equations

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Outline

Hamilton - Jacobi - Bellman equation

➤ Let ϑ be the unique bounded lsc (or continuous) *viscosity* solution of the HJB equation:

$$
\begin{aligned}
&\partial_t \vartheta(x,t) + \sup_{a \in A}(-D_x \vartheta(x,t) \cdot f(x,a)) = 0 \\
&\vartheta(x,0) = \Phi(x).
\end{aligned}
$$

Case Φ continuous: Crandall, P.L Lions, Cappuzo-Dolcetta/Bardi, Barles,...

Case Φ discontinuous: Frankowska, Barron-Jensen, ...

- The lsc solution ϑ is the value function corresponding to the control problem:

$$\begin{aligned} \text{Minimise} \quad & \Phi(y_x(t)) \\ & \dot{y}_x(s) = f(y_x(s), a(s)), \\ & y(0) = x, \\ & a(s) \in A \text{ a.e.} \end{aligned}$$

- Barles-Souganidis'91: general convergene framework for [monotone+regular+consistante] schemes (only when Φ is continuous).

When Φ is continuous, it is proven in (Crandall&Lions'84) that monotone schemes have the following favorable properties:

- Monotone schemes are stable in the L^∞ norm;
- under the monotonicity assumption, the scheme satisfies a *discrete comparison principle*: If u^h and v^h are, respectively, discrete sub- and super-solutions, then $u^h \leq v^h$.
- The error between the numerical solution of a monotone scheme and the exact viscosity solution of the HJ equation, measured in the L^∞ norm, is in general of order $O(h^{1/2})$.

- However, it is an unfortunate fact that linear monotone schemes cannot be higher than first order accurate for smooth solutions.
- Monotone schemes based on “interpolation“ technics are not suitable for the approximation of discontinuous solutions.

Consider the "linear" case,

$$\begin{cases} v_t + f(x) \cdot v_x = 0, \\ v(0,x) = v_0(x) \end{cases}$$

Idea: Approximate the exact average value on each mech interval

$$\widetilde{V}_j^n = \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} v(t_n, x) dx,$$

and not the point value $v(t_n, x_j)$.

Discretization: Uniform mesh x_j, t_n

$$\begin{cases} \dfrac{V_j^{n+1} - V_j^n}{\Delta t} + f(x_j)\, \dfrac{V_{j+1/2}^{n,L} - V_{j-1/2}^{n,R}}{\Delta x} = 0 \\ V_j^0 = \dfrac{1}{\Delta x} \displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} v_0(x)dx \end{cases}$$

- Assume $f > 0$. The upwind scheme is stable but it is diffusive, while the downwind scheme is anti-diffusive but it is unstable
- We denote $\nu_j := f(x_j)\frac{\Delta t}{\Delta x}$ and assume the CFL condition

$$|\nu_j| \leq 1,$$

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Flux definitions for UB-G scheme

- If $\nu_j > 0$, then $V^{n,L}_{j+1/2} := \min(\max(V^n_{j+1}, b_j), B_j)$ (Déprès-Lagoutière)

- If $\nu_j < 0$ then $V^{n,R}_{j-1/2} := \min(\max(V^n_{j-1}, b^-_j), B^-_j)$,

- If $\nu_j \leq 0$ and $\nu_{j+1} \geq 0$, then define

$$V^{n,R}_{j+\frac{1}{2}} := V_{j+1} \quad \text{and} \quad V^{n,L}_{j+\frac{1}{2}} := V_j. \qquad \text{("Downwind choice")}$$

- If $\nu_j \nu_{j+1} > 0$, then define

$$V^{n,R}_{j+\frac{1}{2}} := V^{n,L}_{j+\frac{1}{2}} \text{ (if } \nu_j > 0) \text{ or } V^{n,L}_{j+\frac{1}{2}} := V^{n,R}_{j+\frac{1}{2}} \text{ (if } \nu_{j+1} < 0).$$

