WARNING

THIS IS NOT A TALK
ABOUT FREE PROBABILTY THEORY

WARNING

RIGOR LEVEL: ∞^{-1}

Synchronous MMSE SIR with interference: Diagrams & Replicas

Aris Moustakas (Univ. Athens, Greece)

Collaborators:
M. Debbah (Supelec, France)
R. Kumar, G. Caire (USC, USA)

Introduction

- Why random matrices in communications?
  - Multi-antenna channels $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{z}$
  - Code matrices
    - Random i.i.d.
      - Scrambling codes (+-+-….)
      - Approximated by random Gaussian matrices
    - Orthogonal
      - Hadamard-Walsh codes [++++], [++- -], [+ - + -], [+ - - +]
      - Fourier transform matrices
      - Approximated by unitary Haar matrices

- Two standard functions of matrices
  - $I = Tr \log_2 \left[\mathbf{I} + \rho \mathbf{H}\mathbf{H}^\dagger\right]$
    - Information Capacity
  - $SINR = \rho \mathbf{w}^\dagger \mathbf{H}^\dagger \left[\mathbf{I} + \rho \mathbf{H}\mathbf{U}\mathbf{U}^\dagger \mathbf{H}^\dagger\right]^{-1} \mathbf{H}\mathbf{w}$

    - SINR linear MMSE

Introduction

- Important Statistics of random quantities
  - Mean
  - Variance
  - Higher cumulant moments (vanish for large matrix sizes = CLT)

- Methods
  - Free probability
    - Asymptotic freeness
  - Canonical RMT
    - Stieljes transforms
  - Replicas
    - Gaussian matrices (?)
  - Diagrammatics *(= Free probability??)*
    - Gaussian matrices
    - Unitary matrices
  - Other methods

Cons:

- Non rigorous
- Non-general (Gaussian – Unitary)

Pros:

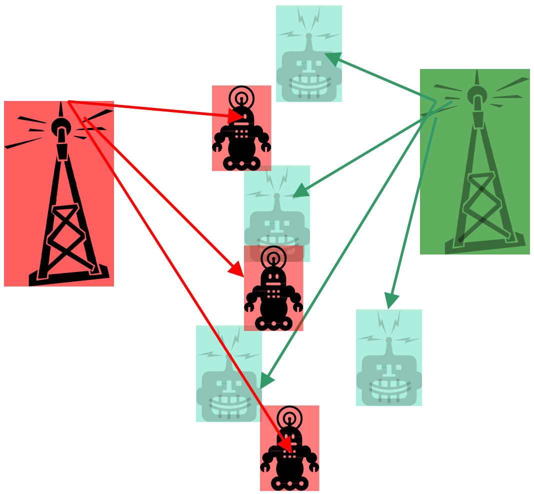
- Back-of-the envelope
- Easy

Methods:

- Diagrammatic Method
  - Important method in high-energy physics since 1930's
    - Applied to mesoscopic systems (1980's)
  - Applies mostly to Gaussian & Unitary matrices
    - Also matrices "*close*" to these
    - Non-hermitian matrices
  - Expand resolvent (Stieljes transform) in powers of random matrix and calculate average and then resum (!)
  - For large N, only a certain type of diagrams survive (planar approximation)
  - Applications: Calculation of mean and variance of resolvent

  - Similar to free probability methods

Application: MMSE SINR for synchronous transmission



- Channel Model:

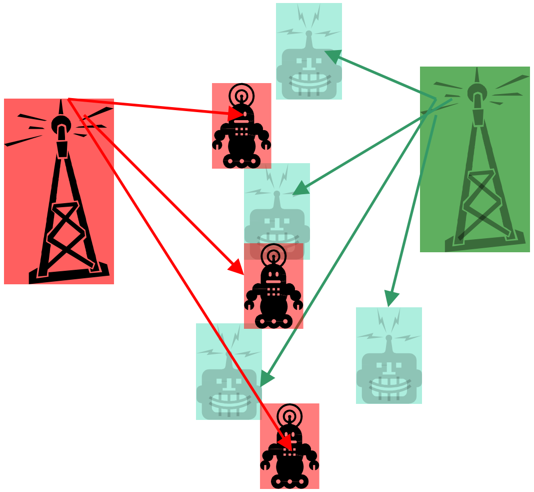
  $$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{H}'\mathbf{x}' + \mathbf{z}$$

  - N time-slots, 2 bases with K & K' users
  - Each user gets a code to transmit

    $$\mathbf{x} = \sum_{k=1}^{K} \mathbf{w}_k d_k$$

  - Synchronous transmission (downlink):
    - Matrix $\mathbf{U} = \left[\mathbf{w}_1\ \mathbf{w}_2 \ldots \mathbf{w}_K \ldots \mathbf{w}_N\right]$ is unitary $\mathbf{U}^\dagger\mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = \mathbf{I}_N$
    - In reality: ***U*** is a Hadamard-Walsh matrix
    - For OFDMA systems ***U*** is the Fourier transform basis matrix
    - Approximate this with Haar-distributed unitary matrices
  - Alternative (uplink): Asynchronous transmission:
    - Elements of U are i.i.d. $\frac{\pm 1}{\sqrt{N}}$
    - Approximate this by Gaussian i.i.d. matrix
  - Assume ***U***, ***U'*** independent

Application: MMSE SINR for synchronous transmission



- Channel Model:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{H}'\mathbf{x}' + \mathbf{z}$$

  - ***H***, ***H'***: Channel matrices
    - Diagonal with independent coefficients:
      - Fast fading (time-variability)
      - Independent frequency channels
    - Toeplitz form
      - Delayed paths
  - ***z***: receiver thermal noise (white)

