For matrix A ($p \times p$) with real eigenvalues, define $F^A$, the empirical distribution function of the eigenvalues of $A$, to be

$$F^A(x) \equiv (1/p) \cdot (\text{number of eigenvalues of } A \leq x).$$

For and p.d.f. $G$ the Stieltjes transform of $G$ is defined as

$$m_G(z) \equiv \int \frac{1}{\lambda - z} dG(\lambda), \quad z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : \Im z > 0\}.$$

Inversion formula

$$G\{[a, b]\} = (1/\pi) \lim_{\eta \to 0^+} \int_a^b \Im\, m_G(\xi + i\eta) d\xi$$

($a, b$ continuity points of G).

Notice

$$m_{F^A}(z) = (1/p) \mathsf{tr}\, (A - zI)^{-1}.$$

Theorem [S. (1995)]. Assume

a) For $n = 1, 2, \ldots$ $X_n = (X_{ij}^n)$, $n \times N$, $X_{ij}^n \in \mathbb{C}$, i.d. for all $n, i, j$, independent across $i, j$ for each $n$, $\mathsf{E}|X_{1\,1}^1 - \mathsf{E}X_{1\,1}^1|^2 = 1$.

b) $N = N(n)$ with $n/N \to c > 0$ as $n \to \infty$.

c) $T_n$ $n \times n$ random Hermitian nonnegative definite, with $F^{T_n}$ converging almost surely in distribution to a p.d.f. $H$ on $[0, \infty)$ as $n \to \infty$.

d) $X_n$ and $T_n$ are independent.

Let $T_n^{1/2}$ be the Hermitian nonnegative square root of $T_n$, and let $B_n = (1/N)T_n^{1/2}X_nX_n^*T_n^{1/2}$ (obviously $F^{B_n} = F^{(1/N)X_nX_n^*T_n}$). Then, almost surely, $F^{B_n}$ converges in distribution, as $n \to \infty$, to a (nonrandom) p.d.f. $F$, whose Stieltjes transform $m(z)$ ($z \in \mathbb{C}^+$) satisfies

$$(*) \qquad m = \int \frac{1}{t(1 - c - czm) - z} dH(t),$$

in the sense that, for each $z \in \mathbb{C}^+$, $m = m(z)$ is the unique solution to $(*)$ in $\{m \in \mathbb{C} : -\frac{1-c}{z} + cm \in \mathbb{C}^+\}$.

We have

$$F^{(1/N)X^*TX} = (1 - \frac{n}{N})I_{[0,\infty)} + \frac{n}{N}F^{(1/N)XX^*T}$$
$$\xrightarrow{a.s.} (1-c)I_{[0,\infty)} + cF \equiv \underline{F}.$$

Notice $m_F$ and $m_{\underline{F}}$ satisfy

$$\frac{1-c}{cz} + \frac{1}{c}m_{\underline{F}}(z) = m_F(z) = \int \frac{1}{-zm_{\underline{F}}t - z}dH(t).$$

Therefore, $\underline{m} = m_{\underline{F}}$ solves

$$z = -\frac{1}{\underline{m}} + c\int \frac{t}{1+t\underline{m}}dH(t).$$

**Facts on F:**

1. The endpoints of the connected components (away from 0) of the support of $F$ are given by the extrema of

$$f(\underline{m}) = -\frac{1}{\underline{m}} + c\int \frac{t}{1+t\underline{m}}dH(t) \quad \underline{m} \in \mathbb{C}$$

[Marčenko and Pastur (1967), S. and Choi (1995)].

2. $F$ has a continuous density away from the origin given by

$$\frac{1}{c\pi}\Im\underline{m}(x) \quad 0 < x \in \text{ support of } F$$

where

$$\underline{m}(x) = \lim_{z\in\mathbb{C}^+\to x} m_{\underline{F}}(z)$$

solves

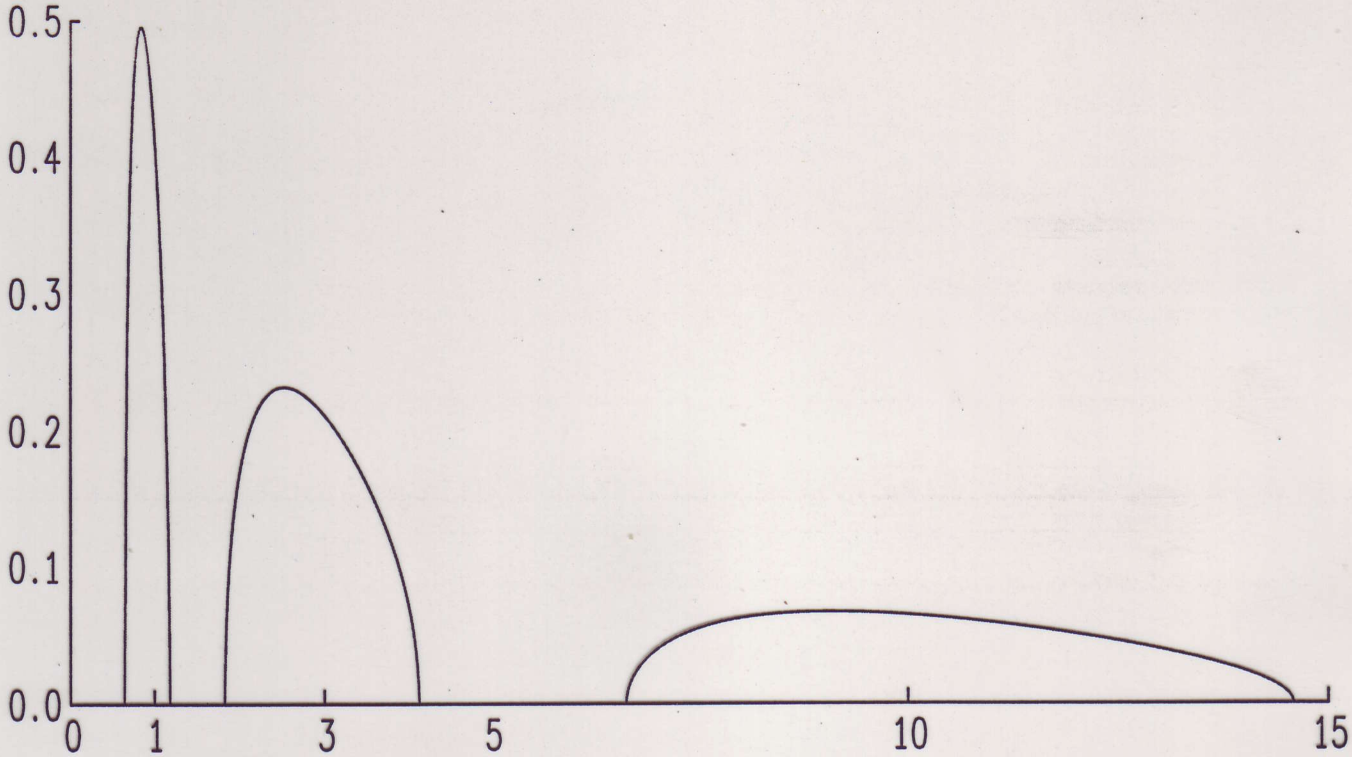
$$x = -\frac{1}{\underline{m}} + c\int \frac{t}{1+t\underline{m}}dH(t).$$