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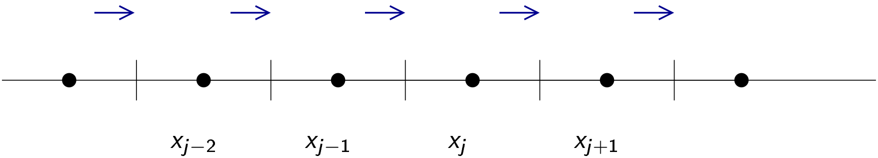
where $\begin{cases} b_j := \mathrm{Max}(V^n_j, V^n_{j-1}) + \frac{1}{\nu_j}(V_j - \mathrm{Max}(V^n_j, V^n_{j-1})), \\ B_j := \mathrm{Min}(V^n_j, V^n_{j-1}) + \frac{1}{\nu_j}(V_j - \mathrm{Min}(V^n_j, V^n_{j-1})), \end{cases}$ $\quad \nu_j = f(x_j)\frac{\Delta t}{\Delta x}$.



Case $\nu_j > 0$, for all j: $\quad V^{n,L}_{j+\frac{1}{2}} =: V^{n,R}_{j+\frac{1}{2}}$

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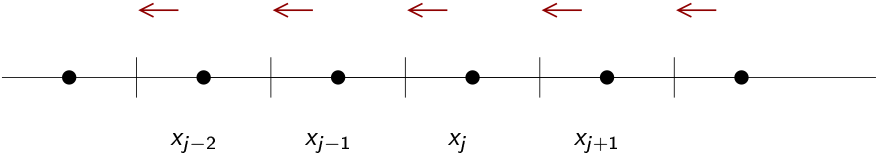

$$\text{Case } \nu_j < 0, \text{ for all } j: \; V^{n,R}_{j+\frac{1}{2}} =: V^{n,L}_{j+\frac{1}{2}}$$

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(H): There is a finite number of points x^* s.t. $f(x^*) = 0$

Theorem (Bokanowski-Z, JSC 2007:)

Under (H) and the CFL condition $|\nu_j| \leq 1$, the scheme is:

(i) *consistent,*

(ii) *L^∞-stable,*

(iii) *TVB, i.e., $\exists C \geq 0$ s.t. $\forall V^0$, $\forall n \geq 0$,*

$$TV(V^n) \leq TV(V^0)\,(1 + C\Delta t)$$

.

(iv) *Moreover, the scheme is convergent*

Rem: The difficulty comes from the points x_j s.t. $f(x_j) < 0$ and $f(x_{j+1}) > 0$

Interesting features of the UB scheme:

- For a constant velocity f, UB advects "exactly" a step function space (Deprès-Lagoutière 2000)
- UB rapidly projects other functions on this space. This is a *conjecture*, yet numerically checked on many examples (for $\nu \neq \frac{1}{2}$.) (Deprès-Lagoutière 2000)
- These properties extend to multi-dimensions ! (with Trotter splitting)
- Simple to implement !

Drawbacks

- It is not monotone

The HJB equation

$$\vartheta_t + \max_{\alpha \in A}(f(x,\alpha)\vartheta_x) = 0,$$
$$\vartheta(x,0) = \Phi(x),$$

where A is a compact set.

Set: $f_m(x) = \min_{\alpha \in A} f(x,\alpha)$, $f_M(x) = \max_{\alpha \in A} f(x,\alpha)$.

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- Step 1: $V^0_{3j} = V^0_{3j+1} = V^0_{3j+2} := \frac{1}{3\Delta x} \int_{x_{3j-\frac{1}{2}}}^{x_{3(j+1)-\frac{1}{2}}} \Phi(x)dx,$

- Step 2: For $n \geq 0$, knowing $V^n_{.}$

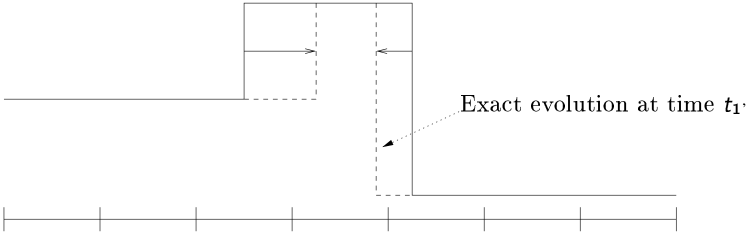
  - For $f \in f_m(x_j), f_M(x_j)$

$$\frac{U_j^{n+1}(f) - V_j^n}{\Delta t} + f \, \frac{U^{n,L}_{j+\frac{1}{2}}(f) - U^{n,R}_{j-\frac{1}{2}}(f)}{\Delta x} = 0,$$