Application: MMSE SINR for synchronous transmission

- Optimal linear receiver for user 1(several caveats):
  - Multiply ***y*** with optimal vector

$$\mathbf{g} = \mathbf{w}_1^{\dagger}\mathbf{H}^{\dagger}\left[\sigma^2\mathbf{I}_N + \mathbf{HUJU}^{\dagger}\mathbf{H}^{\dagger} + \mathbf{H}'\mathbf{U}'\mathbf{J}'\mathbf{U}'^{\dagger}\mathbf{H}'^{\dagger}\right]^{-1}$$

  - ***J***, ***J'*** are input power covariance matrices
  - Resulting SINR

$$\beta = \frac{\eta}{1-\eta}$$

$$\eta = \mathbf{w}_1^{\dagger}\mathbf{H}^{\dagger}\left[\sigma^2\mathbf{I}_N + \mathbf{HUJU}^{\dagger}\mathbf{H}^{\dagger} + \mathbf{H}'\mathbf{U}'\mathbf{J}'\mathbf{U}'^{\dagger}\mathbf{H}'^{\dagger}\right]^{-1}\mathbf{H}\mathbf{w}_1$$

- Aim: Calculate asymptotic properties of β (i.e. evaluate its mean)
- Compare with effective interference
  - Averaged over codes $\mathbf{H}'\mathbf{U}'\mathbf{J}'\mathbf{U}'^{\dagger}\mathbf{H}'^{\dagger} = \mathbf{H}'\mathbf{H}'^{\dagger}Tr\mathbf{J}'/N$
  - Averaged over time & codes $\mathbf{H}'\mathbf{U}'\mathbf{J}'\mathbf{U}'^{\dagger}\mathbf{H}'^{\dagger} = E\left[\mathbf{H}'\mathbf{H}'^{\dagger}\right]Tr\mathbf{J}'/N$

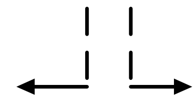
Diagrammatic Approach

- Start with simple problem:

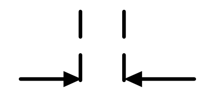
$$g = E\left[tr\left[\mathbf{I} - \mathbf{AUBU}^{\dagger}\right]^{-1}\mathbf{A}\right] = \sum_{n=0}^{\infty} trE\left[\left(\mathbf{AUBU}^{\dagger}\right)^{n}\mathbf{A}\right]$$

  - *tr[.] = Tr[.]/N*

- Represent each matrix in the expansion:
  - U_{ij} as two dashed lines with two external lines U_{ij}



U^*_{ij}



  - Matrices A, B as lines

$\mathbf{G}_0 = \mathbf{A}$

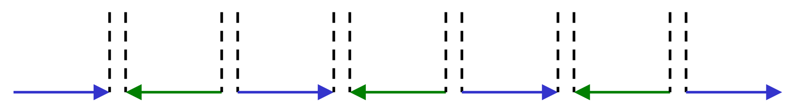
$g_0 = tr\mathbf{G}_0$

$\mathbf{F}_0 = \mathbf{B}$

$f_0 = tr\mathbf{F}_0$

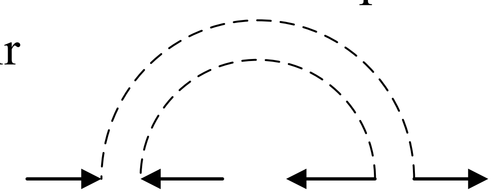
- Trace corresponds connecting solid lines

$$tr\left(\mathbf{AUBU}^{\dagger}\right)^{3}$$



- Averaging over ***U*****:** Connect dashed lines in all possible ways
  - Gives 1/N for each U, U* pair

$$E\left[U^*_{ji}U_{kl}\right]$$



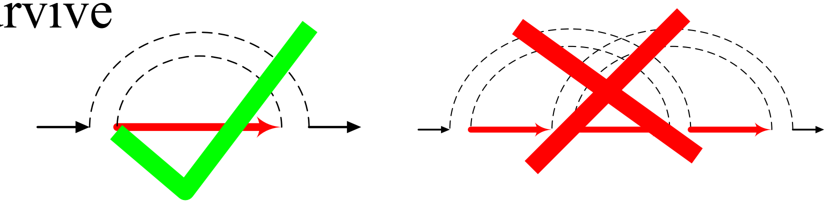
EXACT APPROACH

(Brouwer –Beenakker)

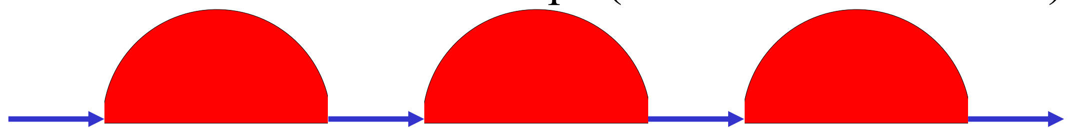
(Argaman – Zee)

Diagrammatic Approach

- In large N limit only planar diagrams survive
  - All crossed (non-planar) diagrams are subleading in N

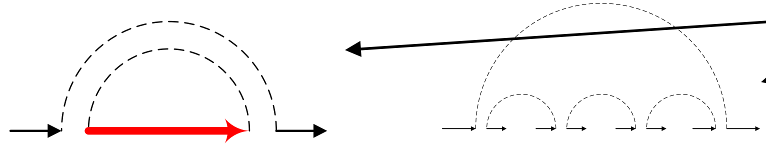


- This allows us to write the trace in disconnected parts
  - no dashed is allowed to escape (not even the other U's)



  - where red blob represents all other terms
    - Self-energy = "R-transform"
    - Leading terms in Weingarten function of each power of ***U***'s