(S. and Choi 1995).

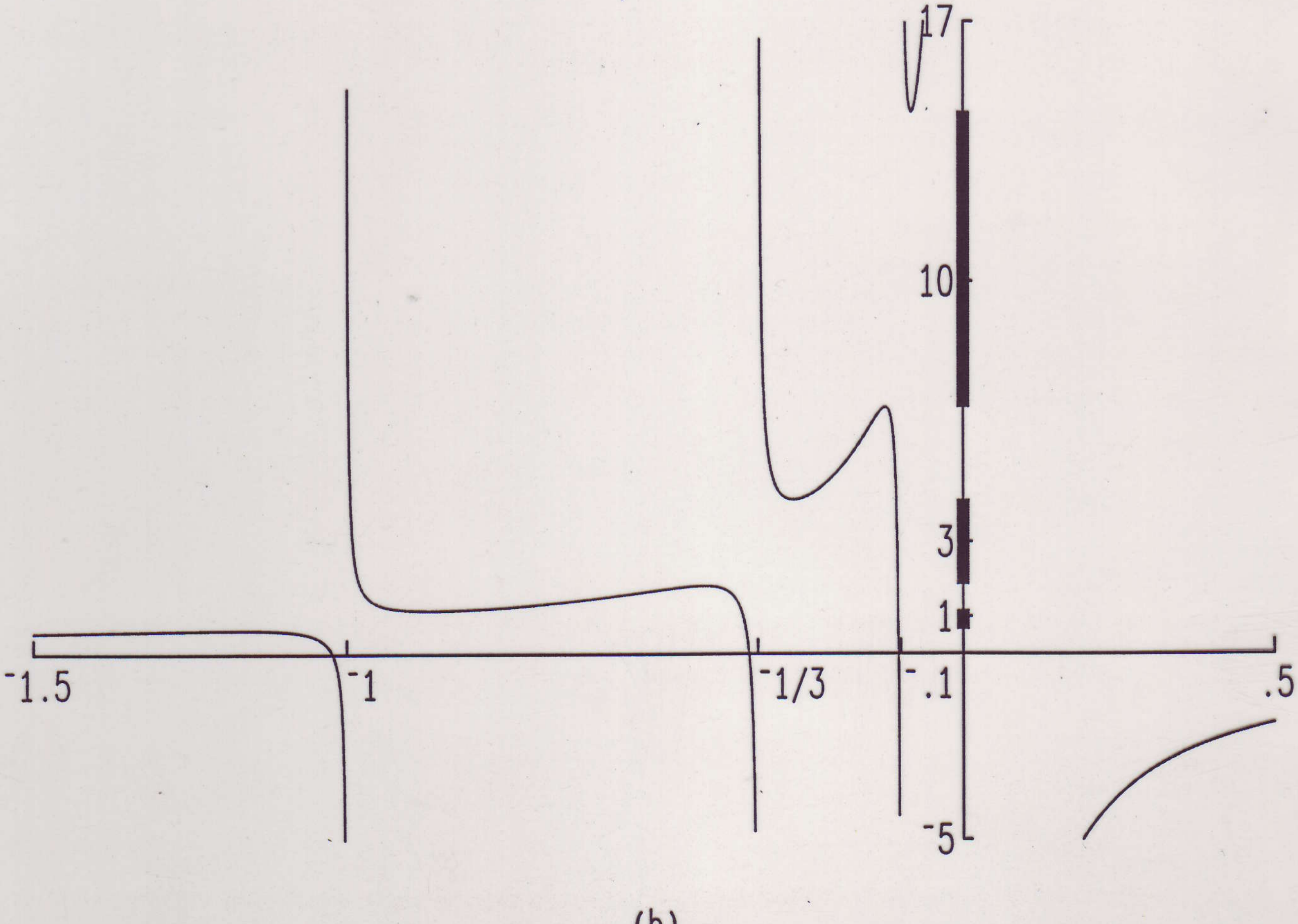
3. $F'$ is analytic inside its support, and when $H$ is discrete, has infinite slopes at boundaries of its support [S. and Choi (1995)].

4. $c$ and $F$ uniquely determine $H$.

