Viscosity related :

Notes on GO (numerics, theory, applications ...)
http://www.trip.caam.rice.edu/downloads/caam641.pdf

J. D. Benamou, S. Luo and H. Zhao, A Compact Upwind Second Order Scheme for the Eikonal Equation, Journal of Computational Mathematics (2010).
$\rightarrow$ *Superconvergence of upwind schemes ????*

Numerical MicroLocal Analysis (NMLA)

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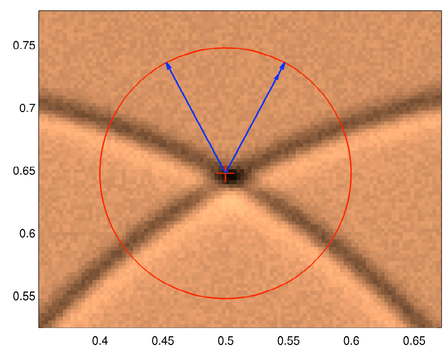
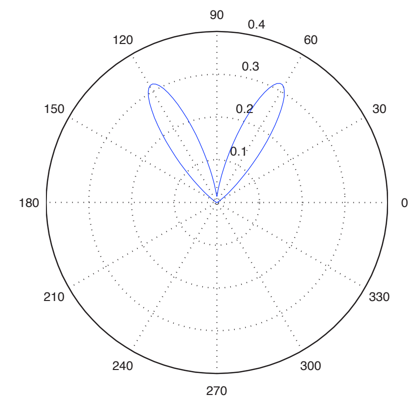
"Given frequency domain wave data, the proposed new algorithm gives a pointwise estimate of the the number of rays, their slowness vectors and corresponding wavefront curvature. With time domain wave data and assuming the source wavelet is given, the method also estimates the traveltime."





NMLA

Geometric optics equations : Find local asymptotic solutions of

$$\frac{\omega^2}{c^2(x)}\widehat{u}(x;\omega) - \Delta\widehat{u}(x;\omega) = 0$$

$\widehat{u}$ replaced by *"ansatz"*

$$\widehat{u} \simeq \widehat{u}^{ray}(x;\omega) = A(x)e^{i\omega\varphi(x)}$$

yields

$$\begin{cases} |\nabla\varphi(x)| = \dfrac{1}{c(x)} \\ 2\nabla\varphi(x)\cdot\nabla A(x) + A(x)\Delta\varphi(x) = 0 \end{cases}$$

Ray equations

$$\begin{cases} \dot{y}(s,x_s) = \nabla\varphi(y(s,x_s)) := p(s,x_s) \\ \dot{p}(s,x_s) = \nabla\frac{1}{c^2(x)}|_{x=y(s,x_s)}, \quad \dot{\varphi}(s,x_s) = \ldots \end{cases}$$

Plane wave approximation

$$\varphi(x) \simeq \varphi(x_0) + (x - x_0) \cdot \nabla\varphi(x_0) + \frac{1}{2}(x - x_0)^T \, H\varphi(x_0)\,(x - x_0) + \dots$$

yields

$$\hat{u}(x;\omega) \simeq B(x_0) e^{i\omega(x-x_0)\cdot\nabla\varphi(x_0)}$$

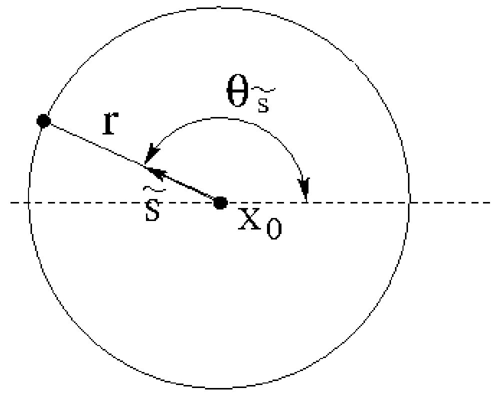
where

$$B(x_0) = A(x_0) e^{i\omega\varphi(x_0)}.$$

The "general" N-rays ansatz :

$$\hat{u}(x;\omega) \simeq \sum_{n=1}^{N} B_n(x_0) e^{i\omega(x-x_0)\cdot\nabla\varphi_n(x_0)} \quad x \text{ near } x_0$$

NMLA observable The observable data ($\tilde{s}$ is on the Sphere $\mathbb{U} = \{\|\tilde{s}\| = 1\}$)



$$U_\alpha(\tilde{s}) = \frac{c(x_0)}{i\omega}\frac{\partial \hat{u}}{\partial r}(x_0 + r\tilde{s}; \omega) + \hat{u}(x_0 + r\tilde{s}; \omega), \qquad r = \frac{\alpha c(x_0)}{\omega}.$$