Differences between Gaussian & NonGaussian



- Resum terms to get final result $g = tr\frac{\mathbf{A}}{\mathbf{I}-\mathbf{A}fm_u}$ $f = tr\frac{\mathbf{B}}{\mathbf{I}-\mathbf{B}gm_u}$
  - Functional form of m encodes statistics of ***U***
    - $m_g=1$ for Gaussian ***U***

$$m_u = \frac{\sqrt{1+4fg}-1}{2fg}$$

Generating function of $(2k)!/(k!^2(2k-1))$

Results

- Generalize to current problem with one interferer

$$g_i = tr\left[\mathbf{H_i}\mathbf{H_i}^\dagger\left(\mathbf{I}+\mathbf{H_1}\mathbf{H_1}^\dagger f_1\bar{m}_{1u}+\mathbf{H_2}\mathbf{H_2}^\dagger f_2\bar{m}_{2u}\right)^{-1}\right]$$

$$f_i = tr\frac{\mathbf{J}_i^\dagger}{\mathbf{I}+\mathbf{J}_i^\dagger g_i\bar{m}_{iu}} \qquad \bar{m}_{iu} = \frac{1-\sqrt{1-4f_ig_i}}{2f_ig_i} \qquad i=1,2$$

$$\frac{\beta}{\beta+1} = \eta = \frac{Nf_1g_1\bar{m}_{1u}}{K}$$

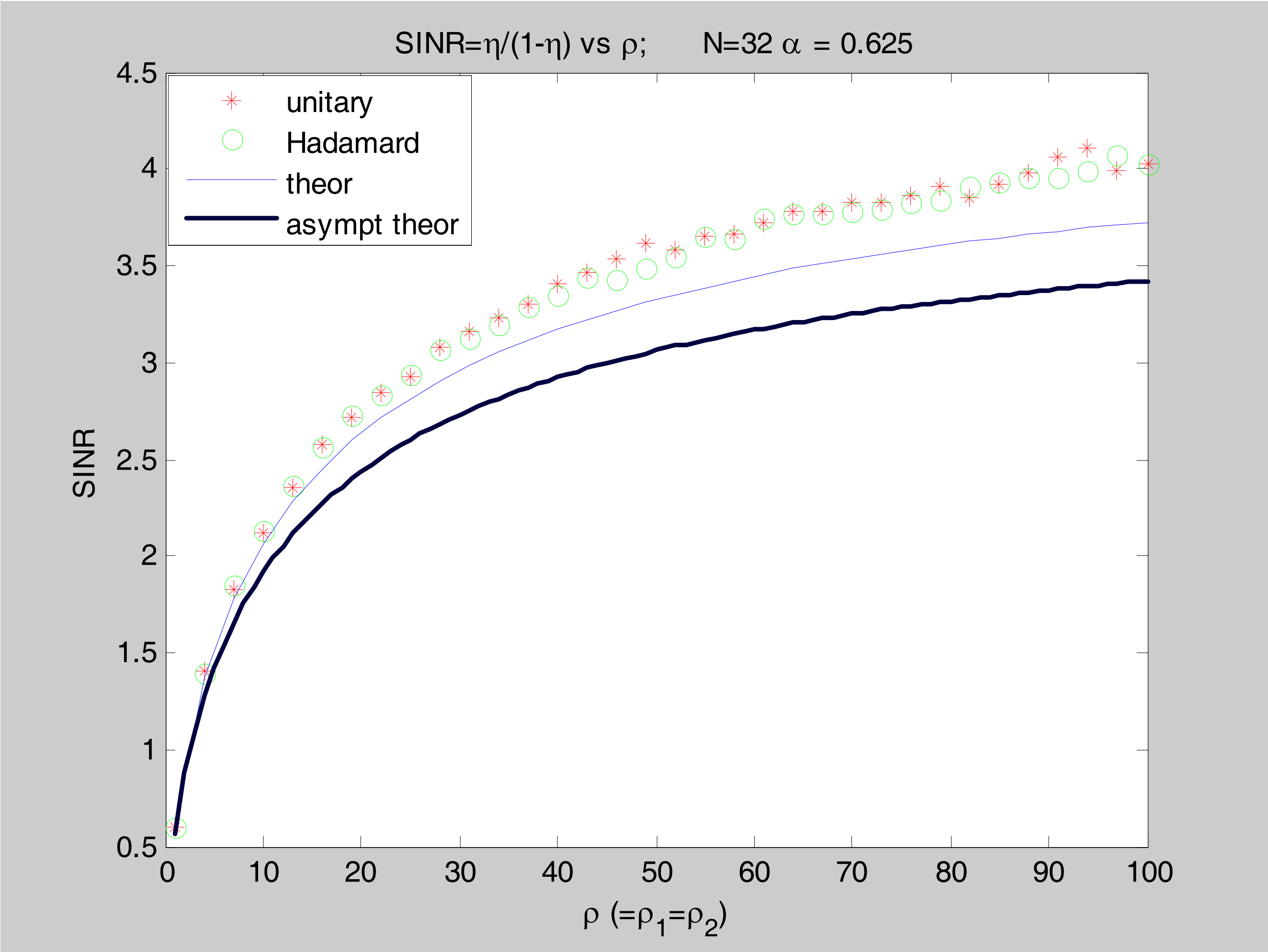
- Note: m_1 does not have to be the same as m_2 (e.g. = 1)

- *"In principle"*, above result cannot be obtained using free probability (?)
  - Depends on relative eigenvector space of H_1, H_2 , not only on their eigenvalues

$$\begin{aligned} r(z) &= trE\left[\left(z-\mathbf{H_1}\mathbf{U_1}\mathbf{J_1}\mathbf{U_1^\dagger}\mathbf{H_1^\dagger}-\mathbf{H_2}\mathbf{U_2}\mathbf{J_2}\mathbf{U_2^\dagger}\mathbf{H_2^\dagger}\right)^{-1}\right] \\ &= tr\left[\left(z-\mathbf{H_1}\mathbf{H_1}^\dagger f_1m_{1u}-\mathbf{H_2}\mathbf{H_2}^\dagger f_2m_{2u}\right)^{-1}\right] \end{aligned}$$