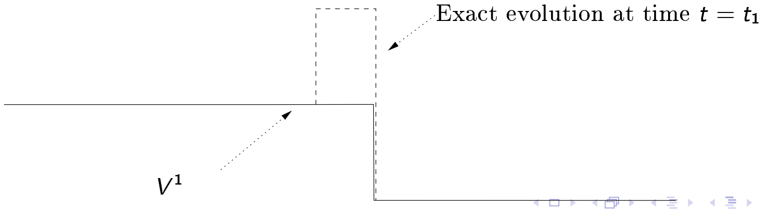
  - Take $V_j^{n+1} := \min_{f=f_m(x_j), f_M(x_j)} U_j^{n+1}(f), \ \forall j.$

  - Truncation step

Initial data



Evolution by the scheme



Theorem (Numerische Math'10, Math Comp'10)

Assume: - f *is L-lipschitz continuous,*
- Φ *is piecewise* C^1 *regular with compact support,*
- *CFL condition:* $\max_{k,x} |f(x,u_k)| \frac{\Delta t}{\Delta x} \leq 1$.

For $T > 0$, $\exists C(L,T,\Phi) > 0$ *s.t.*

$$e^n := \|V^n_\Delta - \overline{V}^n\|_{L^1(\mathbb{R})} \leq C\Delta x \qquad \forall t_n = n\Delta t \leq T,$$

where for $x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$

$$V^n_\Delta(x) = V^n_j,$$

and $$\overline{V}^n(x) = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \vartheta(\xi, t_n)\, d\xi$$

Some notations

- For $a \in \mathbb{R}$, define:

$$\begin{cases} \dot{X}_a^m(t) = f_m(X_a^m(t)) \\ X_a^m(0) = a \end{cases} \qquad \begin{cases} \dot{X}_a^M(t) = f_M(X_a^M(t)) \\ X_a^M(0) = a \end{cases}$$

- For $x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, set:

$$f_m^S(x) := f_m(x_j), \qquad f_M^S(x) := f_M(x_j),$$

and define $X_a^{m,S}$ and $X_a^{M,S}$ by:

$$\begin{cases} \dot{X}_a^{m,S}(t) = f_m^S(X_a^{m,S}(t)) \\ X_a^{m,S}(0) = a \end{cases} \qquad \begin{cases} \dot{X}_a^{M,S}(t) = f_M^S(X_a^{M,S}(t)) \\ X_a^{M,S}(0) = a \end{cases}$$

Idea of the proof: The simple case when $\Phi(x) = 1_{[a,+\infty[}$

- The viscosity solution is given by

$$\vartheta(x,t) = 1_{[X_a^M(t),+\infty[}(x)$$

- The numerical solution satisfies:

$$V_j^n = \frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \vartheta^S(x,t^n)\,dx,$$

where $\vartheta^S(x,t) := 1_{[X_a^{M,S}(t),+\infty[}(x).$

- $||\vartheta^S(t,.) - \vartheta(t,.)||_{L^1(\mathbb{R})} = |X_a^M(t) - X_a^{M,S}(t)|.$

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- $||\vartheta^S(t,.) - \vartheta(t,.)||_{L^1(\mathbb{R})} \leq \frac{1}{2}\, Lt\, e^{Lt}\, \Delta x.$

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- $||\vartheta^S(t,.) - \vartheta(t,.)||_{L^1(\mathbb{R})} = 0$ (Eikonale eq. constant velocity).

Idea of the proof: ϑ_0 has one maximum

We considere here that:

(i) $\vartheta_0(x)$ is an l.s.c. *step function*;