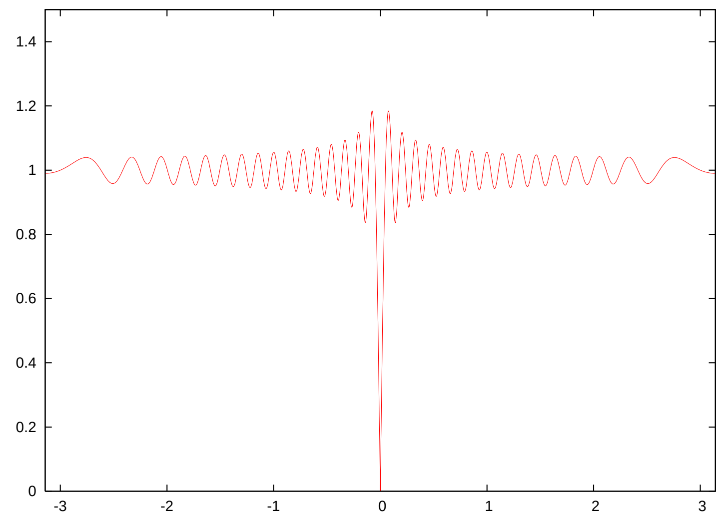
"should" fit the ansatz form

$$U_\alpha(\tilde{s}) \simeq \sum_{n=1}^{N} (\, \tilde{s} \cdot \tilde{d}_n + 1)\, B_n\, e^{i\alpha \tilde{s}\cdot\tilde{d}_n}, \qquad \tilde{d}_n = c(x_0)\, \nabla \varphi_n(x_0)$$

The inverse problem : Given $U_\alpha(\tilde{s})$, find (N, B, d)

$$(B, d) = (B_1, \ldots, B_N; \tilde{d}_1, \ldots, \tilde{d}_N)$$

is not easy ...



$\theta_i \longrightarrow \|e^{i\alpha \cos\theta} - e^{i\alpha \cos(\theta-\theta_i)}\|_{L^2([-\pi,\pi])}$ for $\alpha = 50 \to$ Tons of local minima.

$$\min_{(N,B,d)} \|U(\tilde{s}) - \sum_{n=1}^{N} (\,\tilde{s}\cdot\tilde{d}_n + 1) B_n(x_0) e^{i\alpha\tilde{s}\cdot\tilde{d}_n}\|_V \text{ even HARDER !}$$

Look instead at an infinite dimensional linear problem

Change the unknown (B, d, N) for a density function
$\beta\,:\,\tilde{s} \in \mathbb{U} \mapsto \beta(\tilde{s}) \in \mathbb{C}$

$$\begin{aligned}
U_\alpha(\tilde{s}) &\simeq \sum_{n=1}^{N} (\,\tilde{s}\cdot\tilde{d}_n + 1)\,B_n e^{i\alpha\tilde{s}\cdot\tilde{d}_n} \\
&= \int_{\mathbb{U}} (\,\tilde{s}\cdot\tilde{d} + 1)\,e^{i\alpha\tilde{d}\cdot\tilde{s}}\,\left(\sum_{n=1}^{N} B_n\delta(\tilde{d}-\tilde{d}_n)\right) d\sigma(\tilde{d}) \\
&= \int_{\mathbb{U}} (\,\tilde{s}\cdot\tilde{d} + 1)\,e^{i\alpha\tilde{d}\cdot\tilde{s}}\,\beta(\tilde{d})\,d\sigma(\tilde{d}) \\
&= K_\alpha\,\beta(\tilde{s}).
\end{aligned}$$

- K_α can be diagonalized on the Fourier basis $\left\{ e_l(\tilde{s}) = \frac{1}{\sqrt{2\pi}} e^{il\theta_{\tilde{s}}} \right\}_{l \in \mathbb{Z}}$ and Eigenvalues depend on *Bessel functions*

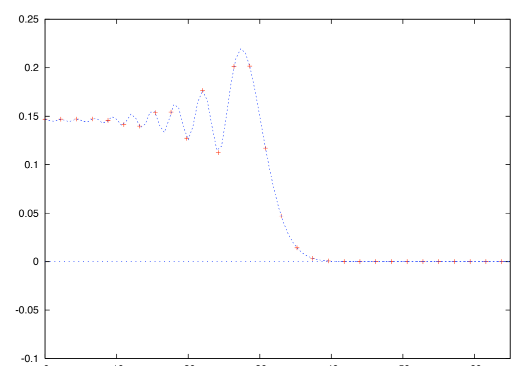
$$K_\alpha e_l = D_l(\alpha)\, e_l, \quad D_\ell(\alpha) = 2\pi i^\ell (J_\ell(\alpha) - i\ J'_\ell(\alpha))$$

$J_l(\alpha)$ *is the Bessel function* of order l and argument α.