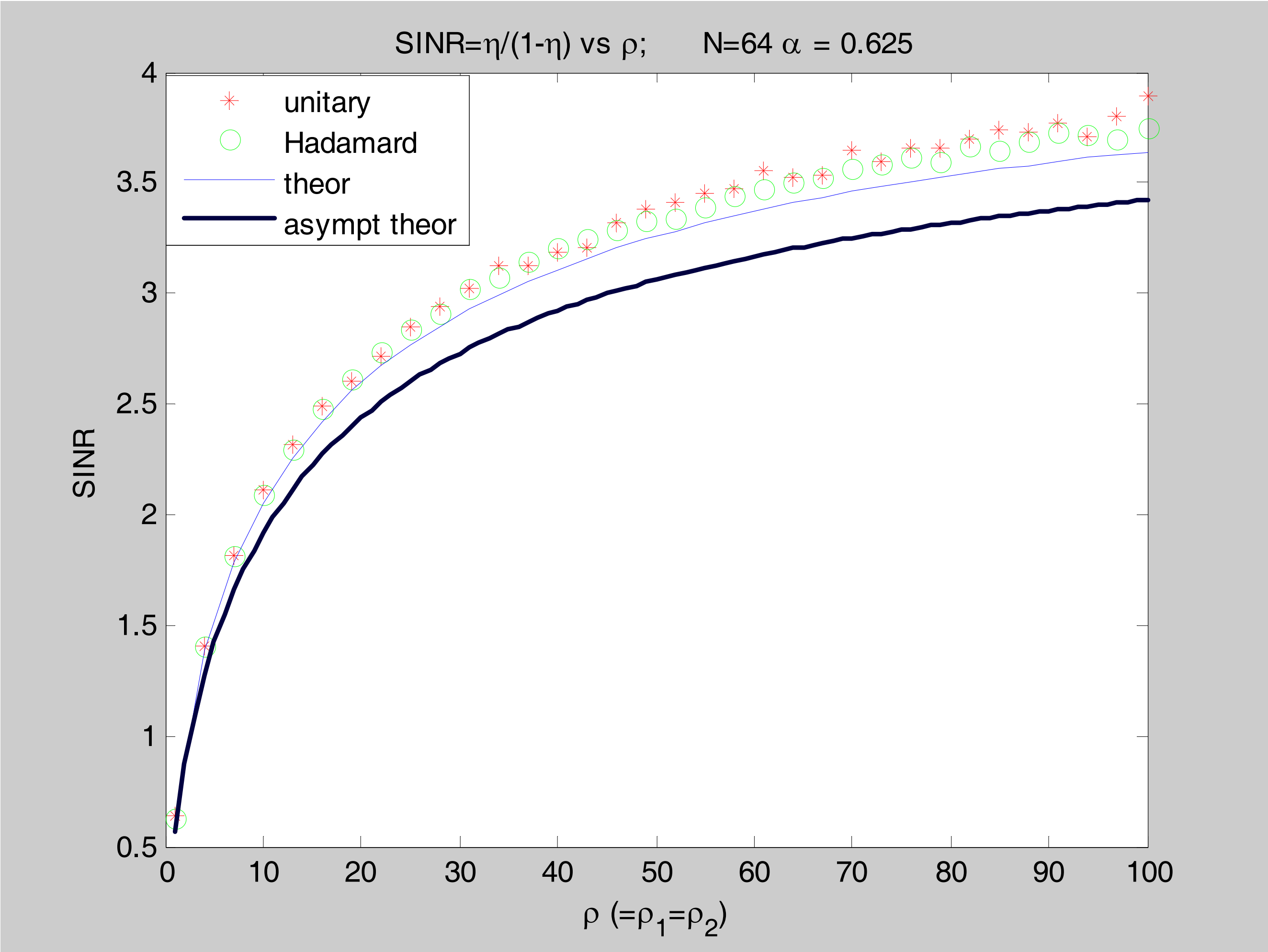
Results

- $\alpha = K_1/N = K_2/N$ $\rho = 1/\sigma^2$
- Asymptotic theory: introduce fast fading on each channel