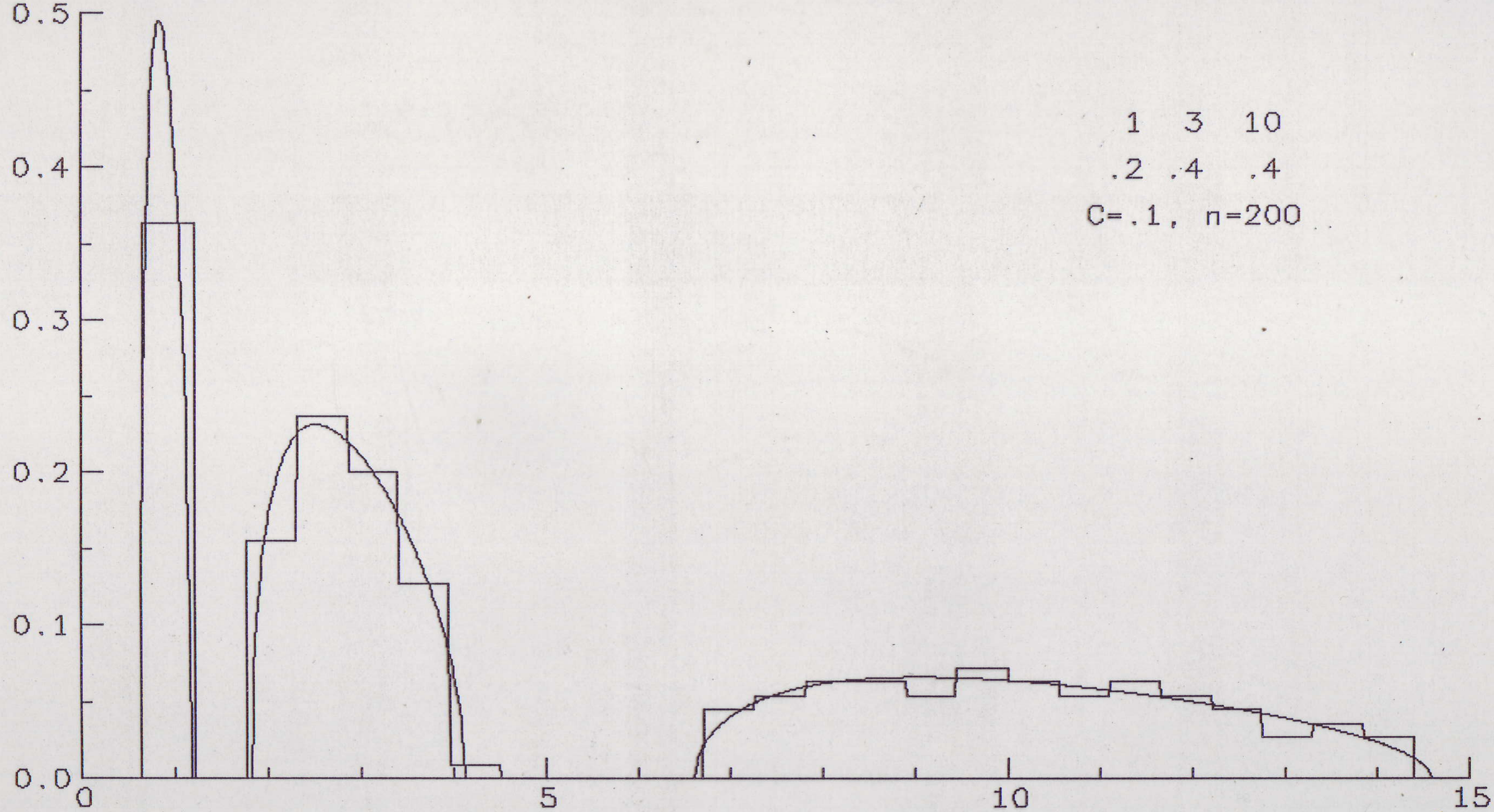
5. $F \xrightarrow{D} H$ as $c \to 0$ (complements $B_n \xrightarrow{a.s.} T_n$ as $N \to \infty$, $n$ fixed).




(a)




(b)



0.5
0.4
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0
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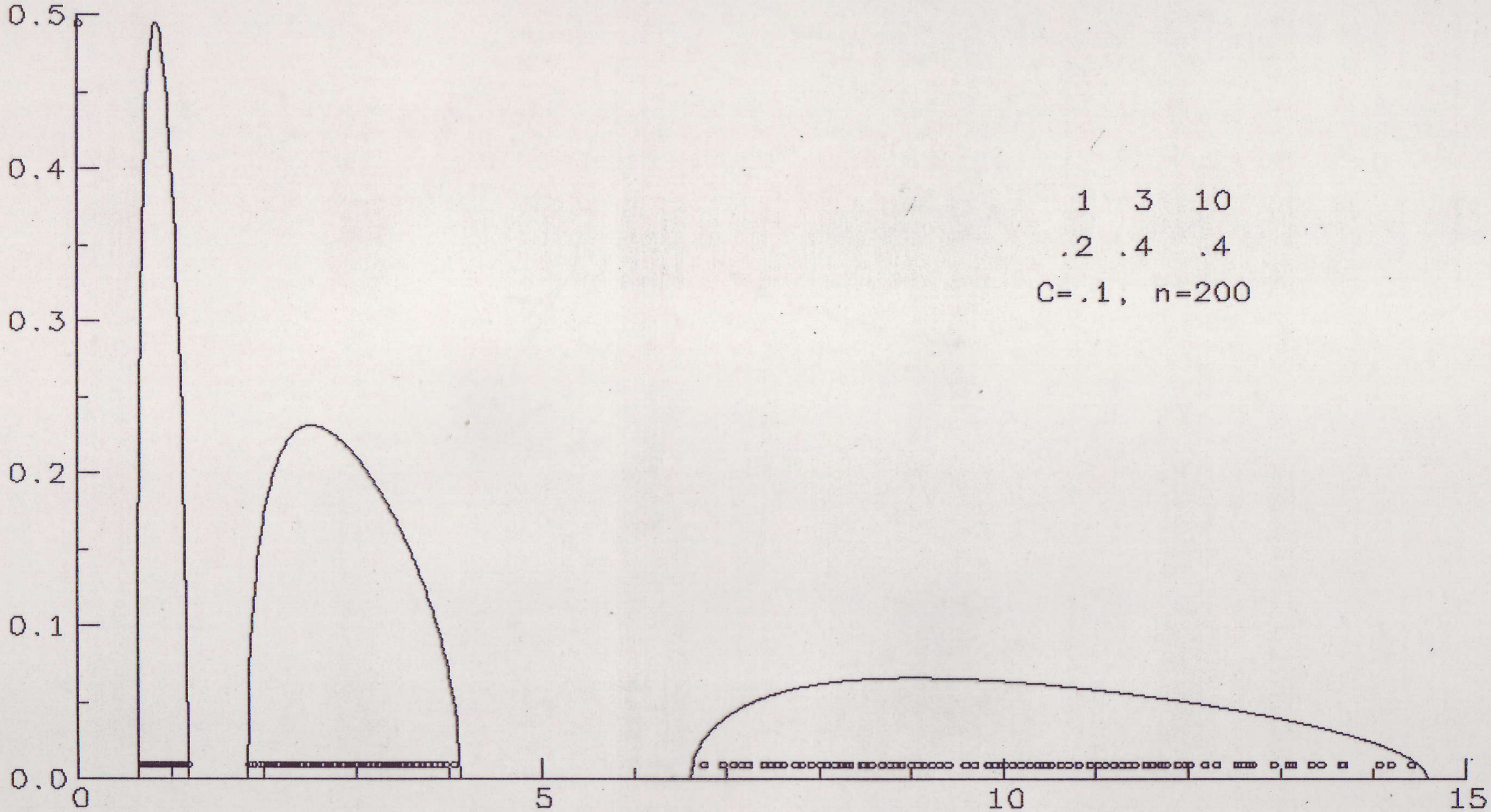
15
1 3 10
.2 .4 .4
C=.1, n=200