- Solution of the linear problem is formally given as

$$\beta := \mathcal{F}^{-1}(\{\widehat{\beta}_\ell\}), \quad \widehat{\beta}_\ell = D_\ell^{-1} \mathcal{F}(\{U_\alpha\})_\ell$$

$J_\ell(\alpha) - i\ J'_\ell(\alpha)$ for $\alpha = 30$:



The bounded (stable) normalized inverse operator :

$$\text{(NMLA filter)}\ \beta := \frac{1}{2\,L(\alpha)+1}\mathcal{F}^{-1}(\{\widehat{\beta}_\ell\}), \qquad \widehat{\beta}_\ell = H_\ell \mathcal{F}(\{U_\alpha\})_\ell$$

with

$$\begin{aligned} H_\ell &= D_\ell^{-1} \text{ if } |\ell| < L(\alpha) \\ &= 0 \text{ else.} \end{aligned}$$

where $L(\alpha) = \min\{\alpha, \alpha + \alpha^{1/3} - 2.5\} \to \|K'^{-1}_\alpha\| < 3$

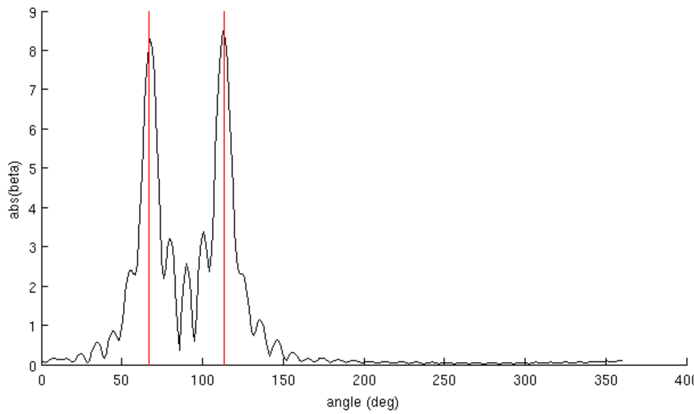
Discretization

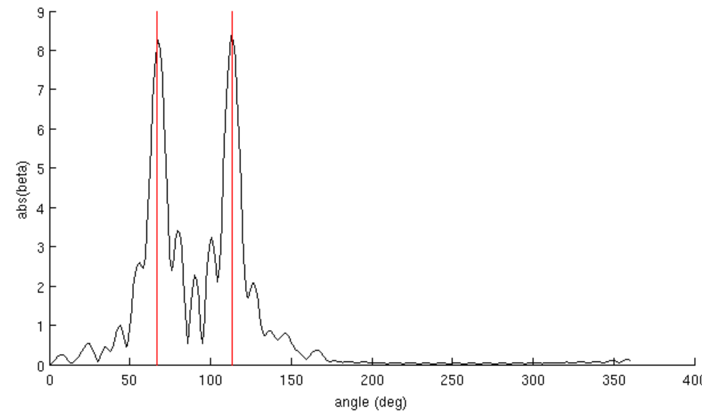
$$\{B_m\} := \frac{1}{2\,L(\alpha)+1}\mathcal{F}^{-1}(\widehat{\beta}_m\})\ \ \widehat{\beta}_\ell = H_\ell \mathcal{F}(\{U_{m'}\})_\ell$$

$$m^\star \text{ such that } \quad |B_{m^\star}| = \max_m |B_m|$$

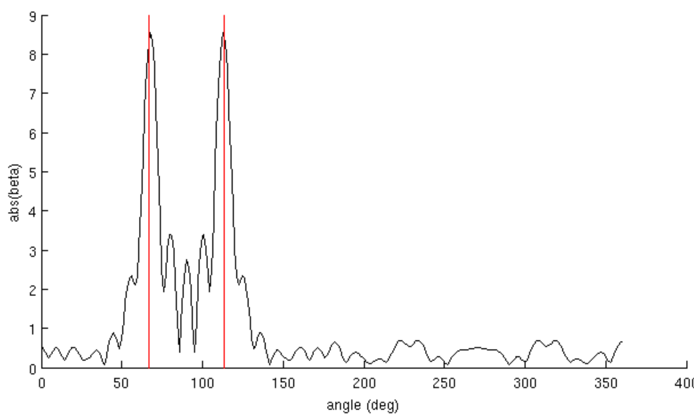
Test 2 sources , homogeneous medium NMLA stability.