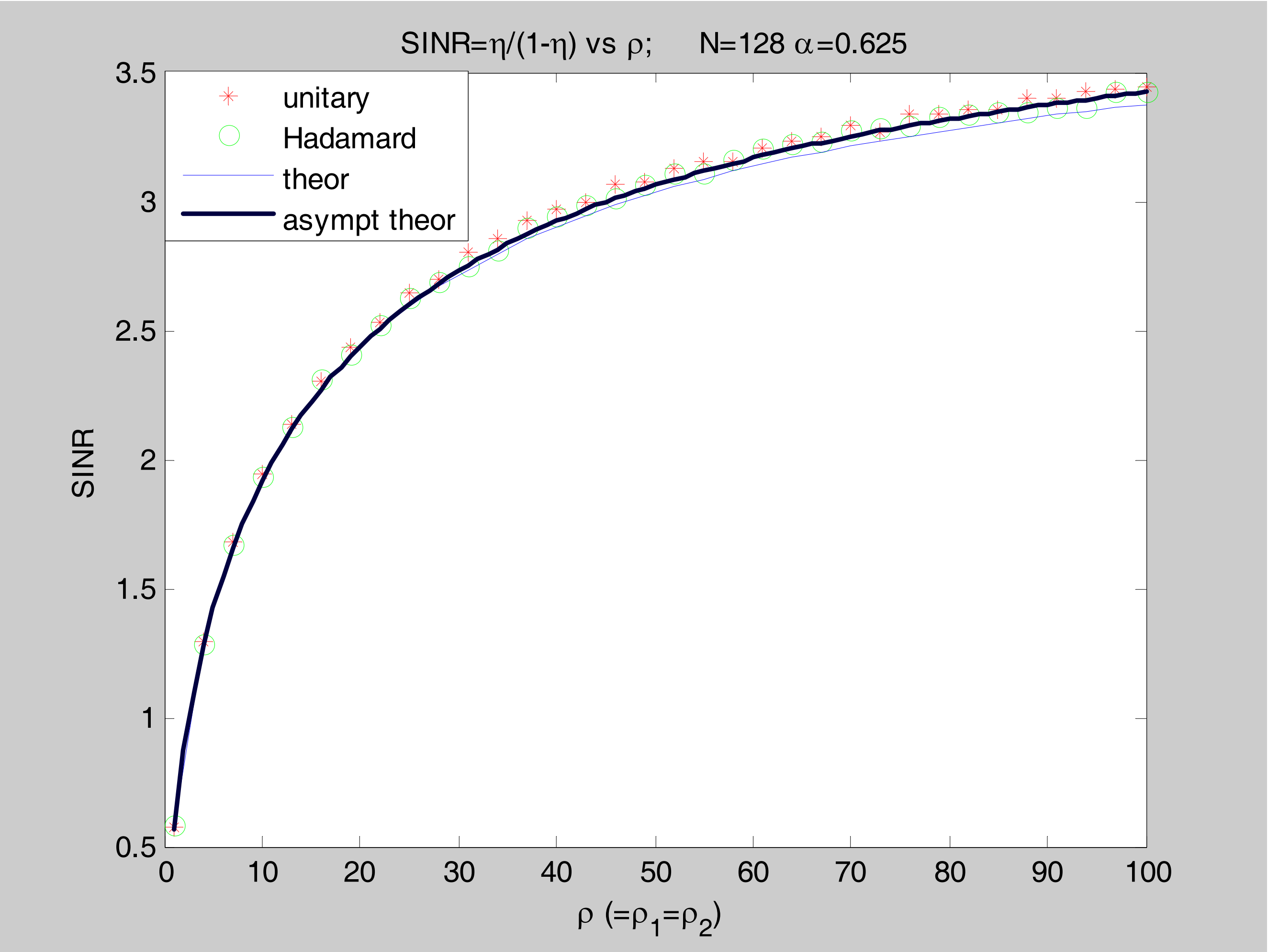
Results

- $\alpha = K_1/N = K_2/N$ $\rho = 1/\sigma^2$
- Asymptotic theory: introduce fast fading on each channel