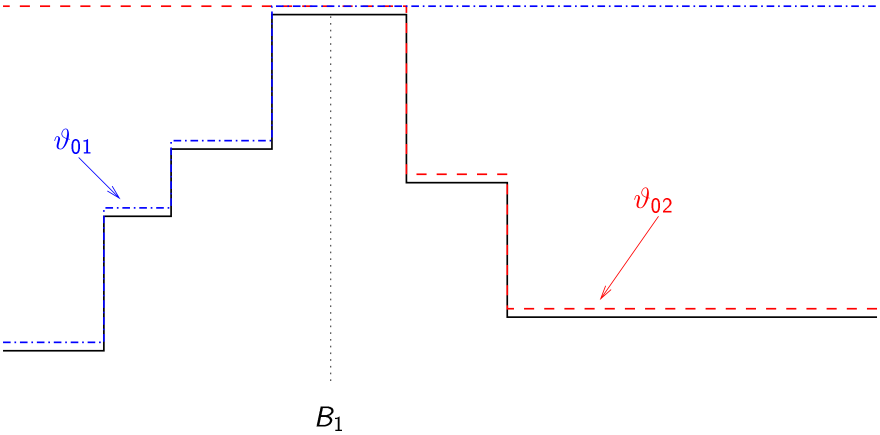
(ii) $\exists\ B_1 \in \mathbb{R}$, s.t. $\vartheta_0(x) \nearrow$ for $x \leq B_1$ and $\vartheta_0 \searrow$ for $x \geq B_1$.

(ii) f_m and f_M are increasing functions

Decomposition Lemma

- $\exists\ \vartheta_{01}, \vartheta_{02}$ s.t. $\vartheta_0 = \min(\vartheta_{01}, \vartheta_{02})$ with $\vartheta_{01} \nearrow$, $\vartheta_{02} \searrow$.
- Then $\vartheta(t,x) = \min(\vartheta_1(t,x), \vartheta_2(t,x))$ where

$$\vartheta_1(t,x) := \vartheta_{01}(X_x^M(-t)) \quad \text{and} \quad \vartheta_2(t,x) := \vartheta_{02}(X_x^m(-t)).$$



ϑ_0 and its decomposition into ϑ_{01} and ϑ_{02}.

Idea of the proof: ϑ_0 has one maximum

$$\vartheta(x,t)=\min\left(\vartheta_{01}(X_x^M(-t)),\vartheta_{02}(X_x^m(-t))\right)=\min\left(\vartheta_1(x,t),\vartheta_2(x,t)\right)$$

$$\vartheta^S(x,t)=\min\left(\vartheta_{01}(X_x^{M,S}(-t)),\vartheta_{02}(X_x^{m,S}(-t))\right)=\min\left(\vartheta_1^S(x,t),\vartheta_2^S(x,t)\right)$$

- $\|V^n-\overline{\vartheta}^S(\cdot,t_n)\|\leq C\Delta x.$
- $||\vartheta(.,t_n)-\vartheta^S(.,t_n)||_{L^1(\mathbb{R})}$

$$\leq \frac{1}{2}\,Lt\,e^{Lt}\;TV(\vartheta_{01})\Delta x+\frac{1}{2}\,Lt\,e^{Lt}\;TV(\vartheta_{02})\Delta x$$

$$\leq \frac{1}{2}\,Lt\,e^{Lt}\;TV(\vartheta_0)\Delta x$$

Idea of the proof: ϑ_0 has one maximum

Define $\overline{\vartheta}^S(t_n, x) = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \vartheta^S t_n, y)dy,$ for $x \in]x_{j-\frac{1}{2}}, _{j+\frac{1}{2}}[$.

- $\|V^n - \overline{\vartheta}^S(\cdot, t_n)\| \leq C\Delta x.$
- $||\vartheta(., t_n) - \vartheta^S(., t_n)||_{L^1(\mathbb{R})}$
$$\leq \frac{1}{2} Lt\ e^{Lt}\ TV(\vartheta_{01})\Delta x + \frac{1}{2} Lt\ e^{Lt}\ TV(\vartheta_{02})\Delta x$$
$$\leq \frac{1}{2} Lt\ e^{Lt}\ TV(\vartheta_0)\Delta x$$

Idea of the proof: A general case

Suppose $\begin{cases} \vartheta_0(x) \text{ is a step function;} \\ \vartheta_0 \text{ has } q \text{ local maxima (denoted } B_1, \ldots, B_q). \end{cases}$

Let $\quad \vartheta_0^{(i)}(x) := \min_{y\in[x;B_i]} \vartheta_0(y), \quad i = 1, \ldots, q.$

By the first decomposition Lemma,

$$\vartheta_0^{(i)} = \min(\vartheta_{01}^{(i)}, \vartheta_{02}^{(i)}), \qquad \text{with } \vartheta_{01}^{(i)} \nearrow, \ \vartheta_{02}^{(i)} \searrow.$$