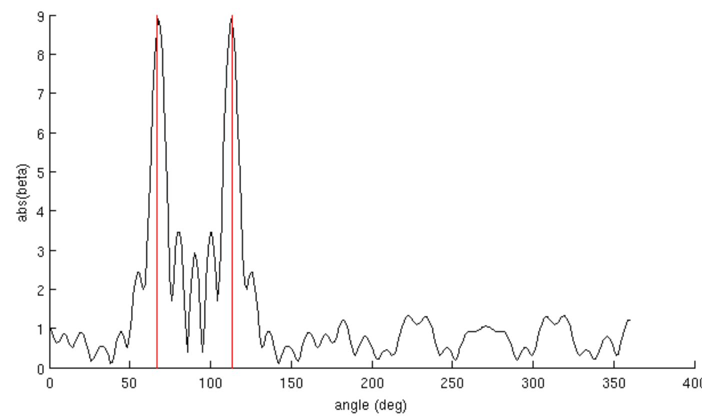
$|\beta(\theta_{\hat{s}})|$ White noise (20%-40%)





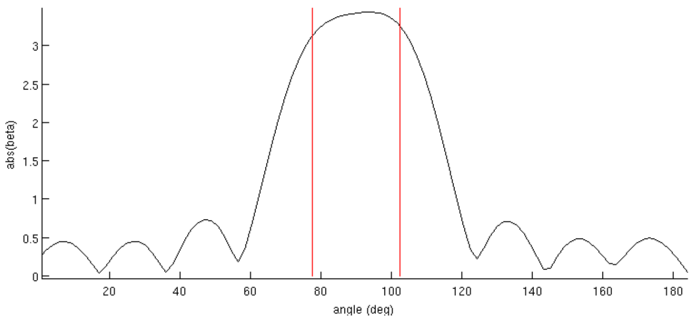
Correlated noise (20%-40%)

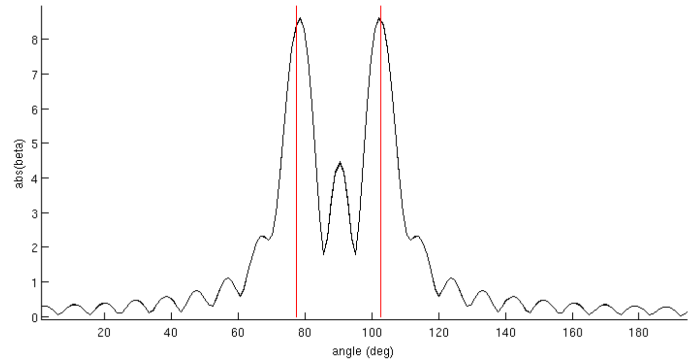


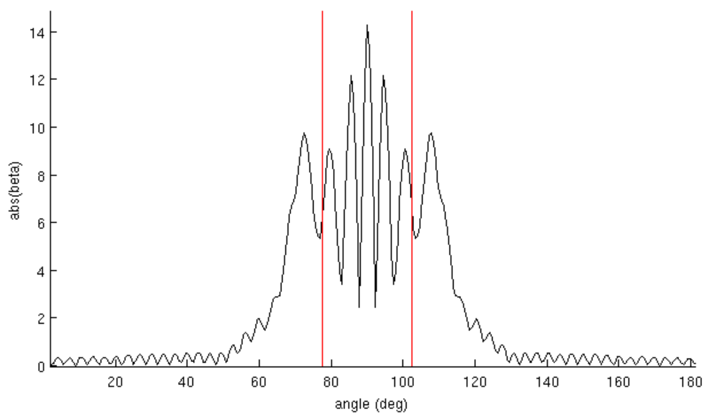


Red lines : exact ray angles.

Varying α : 10, 20, 50







NMLA 2^{nd} order

- α bounds the number of Fourier modes, while we hope to recover dirac masses ...

- Cannot increase α because of the plane wave approximation.

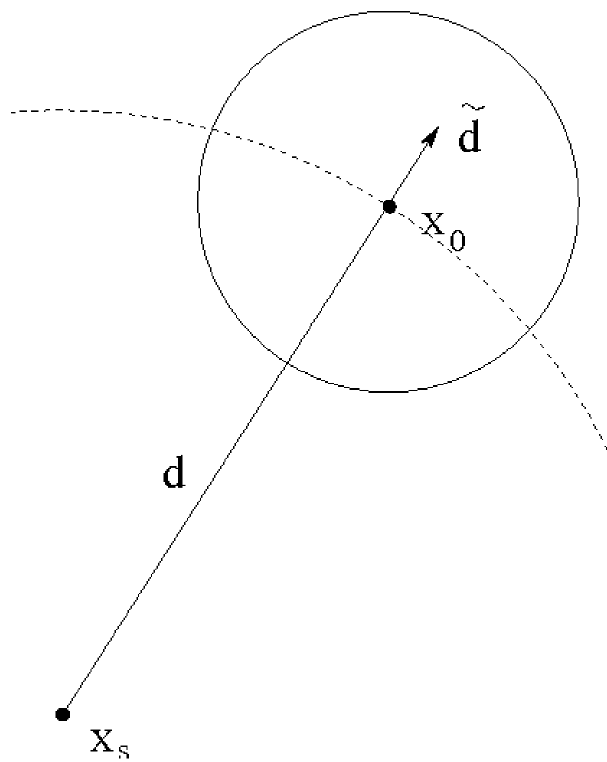
  $$\varphi(x) \simeq \varphi(x_0)+(x-x_0)\cdot\nabla\varphi(x_0)+\frac{1}{2}(x-x_0)^T\, H\varphi(x_0)\,(x-x_0)+\ldots$$

  Recall $x - x_0 = \alpha\frac{c(x_0)}{\omega}$.