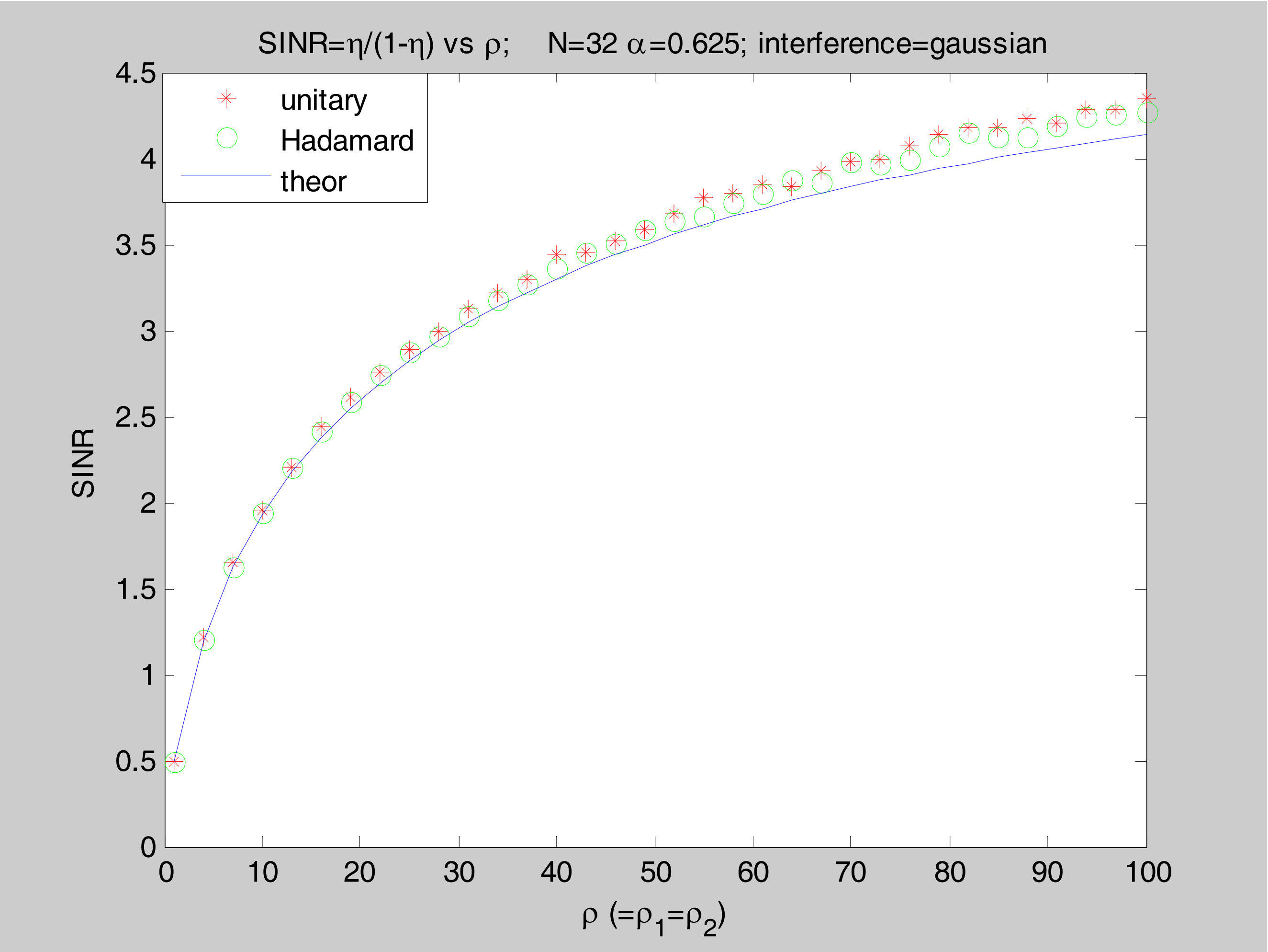
Results

- $\alpha=K_1/N=K_2/N$ $\rho=1/\sigma^2$
- Asymptotic theory converges Hadamard=good approximation