General Decomposition Lemma

(i) $\vartheta_0 = \max\limits_{i=1,\ldots,q} \min\left(\vartheta_{01}^{(i)},\ \vartheta_{02}^{(i)}\right), \quad$ with $\vartheta_{01}^{(i)} \nearrow,\ \vartheta_{02}^{(i)} \searrow$

(ii) $\vartheta(t,x) = \max\limits_{i=1,\ldots,q} \min\left(\vartheta_{01}^{(i)}(X_x^m(-t)),\ \vartheta_{02}^{(i)}(X_x^M(-t))\right)$

(iii) $\vartheta^S(t,x) := \max \min\left(\vartheta_{01}^{(i)}(X_x^{m,S}(-t)),\ \vartheta_{02}^{(i)}(X_x^{M,S}(-t))\right)$

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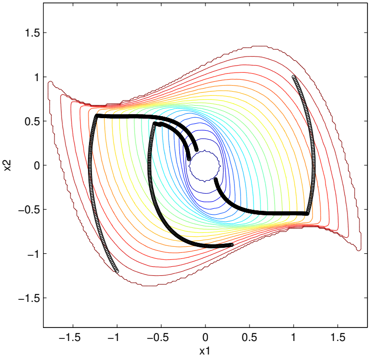
(iv) $\|V^n - \overline{\vartheta}^S(\cdot, t_n)\| \leq C\Delta x.$

Van der Pol Problem :

$$\begin{cases} \dot{y}_1(t) = y_2 \\ \dot{y}_2(t) = -y_1 + y_2(1 - y_1^2) + a(t) \\ a(t) \in [-1, 1] \end{cases}$$

$$\Phi(y) = 1 - \mathbf{1}_{|y| \leq r_0}$$



- The value function needs to be computed only in a neighborhood of the front: Narrow band implementation for front propagation problems (with $\Phi(x) \in \{0, 1\}$).

Small target problem

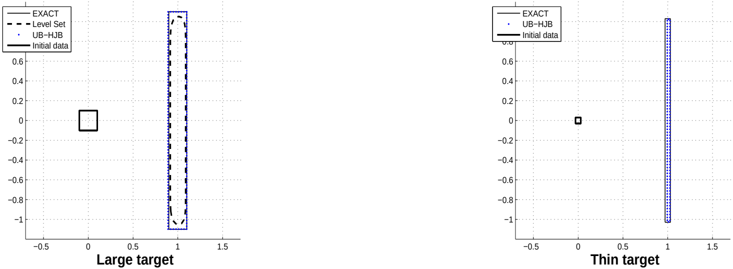
- Consider

$$\begin{cases} \vartheta_t(t,x) + \vartheta_{x_1}(t,x) + |\vartheta_{x_2}(t,x)| = 0, & t \in [0,T],\ x = (x_1,x_2) \in \mathbb{R}^2. \\ \vartheta(0,x) = \varphi_r(x), \end{cases}$$

where the initial data is given by

$$\varphi_r(x) := \begin{cases} -1 & \text{if } ||x||_\infty \leq r, \\ 1 & \text{otherwise} \end{cases}$$

- Two types of target:
  - $r = 0.1$: **large** target case.
  - $r = \Delta x$: **thin** target case.


EXACT
Level Set
UB−HJB
Initial data
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−0.5
0
0.5
1
1.5
Large target
EXACT
UB−HJB
Initial data
Thin target

Large target ($r = 0.1$)

N^2	UB-HJB		Level set	
	L^1 error	Haus.	L^1 error	Haus.
51^2	0.178	0.052	-	-
101^2	0.105	0.022	0.101	0.094
201^2	0.044	0.011	0.008	0.047
401^2	0.022	0.006	0.006	0.027

Thin target ($r = \Delta x$)

N^2	UB-HJB		Level set	
	L^1 error	Haus.	L^1 error	Haus.
51^2	0.166	0.043	-	-
101^2	0.080	0.031	-	-
201^2	0.040	0.016	-	-
401^2	0.020	0.008	-	-

2-dimensional deformation of a half plane

• Consider a front propagation problem, where the initial front Γ_0 is given by: $\Gamma_0 := \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0\}$.
The velocity of the front evolution is given by

$$f(t, x_1, x_2) = -sign(\frac{T}{2} - t) \begin{pmatrix} -2\pi\, x_2 \\ 2\pi\, x_1 \end{pmatrix} \max(1 - \sqrt{x_1^2 + x_2^2}, 0).$$