$\rightarrow$ Need to estimate 2^{nd} order terms.

The simplest 2^{nd} order approximation

Assume only one ray in the solution.
Constant curvature HF asymptotics

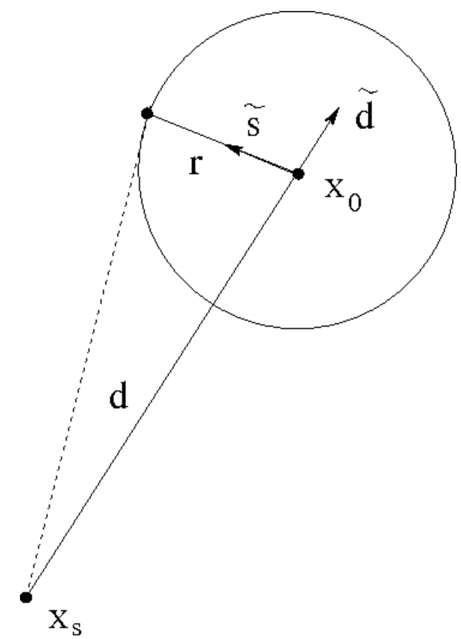


corresponds to Hankel function fundamental solution $H_0^1(\frac{\omega}{c}|x|)$.

Approximate NMLA data with H_0^1

$$U_\alpha(\tilde{s}) \simeq \frac{A_0(x_0)}{|\frac{i}{4}H_0^{(1)}\left(\frac{\omega}{c}|d\tilde{d}+r\tilde{s}|\right)|} e^{i\omega(\varphi(x_0)-d)} \frac{i}{4}H_0^{(1)}\left(\frac{\omega}{c}|d\tilde{d}+r\tilde{s}|\right)$$

$\tilde{d}$ and d yet to be found



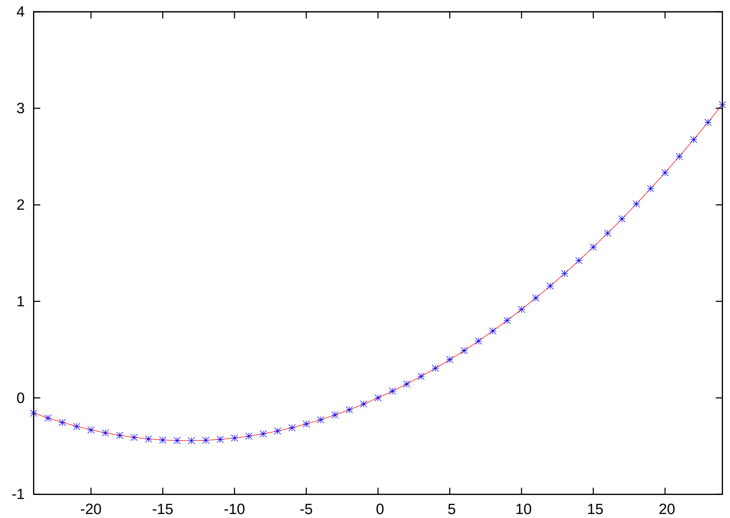
$\theta_{\tilde{d}}$ and $\frac{1}{d}$ are the local

ray direction and mean curvature.

Use FMM type asymptotic expansions $\gamma = \frac{\omega\, d}{c(x_0)}$ is the large parameter. This "new ansatz" yields a curvature correction to the NMLA Fourier modes

$$\widehat{\beta}_\ell \simeq A e^{i\omega\varphi(x_0)} e^{i(\ell\theta_{\tilde{d}} + \frac{(\ell^2 - \frac{1}{4})}{2\gamma})}$$

$\frac{1}{i}\log(\frac{\widehat{\beta}_\ell}{\widehat{\beta}_0})$ versus ℓ



Nota : Plane wave approximation Fourier modes were

$$\widehat{\beta}_\ell \simeq A e^{i\omega\varphi(x_0)} e^{i\ell\theta_{\tilde{d}}}$$