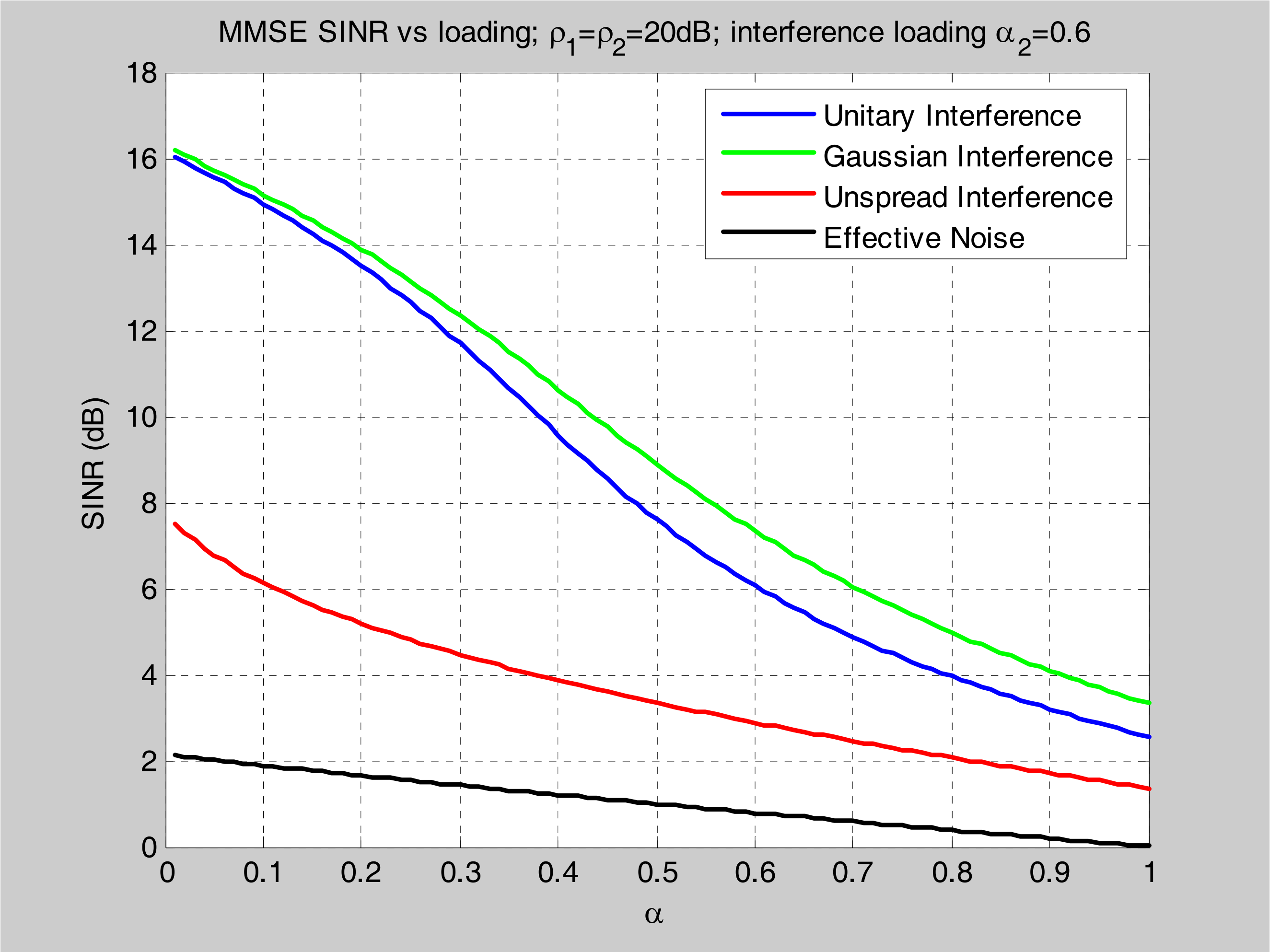


Results

- $\alpha=K_1/N=K_2/N$ $\rho=1/\sigma^2$
- Gaussian interference

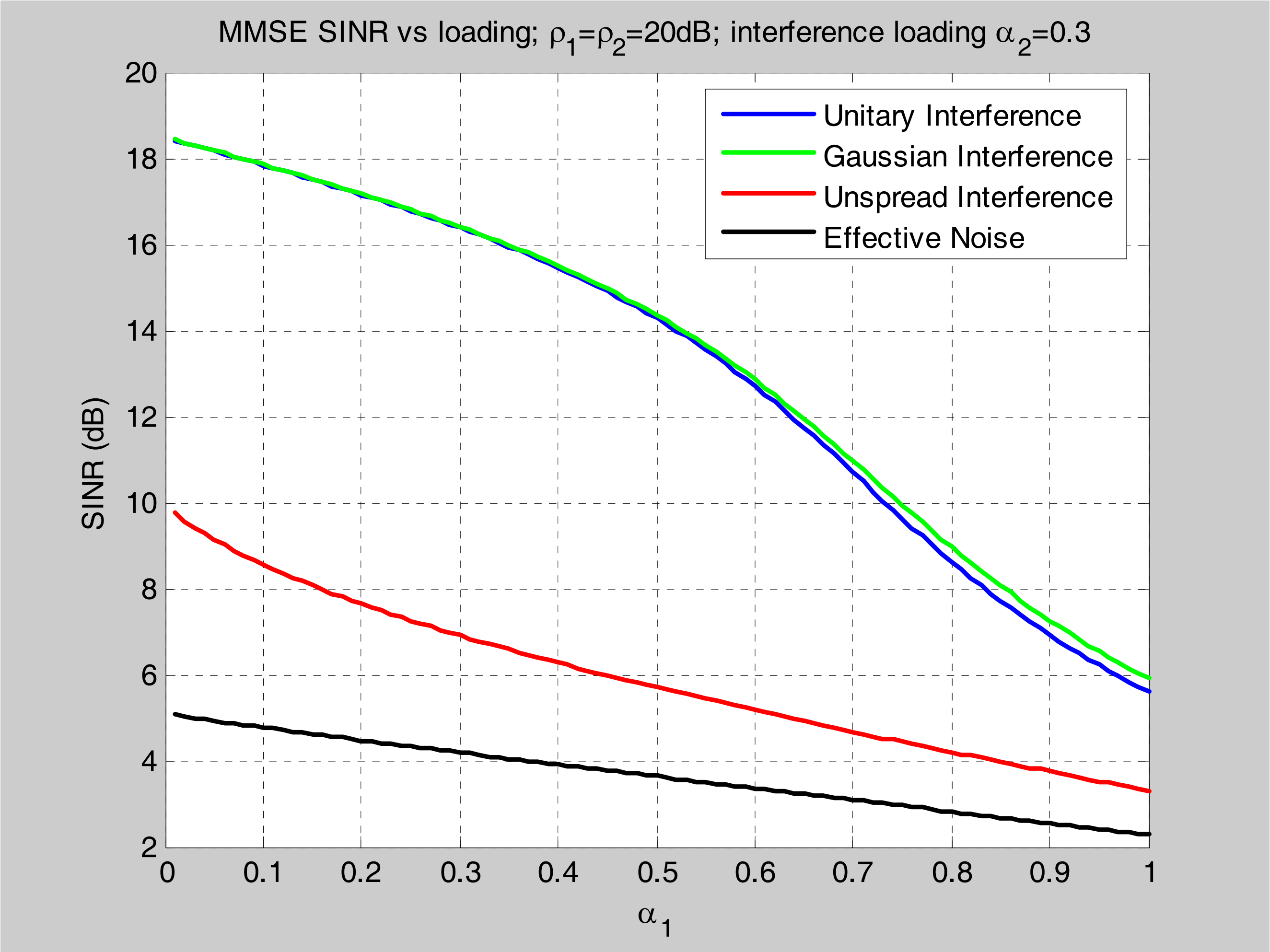


Results

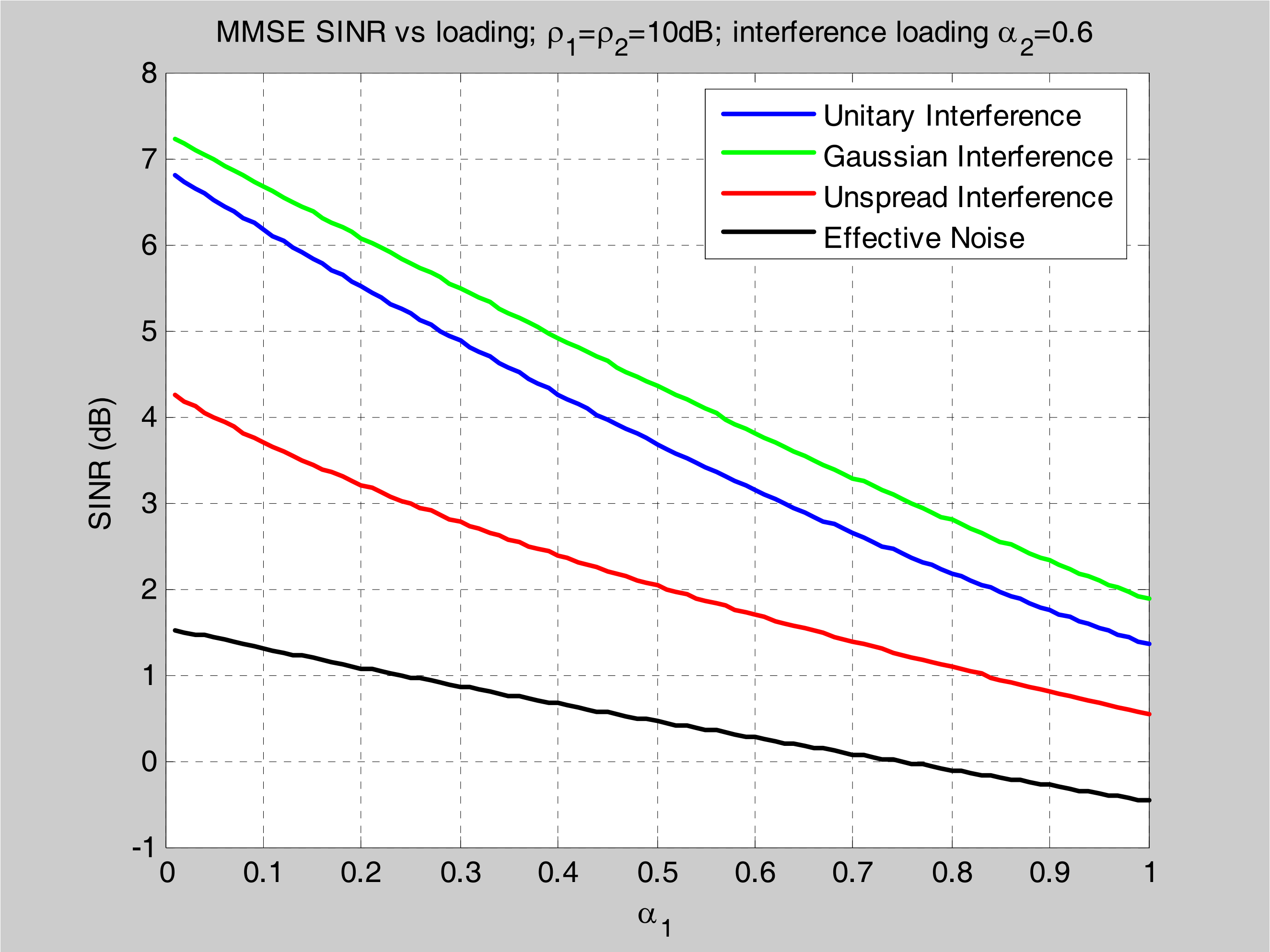


Blue line "interpolates" between green and red with α2 (α2=1 Interference cancels unitary matrices – α2<<1 Unitary looks Gaussian

Results


MMSE SINR vs loading; ρ1=ρ2=20dB; interference loading α 2=0.3
SINR (dB)
α 1
20
18
16
14
12
10
8
6
4
2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Unitary Interference
Gaussian Interference
Unspread Interference
Effective Noise

Results



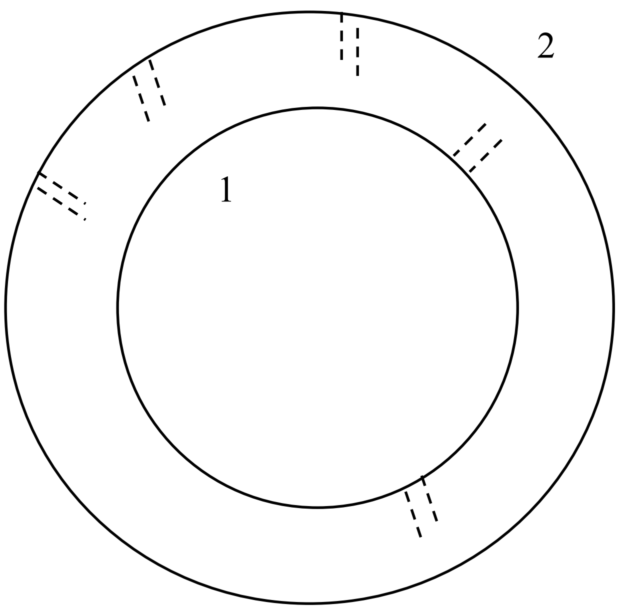
MMSE SINR vs loading; ρ1=ρ2=10dB; interference loading α2=0.6
SINR (dB)
α1
Unitary Interference
Gaussian Interference
Unspread Interference
Effective Noise