0.5
0.4
0.3
0.2
0.1
0.0
0
5
10
15
1 3 10
.2 .4 .4
C=.1, n=200

$T_n = I_n \Longrightarrow F = F_c$, where, for $0 < c \leq 1$, $F_c'(x) = f_c(x) =$

$$\frac{1}{2\pi c x}\sqrt{(x-b_1)(b_2-x)} \quad b_1 < x < b_2,$$

0 otherwise, where

$$b_1 = (1-\sqrt{c})^2 \text{ and } b_2 = (1+\sqrt{c})^2,$$

and for $1 < c < \infty$,

$$F_c(x) = (1-(1/c))I_{[0,\infty)}(x) + \int_{b_1}^{x} f_c(t)dt.$$

Marčenko and Pastur (1967)

Grenander and S. (1977)

Let, for any $d > 0$ and d.f. $G$, $F^{d,G}$ denote the limiting spectral d.f. of $(1/N)X^*TX$ corresponding to limiting ratio $d$ and limiting $F^{T_n}$ $G$.

Theorem [Bai and S. (1998)]. Assume:

a) $X_{ij}$, $i, j = 1, 2, ...$ are i.i.d. random variables in $\mathbb{C}$ with $\mathsf{E}X_{11} = 0$, $\mathsf{E}|X_{11}|^2 = 1$, and $\mathsf{E}|X_{11}|^4 < \infty$.

b) $N = N(n)$ with $c_n = n/N \to c > 0$ as $n \to \infty$.

c) For each $n$ $T_n$ is an $n \times n$ Hermitian nonnegative definite satisfying $H_n \equiv F^{T_n} \xrightarrow{D} H$, a p.d.f.

d) $\|T_n\|$, the spectral norm of $T_n$ is bounded in $n$.

e) $B_n = (1/N)T_n^{1/2}X_nX_n^*T_n^{1/2}$, $T_n^{1/2}$ any Hermitian square root of $T_n$, $\underline{B}_n = (1/N)X_n^*T_nX_n$, where $X_n = (X_{ij})$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, N$.

f) The interval $[a, b]$ with $a > 0$ lies in an open interval outside the support of $F^{c_n,H_n}$ for all large $n$.

Then

$$\mathsf{P}(\text{ no eigenvalue of } B_n \text{ appears in } [a, b] \text{ for all large } n\ ) = 1.$$

Theorem [Bai and S. (1999)]. Assume (a)–(f) of the previous theorem.

1) If $c[1 - H(0)] > 1$, then $x_0$, the smallest value in the support of $F^{c,H}$, is positive, and with probability one $\lambda_N^{B_n} \to x_0$ as $n \to \infty$. The number $x_0$ is the maximum value of the function

$$z(m) = -\frac{1}{m} + c \int \frac{t}{1+tm} dH(t)$$

for $m \in \mathbb{R}^+$.

2) If $c[1 - H(0)] \leq 1$, or $c[1 - H(0)] > 1$ but $[a, b]$ is not contained in $[0, x_0]$ then $m_{F^{c,H}}(b) < 0$. Let for large $n$ integer $i_n \geq 0$ be such that

$$\lambda_{i_n}^{T_n} > -1/m_{F^{c,H}}(b) \qquad \text{and} \qquad \lambda_{i_n+1}^{T_n} < -1/m_{F^{c,H}}(a)$$

(eigenvalues arranged in non-increasing order). Then

$$\mathsf{P}(\lambda_{i_n}^{B_n} > b \quad \textit{and} \quad \lambda_{i_n+1}^{B_n} < a \quad \text{for all large } n \;) = 1.$$

Theorem (Baik and S. (2006)) Assume the conditions in Bai and S. (1998). Suppose a fixed number of the eigenvalues of $T_n$ are different than 1, positive, and remain the same value for all $n$ large. Assume the $i_n^{\text{th}}$ largest eigenvalue, $\lambda_{i_n}^{T_n}$, is one of these numbers, say $\alpha$. Then with probability one

$$\lambda_{i_n}^{B_n} \to \begin{cases} \alpha + \frac{c\alpha}{\alpha-1} & \text{if } |\alpha - 1| > \sqrt{c} \\ (1+\sqrt{c})^2 & \text{if } 1 < \alpha \leq 1 + \sqrt{c} \\ (1-\sqrt{c})^2 & \text{if } 1 - \sqrt{c} \leq \alpha < 1. \end{cases}$$