Curvature correction Algorithm

1. Get an estimated $\theta_{m^\star}$ of $\theta_{\tilde{d}}$ (using NMLA).

$$\{B_m\} := \frac{1}{2\,L(\alpha)+1}\mathcal{F}^{-1}(\widehat{\beta}_m\}) \;\; \widehat{\beta}_\ell = H_\ell \mathcal{F}(\{U_{m'}\})_\ell$$

$$m^\star \text{ such that } \quad |B_{m^\star}| = \max_m |B_m|$$

2. Estimate Curvature γ and angle $\delta\theta = \theta_{\tilde{d}} - \theta_{m^\star}$ corrections through parabolic fitting of the phase of

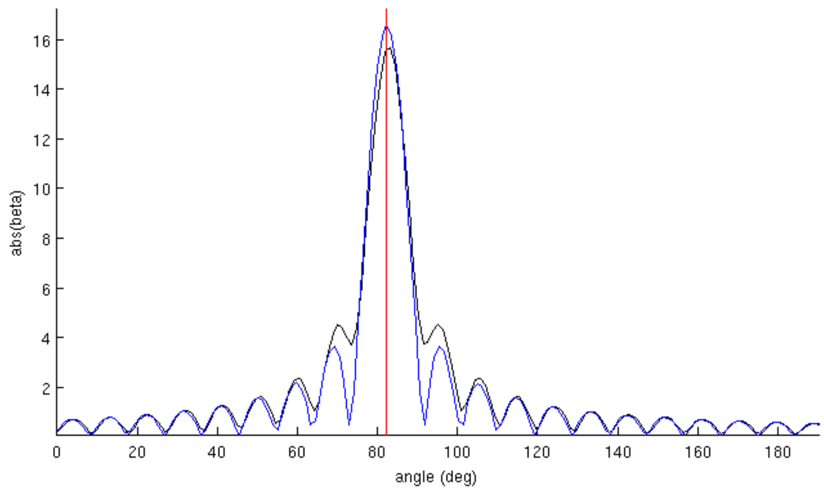
$$\frac{\widehat{\beta}_\ell}{\widehat{\beta}_0} e^{-i\ell\theta_{m^\star}} = e^{i(\ell\delta\theta + \frac{(\ell^2 - \frac{1}{4})}{2\gamma})}$$

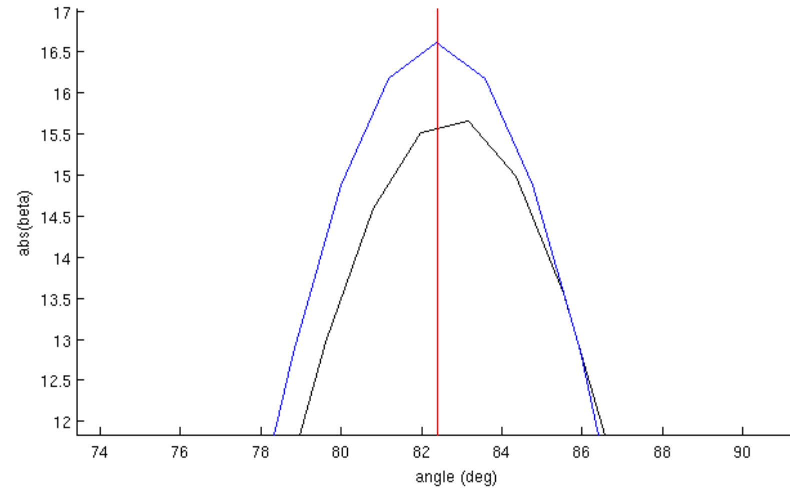
3. Correct NMLA amplitudes

$$\{B'_m\} := \frac{1}{2\,L(\alpha)+1}\mathcal{F}^{-1}(\widehat{\beta}_\ell e^{-i(\ell\delta\theta + \frac{(\ell^2 - \frac{1}{4})}{2\gamma})}\})$$

Test 1 source , homogeneous medium

Black : NMLA - Blue : NMLA 2nd order - Red : exact direction.