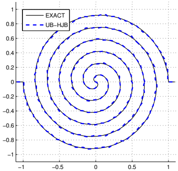
• Hence the evolution is driven by

$$\begin{cases} \vartheta_t(t, x) + f(t, x) \cdot \nabla\vartheta(t, x) = 0 \quad x \in \mathbb{R}^2,\ t \in [0, T], \\ \vartheta(0, x) = \varphi(x) \end{cases} \tag{1}$$

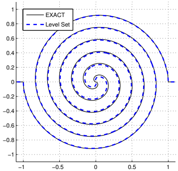
and with (for the UltraBee scheme):

$$\varphi(x_1, x_2) := \begin{cases} -1 & x_2 \leq 0 \\ 1 & \text{otherwise} \end{cases} \tag{2}$$

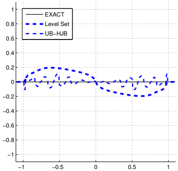
N = 100



UB-HJB, $t = 3$

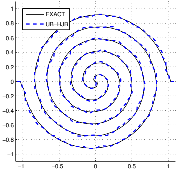


Level set, $t = 3$

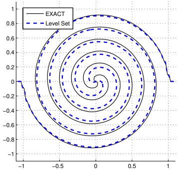


$t = 6$

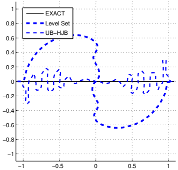
N = 50



UB-HJB, $t = 3$



Level set, $t = 3$



$t = 6$

$T = 3$

N^2	UB-HJB		Level set	
	L^1 error	Haus.	L^1 error	Haus.
50^2	0.170	0.035	0.584	0.086
100^2	0.092	0.019	0.136	0.028
200^2	0.057	0.013	0.047	0.008

$T = 6$

N^2	UB-HJB		Level set	
	L^1 error	Haus.	L^1 error	Haus.
50^2	0.193	0.308	0.995	0.639
100^2	0.073	0.107	0.282	0.195
200^2	0.041	0.064	0.079	0.053

3-dimensional rotation problem

- 3d advection:

$$\begin{cases} \vartheta_t(t,x) + f(x)\cdot\nabla\vartheta(t,x) = 0, & t\in[0,\ T],\ x\in[-2,2]^3, \\ \vartheta(0,x) = \varphi(x) \end{cases}$$

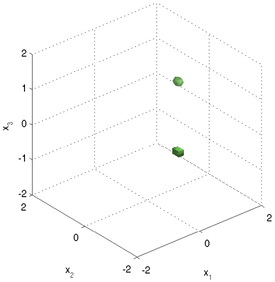
- Corresponding target problem:

$$\begin{cases} \vartheta_t(t,x) + \max(0,\ f(x)\cdot\nabla\vartheta(t,x)) = 0, & t\in[0,\ T],\ x\in[-2,2]^3, \\ \vartheta(0,x) = \varphi(x) \end{cases}$$

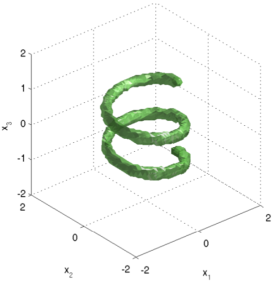
with

$$f(x_1,x_2,x_3) = (-2\pi x_2, 2\pi x_1, -1)^{\mathsf{T}}$$

- The initial data corresponds to a sphere centered at $(-1,0,1)$ and with radius $r=0.1$.