**Theorem** (Baik and S.). Assume when $X_{1\,1}$ is complex, $\mathsf{E}X_{1\,1}^2 = 0$. Suppose $\alpha > 0$ is an eigenvalue of $T_n$ with multiplicity $k$ satisfying

$$(**) \qquad |\alpha - 1| > \sqrt{c}$$

and eigenspace formed from $k$ elements of the canonical basis set $\mathcal{B}_n = \{(1,0,\ldots,0)^*, (0,1,\ldots,0)^*, \ldots, (0,\ldots,1)^*\}$ in $\mathbb{R}^n$. Let $\lambda_i$, $i = 1,\ldots,k$ be the eigenvalues of $B_n$ corresponding to $\alpha$. Let

$$\lambda_n = \alpha + \frac{n}{N}\frac{\alpha}{\alpha - 1}.$$

Then $\sqrt{n}(\lambda_i - \lambda_n)$ $i = 1,\ldots,k$ converge weakly to the eigenvalues of a mean zero Gaussian $k \times k$ Hermitian matrix containing independent random variables on and above the diagonal. The diagonal entries have variance

$$= (\mathsf{E}|x_1|^4 - 1 - t/2)\frac{\alpha^2((\alpha-1)^2 - c)^2}{(\alpha-1)^4} + (t/2)\frac{\alpha^2((\alpha-1)^2 - c)}{(\alpha-1)^2}$$

where $t = 4$ when $X_{1\,1}$ is real, $t = 2$ when $X_{1\,1}$ is complex. In the complex case the real and imaginary parts of the off-diagonal entries are i.i.d. The real part of the off-diagonal elements (in either case) has variance

$$(t/4)\frac{\alpha^2((\alpha-1)^2 - c)}{(\alpha-1)^2}.$$

Moreover, the weak limit of eigenvalues corresponding to different positive $\alpha$'s satisfying $(**)$ with eigenvectors in $\mathcal{B}_n$ are independent.

Theorem (Rao and S.). Assume the conditions in Bai and S. (1998), and additionally

a) There are $r$ positive eigenvalues of $T_n$ all converging uniformly to $t'$, a positive number. Denote by $\hat{H}_n$ the e.d.f. of the $n-r$ other eigenvalues of $T_n$.

b) There exists positive $t_a < t_b$ contained in an interval $(\alpha, \beta)$ with $\alpha > 0$ which is outside the support of $\hat{H}_n$ for all large $n$, such that for these $n$

$$\frac{n}{N}\int \frac{\lambda^2}{(\lambda - t)^2}d\hat{H}_n(\lambda) \leq 1$$

for $t = t_a, t_b$.

c) $t' \in (t_a, t_b)$.

Suppose $\lambda_{i_n}^{T_n}, \ldots, \lambda_{i_n+r-1}^{T_n}$ are the eigenvalues stated in a). Then, with probability one

$$\lim_{n\to\infty} \lambda_{i_n}^{B_n} = \cdots = \lim_{n\to\infty} \lambda_{i_n+r-1}^{B_n} = z(-1/t')$$

$$= t'\left(1 + c\int \frac{\lambda}{t' - \lambda}dH(\lambda)\right).$$

Application to *array signal processing*:

Consider $m$ i.i.d. samples of an $n$ dimensional Gaussian (real or complex) distributed random vectors $x_1, \ldots, x_N$, modeling the recordings ($N$ "snapshots") taken from a bank of $n$ antennas due to signals emitting from an unknown number, $k$, of sources, and through a noise-filled environment. It is typically assumed that $x_1$ is mean zero, with covariance matrix $R = \Psi + \Sigma$, where $\Psi$ is the covariance matrix attributed to the signals and would be of rank $k$, and $\Sigma$ is the covariance of the additive noise, assumed to be of full rank. Then the matrix

$$R_\Sigma = \Sigma^{-1} R = \Sigma^{-1} \Psi + I$$

would have $n - k$ eigenvalues, attributed to the noise, equal to 1, the remaining $k$ eigenvalues all strictly greater than 1.

When it is possible to sample the purely additive noise portion along the antennas, say $z_1, \dots, z_{N'}$ with $N' > n$, then one would estimate $R_\Sigma$ with $\widehat{R}_{\widehat{\Sigma}} \equiv \widehat{\Sigma}^{-1}\widehat{R}$, where

$$\widehat{R} = \frac{1}{N}\sum_{i=1}^{N} x_i x_i^* \quad \text{and} \quad \widehat{\boldsymbol{\Sigma}} = \frac{1}{N'}\sum_{j=1}^{N'} z_j z_j^*,$$

and seek to identify eigenvalues of $\widehat{R}_{\widehat{\Sigma}}$ which are near one, the number of the remaining eigenvalues (presumably all greater than these) would be an estimate of $k$, known in the literature as the *detection* problem.

Theorem (Rao and S.) Denote the eigenvalues of $R_\Sigma$ by $\lambda_1 \geq \lambda_2 > \ldots \geq \lambda_k > \lambda_{k+1} = \ldots \lambda_n = 1$. Let $l_j$ denote the $j$-th largest eigenvalue of $\widehat{R}_{\widehat{\Sigma}}$. Then as $n, N(n), N'(n) \to \infty$, $c_n = n/N \to c > 0$, $c^1_{N'} = n/N' \to c_1 < 1$, with probability 1, $l_j$ converges to:

$$\lambda_j \left( 1 - c - c \frac{-c_1\lambda_j - \lambda_j + 1 + \sqrt{{c_1}^2{\lambda_j}^2 - 2c_1{\lambda_j}^2 - 2c_1\lambda_j + {\lambda_j}^2 - 2\lambda_j + 1}}{2c_1\lambda_j} \right)$$

if $\lambda_j > \tau(c, c_1)$,

$$\left( \frac{1 + \sqrt{1 - (1-c)(1-c_1)}}{1 - c_1} \right)^2$$

if $\lambda_j \leq \tau(c, c_1)$

for $j = 1, \ldots, k$. The threshold $\tau(c, c_1) =$

$$\frac{({c_1}^2 + c_1\sqrt{c + c_1 - c_1c} - \sqrt{c + c_1 - c_1c} - 1)c - {c_1}^2 - 2c_1\sqrt{c + c_1 - c_1c} - c_1}{((c_1 - 1)\,c - c_1)\,(1 - c_1)^2}.$$