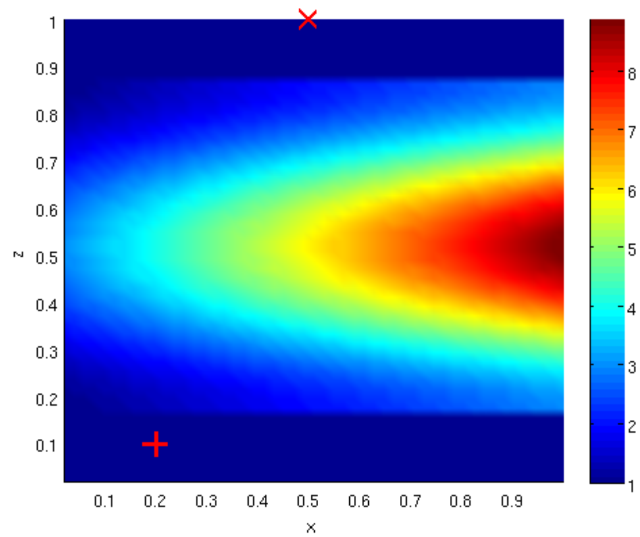


. zoom

Numerical illustration

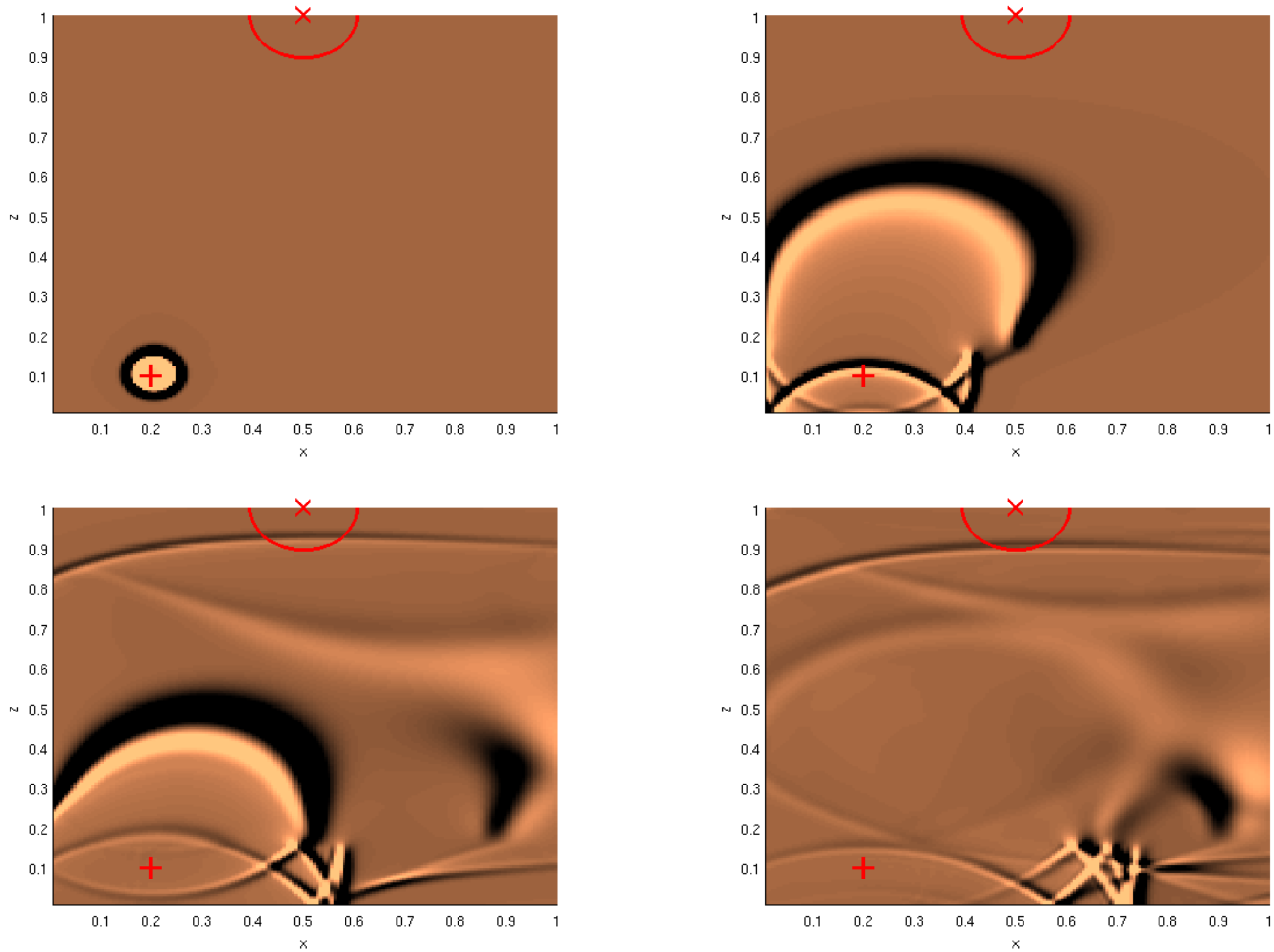
Source point discovery in Heterogeneous medium.