Advection



Target problem

50^3 mesh points

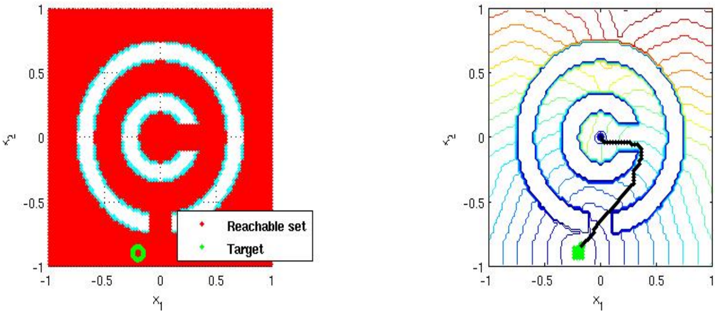
Advection

N^3	CPU		L^1 error	Hausdorff
50^3	0.22		4.1e-3	2.6e-1
100^3	1.00	4.6	2.2e-3	8.0e-2
200^3	7.19	7.2	6.2e-4	4.4e-2

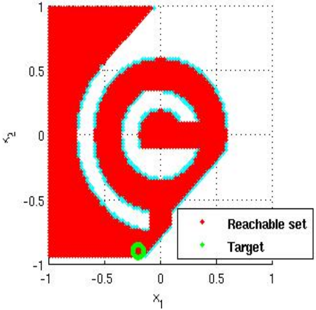
Target problem

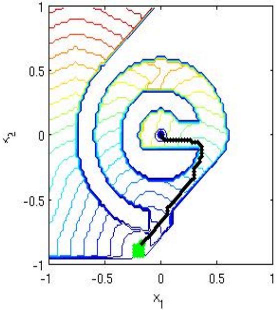
N^3	CPU		L^1 error	Hausdorff
50^3	3.61		1.3e-1	1.9e-1
100^3	17.9	5.2	6.4e-2	8.0e-2
200^3	128.8	7.2	5.6e-2	4.9e-2

Controllable case (8 directions)



Non-controllable case (4 directions: $\swarrow, \downarrow, \searrow, \rightarrow$)



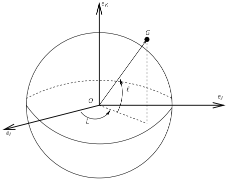


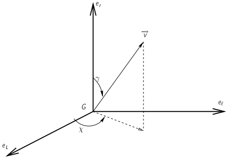
Ariane V

GOAL

For a given payload M_{CU}, minimize the consumption needed to steer the launcher to the GTO.

- The physical model involves 7 state variables, the position $\overrightarrow{OG}$ of the rocket in the 3D space, its velocity $\overrightarrow{v}$ and its mass m.





Projection of $\overrightarrow{v}$ on the frame (e_r, e_L, e_ℓ)

- The forces acting on the rocket are: Gravity $\overrightarrow{P}$, Drag $\overrightarrow{F_D}$, Thrust $\overrightarrow{F_T}$, and Coriolis $\overrightarrow{\Omega}$.
- Newton Law:

$$m\frac{d\overrightarrow{v}}{dt} = \overrightarrow{P} + \overrightarrow{F_D} + \overrightarrow{F_T} - 2m\overrightarrow{\Omega} \wedge \overrightarrow{v} - m\overrightarrow{\Omega} \wedge (\overrightarrow{\Omega} \wedge \overrightarrow{OG}),$$

The related equation

State variables:

r=altitude

v=modulus of the velocity

γ=angle between the direction earth-rocket and the direction of the rocket's velocity.

L= latitude

ℓ= longitude

χ= azimuth

m= masse of the engine

Control:

α=angle between the thrust direction and the direction of the rocket's velocity.

$$\dot{r} = v\cos\gamma$$

$$\dot{v} = -g(r)\cos\gamma - \frac{F_D(r,v)}{m} + \frac{F_T(r,v,a)}{m}\cos\alpha$$
$$\Omega^2 r\cos\ell(\cos\gamma\cos\ell - \sin\gamma\sin\ell\sin\chi)$$

$$\dot{\gamma} = \sin\gamma\left(\frac{g(r)}{v} - \frac{v}{r}\right) - \frac{F_T(r,v,a)}{vm}\sin\alpha$$
$$-2\Omega\cos\ell\cos\chi - \Omega^2\tfrac{r}{v}\cos\ell(\sin\gamma\cos\ell - \cos\gamma\sin\ell\sin\chi)$$

$$\dot{L} = \frac{v}{r}\frac{\sin\gamma\cos\chi}{\cos\ell}$$

$$\dot{\ell} = \frac{v}{r}\sin\gamma\sin\chi$$

$$\dot{\chi} = -\frac{v}{r}\sin\gamma\tan\ell\cos\chi - 2\Omega(\sin\ell - \mathrm{cotan}\gamma\cos\ell\sin\chi) +$$
$$\Omega^2\frac{r}{v}\frac{\sin\ell\cos\ell\cos\chi}{\sin\gamma}$$