Theorem (Dozier and S. (2007)). Assume

a) For $n = 1, 2, \ldots$ $X_n = (X_{ij}^n)$, $n \times N$, $X_{ij}^n \in \mathbb{C}$, i.d. for all $n, i, j$,, independent across $i, j$ for each $n$, $\mathsf{E}|X_{1\,1}^1 - \mathsf{E}_{1\,1}|^2 = 1$.

b) $N = N(n)$ with $n/N \to c$ as $n \to \infty$.

c) $R_n$ is $n \times N$, independent of $X_n$ with $F^{(1/N)R_n R_n^*} \xrightarrow{D} H$ a.s.

Let, for $\sigma > 0$ $C_n = (1/N)(R_n + \sigma X_n)(R_n + \sigma X_n)^*$. Then, almost surely, $F^{C_n}$ converges in distribution, as $n \to \infty$ to a non-random p.d.f. $F$ whose Stieltjes transform $m(z)$ $(z \in \mathbb{C}^+)$ uniquely satisfies

$$m = \int \frac{1}{\dfrac{t}{1 + \sigma^2 cm} - (1 + \sigma^2 cm)z + \sigma^2(1 - c)} dH(t). \qquad (*)$$

**Facts on F** (Dozier and S. (2007)):

1. $F$ has a contiuous density $f$ away from the origin given by

$$f(x) = \frac{1}{\pi}\Im m(x) \quad 0 < x \in \text{ support of } F$$

where

$$m(x) = \lim_{x \in \mathbb{C}^+ \to x} m(z)$$

solves $(*)$ for $z = x$.

2. $f$ is analytic inside its support, and when $H$ is discrete, has infinite slopes at boundaries of its support.

3. $c$ and $F$ uniquely determine $H$.

4. $F(x) \xrightarrow{D} H(x - \sigma^2)$ as $c \to 0$ (complements $C_n \xrightarrow{a.s.} \lim_{N \to \infty} (1/N) R_n R_n^* + \sigma^2 I_n$, $n$ fixed ).

5. Let $b(z) = 1 + \sigma^2 cm(z)$ and $w(z) = b^2(z)z - b(z)\sigma^2(1-c)$. Then $(*)$ can be rewritten as

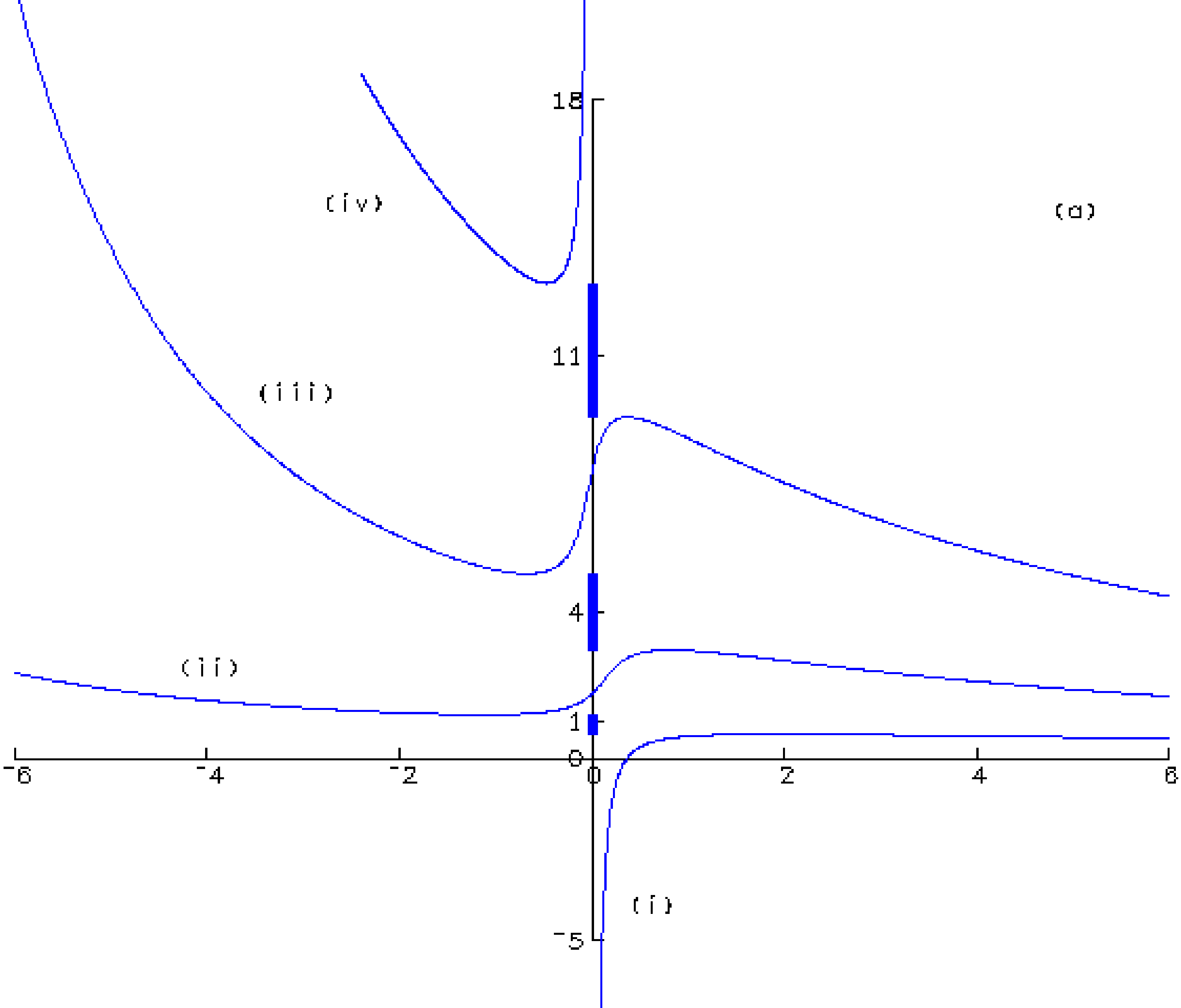
$$\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right) = \int \frac{dH(t)}{t - w(z)}.$$