Acoustic speed



$+$: source point, X : observation point

Synthetic data numerical simulation (snapshots)



Generated using standard FDTD + ABCs

Microlocal Analysis at observation

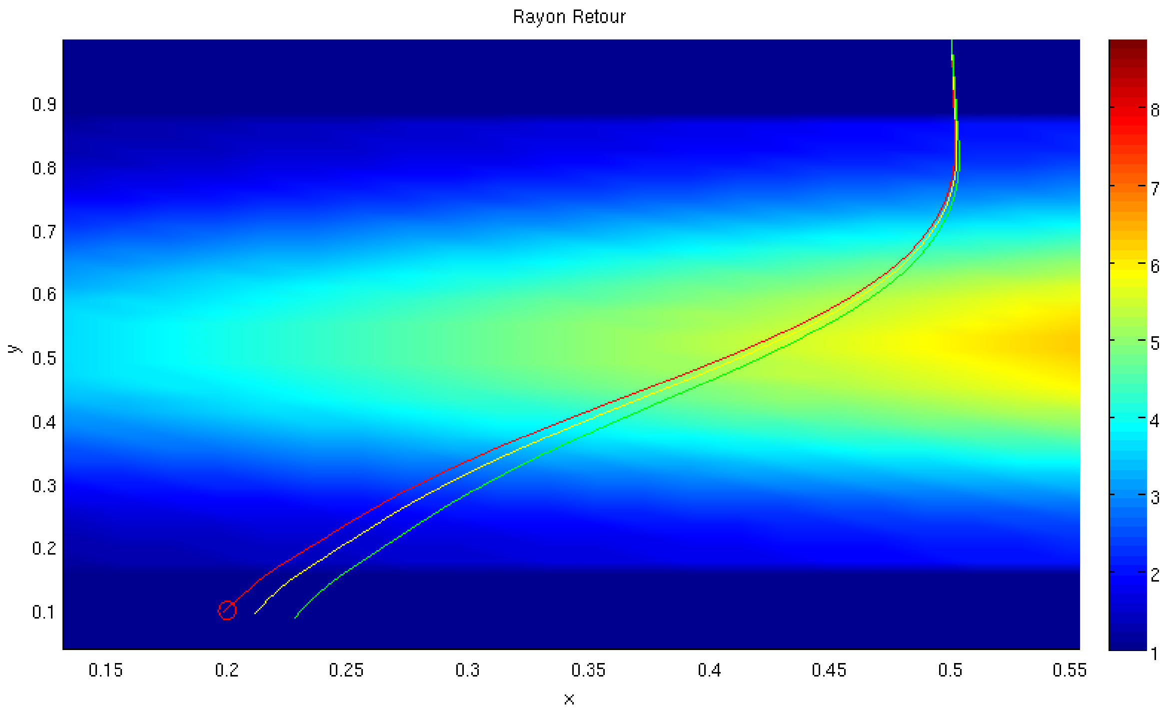
NMLA basic : ray take-off angle at observation point : 90.2970^o

NMLA 2nd order (red) : 90.5503^o traveltime : $0.4602s$.

Radon (green) : 91.002^o

PWD (yellow) : 90.7418^o

Ray backward propagation



Summary/Conclusion* - Robust local HF components analysis tool.

- Based on the "true" HF model.
- Completely automatic, no tuning parameters.
- Possible extensions : 3D, Elastic Waves.
- Possible target applications in wave modeling : hybrid HF/FD-FE methods.
- Possible target applications (Geophysics) : RTM angle gathers, Data analysis and cleaning.
- Possible target applications (Electromagnetics) : Antenna DOA, ISAR bright source points SER reduced models.

*Jean-David Benamou, Francis Collino, and Simon Marmorat. 1)Source point discovery through High Frequency Asymptotic Time Reversal. 2) Numerical MicroLocal Analysis of 2-D Noisy Harmonic Plane Wave and Source Point Wavefields. *Submitted*. Available at `http://hal.archives-ouvertes.fr/inria-00558881/fr/`

Numerical algorithm

- Discretization : $\theta_m = m\,\frac{2\,\pi}{M}$, $m = 1..M$
- Data : $U_{\alpha,m} = U_\alpha(\theta_m)$ Generated from time domain seismogram using Time FT and One way extrapolation.
- $\mathcal{F}$ is the 1D FFT

NMLA :

$$\{B_m\} := \frac{1}{2\,L(\alpha)+1}\mathcal{F}^{-1}(\{\widehat{\beta}_\ell\}), \quad \widehat{\beta}_\ell = H_\ell \mathcal{F}(\{U_{\alpha,m}\})_\ell$$

$\{B_m\}$ is the Fourier interpolant of β. Choose $M > L(\alpha)$ then accuracy is only dependent α

Many pre-post processing options ... Simplest is
$|B_{m^\star}| = \max_m |B_m|$ then $\theta_{m^\star}$ most energetic ray direction ...

Traveltime Computation with corrected amplitudes Based on the "ansatz" we expect

$$B'_{m^\star}(\omega) \simeq A_0 \exp^{i\,\omega\,\varphi(x_0)}.$$

$$\frac{\partial_\omega B'_{m^\star}}{i\,B'_{m^\star}} \simeq \varphi(x_0).$$

In the Heterogeneous test case traveltime is approx. 0.46. We use a 0.1s time window on the seismogram.

$\varphi(x_0)$ versus center of time window