➢ The plane of motion is the equatorial plane $\ell \equiv 0$, and $\chi \equiv 0$.

$$\dot{r} = v\cos\gamma$$

$$\dot{v} = -g(r)\cos\gamma - \frac{F_D(r,v)}{m} + \frac{F_T(r,v,a)}{m}\cos\alpha + \Omega^2 r\cos\gamma$$

$$\dot{\gamma} = \sin\gamma\left(\frac{g(r)}{v} - \frac{v}{r}\right) - \frac{F_T(r,v,a)}{vm}\sin\alpha - 2\Omega - \Omega^2\frac{r}{v}\sin\gamma$$

$$\dot{L} = \frac{v}{r}\sin\gamma$$

$$\dot{m} = -b(m(t))$$

➢ The plane of motion is the equatorial plane $\ell \equiv 0$, and $\chi \equiv 0$.

$$
\begin{aligned}
\dot r &= v\cos\gamma \\
\dot v &= -g(r)\cos\gamma - \frac{F_D(r,v)}{m} + \frac{F_T(r,v,a)}{m}\cos\alpha + \Omega^2 r\cos\gamma \\
\dot\gamma &= \sin\gamma\left(\frac{g(r)}{v} - \frac{v}{r}\right) - \frac{F_T(r,v,a)}{vm}\sin\alpha - 2\Omega - \Omega^2\frac{r}{v}\sin\gamma \\
\dot L &= \frac{v}{r}\sin\gamma \\
\dot m &= -b(m(t))
\end{aligned}
$$

The rocket's mass

- The evolution of the mass can be summarized as follows

Phase 0 & 1	Phase 2	Phase 3
$\dot{m}_1(t) = -\beta_{EAP}$	$\dot{m}_1(t) = 0$	$\dot{m}_1(t) = 0$
$\dot{m}_2(t) = -\beta_{E1}$	$\dot{m}_2(t) = -\beta_{E1}$	$\dot{m}_2(t) = 0$
$\dot{m}_3(t) = 0$	$\dot{m}_3(t) = 0$	$\dot{m}_3(t) = -\beta_{E2}$

where β_{EAP}, β_{E1} and β_{E2} are the mass flow rates for the boosters, the first and the second stage.

- At the changes of phases, we have a (not negligible) discontinuity in the rocket's mass.

The control problem can be formulated as (for a fixed payload)

Minimize t_f

$(r, v, \gamma, m, \alpha)$ satisfy the state equation

$$\alpha(t) \in [0, \pi/2] \quad \text{a.e. } t \in (0, t_f),$$

$$(r(t_f), v(t_f), \gamma(t_f)) \in \mathcal{C},$$

$$Q(r(t), v(t))\alpha(t)) \leq C_s \quad \text{for } t \in (0, t_f),$$
$$m(t_f) = M_p.$$

where the target $\mathcal{C}$ corresponds to the GTO orbit, and the function Q is the dynamic pressure.

- The Capture Basin is wide
  - We introduce "physical" state constraints to define the computational domain
- Due to the CFL condition, the time step is very small
  - Adaptative time discretization
- "Different scales" for the state variables:
  - Change of variable: $\begin{cases} r = r_0(e^x - 1) + r_T \\ v = v_0(e^y - 1) + v_T \end{cases}$

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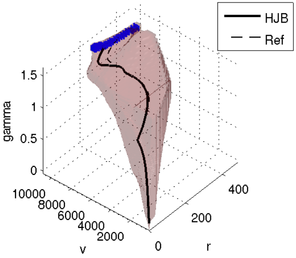
GTO target (comparison with the reference trajectory of CNES)



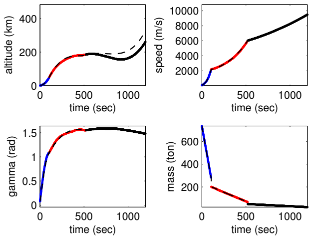


Figure: Full trajectory using the HJB minimal time value function

Reference trajectory, final mass: $m_T = 21.57$ (t)

HJB trajectory, final mass (after reconstruction): $m_T = 22.50$ (t)

... *Thank you for your attention.*