Any interval $I \subset \mathbb{R}$ outside the support of $F$ implies $w(I) \subset \mathbb{R}$ is outside the support of $H$. The inverse function

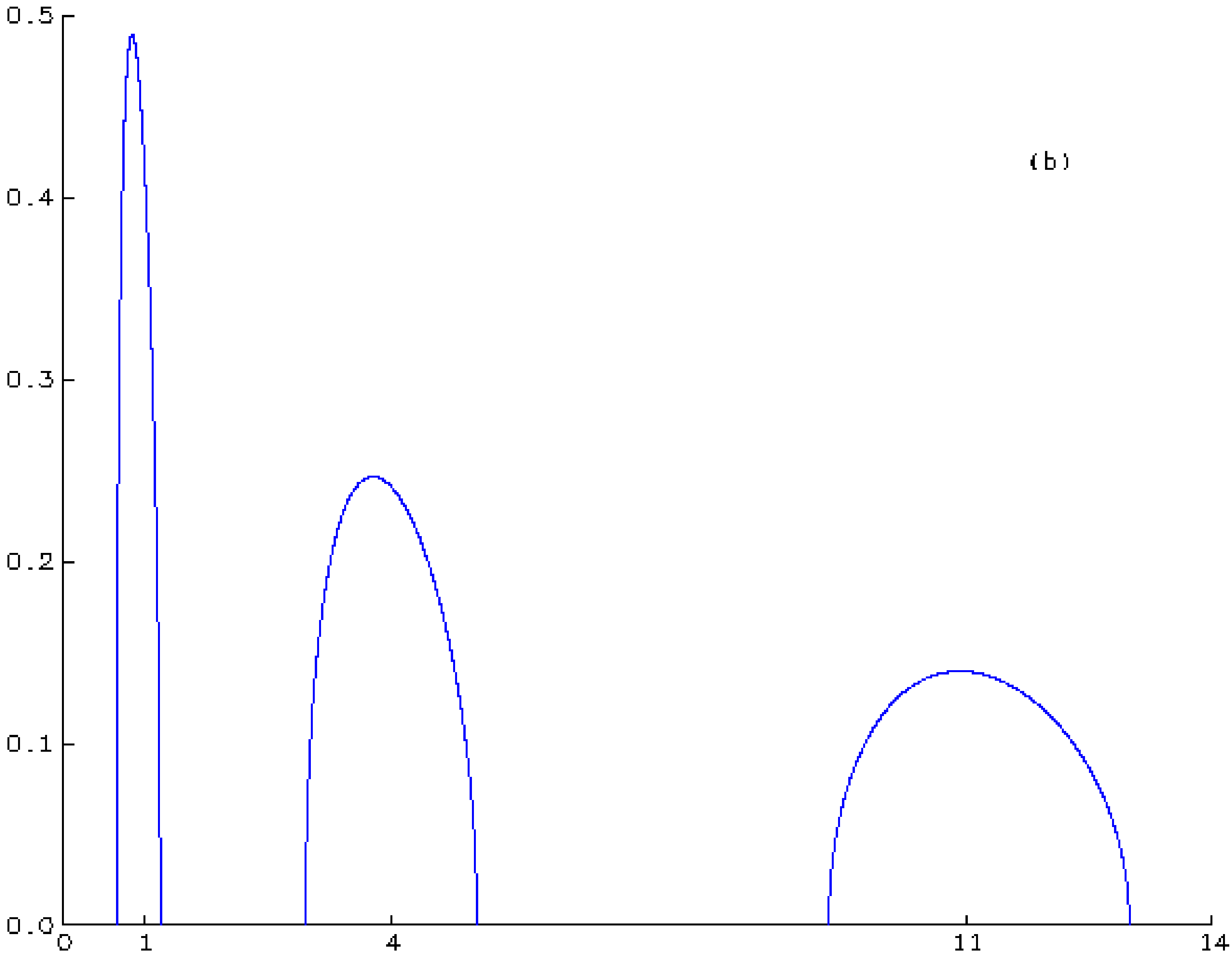
$$x(b) = \frac{1}{b^2} m_H^{-1}\left(\frac{1}{\sigma^2 c}\left(1 - \frac{1}{b}\right)\right) + \frac{1}{b}\sigma^2(1-c)$$

is increasing on $b(I)$. Moreover, all intervals outside the support of $F$ correspond to intervals outside the support of $H$, resulting in an inverse function.

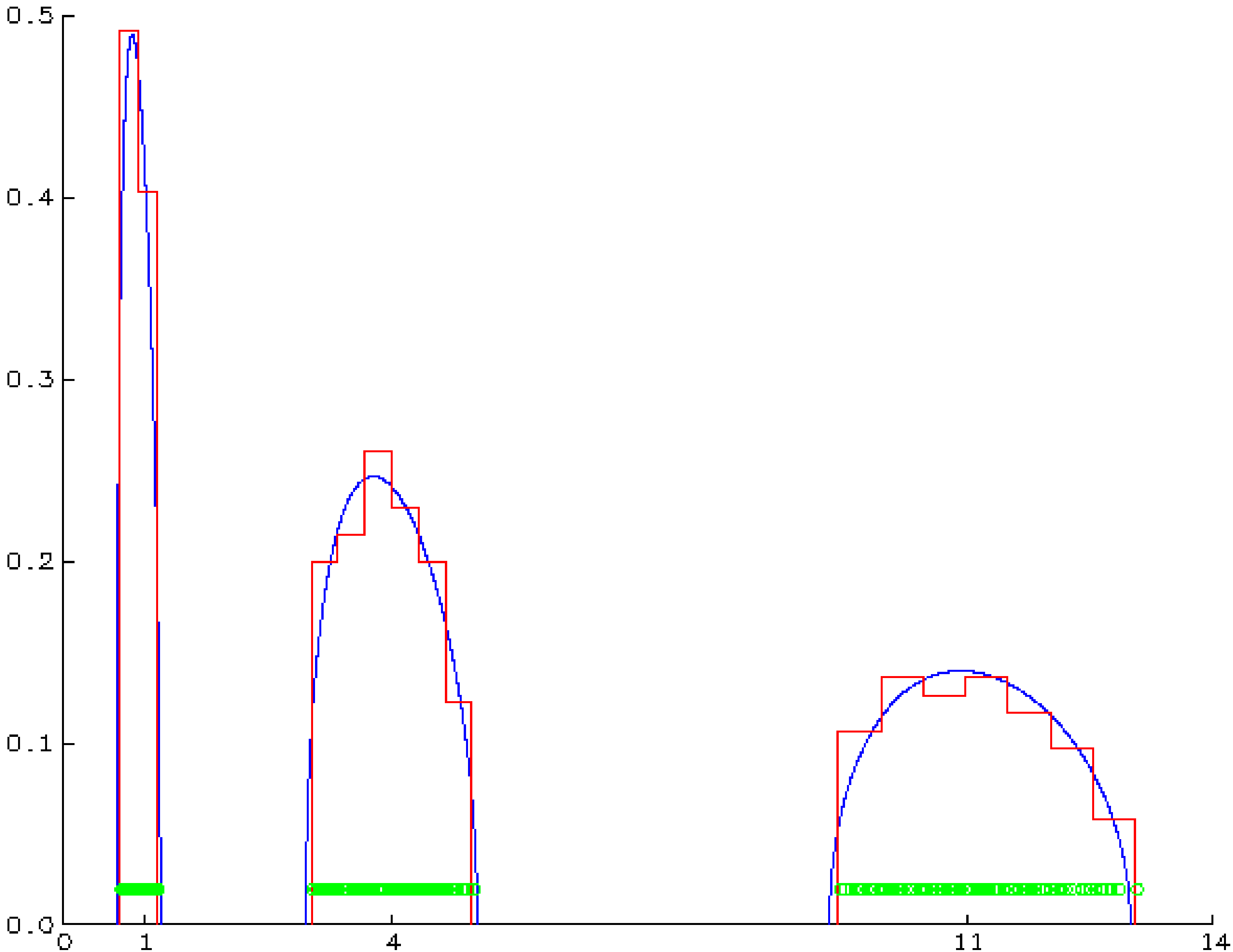
Example: $c = .1$ $\sigma^2 = 1$, $H$ places mass .2, .4, .4 at 0,3,10, respectively. The complement of the support consists of four intervals $I_{(i)} = (-\infty, 0)$, $I_{(ii)} = (0, 3)$, $I_{(iii)} = (3, 10)$, and $I_{(iv)} = (10, \infty)$.



(a)
(i)
(ii)
(iii)
(iv)
18
11
4
1
0
−5
−6
−4
−2
0
2
4
6



0.5
0.4
0.3
0.2
0.1
0.0
0
1
4
11
14
(b)


0.5
0.4
0.3
0.2
0.1
0.0
0
1
4
11
14

Matrix used in MIMO (multiple-input-multiple-output) systems:

$$D_n = (1/N)A_n^{1/2}X_nB_nX_n^*A_n^{1/2}$$

where $X_n$ is as above, $A_n$ $n \times n$, $B_n$ is $N \times N$, both Hermitian nonnegative definite, independent of $X_n$, and $A_n^{1/2}$ is the Hermitian nonnegative square root of $A_n$.

Theorem (Zhang (2006), Paul and S.) Assume, almost surely, $F^{A_n} \xrightarrow{D} F^A$, $F^{B_n} \xrightarrow{D} F^B$, both limits nonrandom d.f.'s, as $n \to \infty$, and $c_n = n/N \to c > 0$. Then, with probability one $F^{D_n} \xrightarrow{D} F$ as $n \to \infty$ where $F$ is nonrandom having Stieltjes transform $m(z)$ satisfying for $z \in \mathbb{C}^+$

$$(*) \qquad m(z) = \int \frac{1}{a\int \frac{b}{1+cbe}dF^B(b) - z} dF^A(a),$$

*where $e$ has positive imaginary part and satisfies*

$$(**) \qquad e = \int \frac{a}{a\int \frac{b}{1+cbe}dF^B(b) - z} dF^A(a).$$

It is the only solution with positive imaginary part.

Theorem (Paul and S.) In addition to the assumptions in the previous theorem, assume the conditions as in Bai and S. (1998) on the entries of $X_n$, $B_n$ is diagonal and both $\|A_n\|$ and $\|B_n\|$ are nonrandom and bounded in $n$. Let $F^{c_n,A_n,B_n}$ denote the d.f. defined by $(*)$ $(**)$ with $c$, $F^A$, $F^B$ replaced, respectively, by $c_n$, $F^{A_n}$,$F^{B_n}$. Assume the interval $[a,b]$ with $a > 0$ lies in an open interval outside the support of $F^{c_n,A_n,B_n}$ for all large $n$.

Then,

$$\mathsf{P}(\text{ no eigenvalues of } D_n \text{ appears in } [a,b] \text{ for all } n \text{ large}) = 1.$$