8
7
6
5
4
3
2
1
0
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Second Order Moments using Diagrammatics

- Calculate second order statistics of eigenvalues (=differentiate below)

$$v_{12}(z) = E\left[Tr\log\left[z\mathbf{I} - \mathbf{A_1}\mathbf{U}\mathbf{B_1}\mathbf{U}^\dagger\right] Tr\log\left[z\mathbf{I} - \mathbf{A_2}\mathbf{U}\mathbf{B_2}\mathbf{U}^\dagger\right]\right]_c$$

- Two traces – two lines (closed)
- Apply same principles (more diagrams)
  - Intra-circle
  - Cross-circle
  - Given x-circle connections
    - Can sum over all possible intra-circle ones
    - Then can sum over all cross-circle positions
    - Thus get from Go -> G and Fo->F
- Variance is O(1)



Diagrammatic Approach

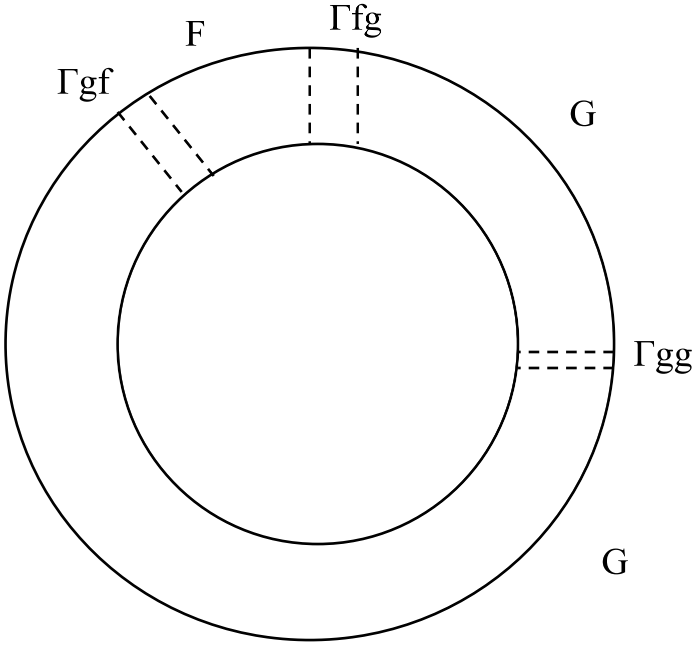
- The x-circle connections characterized by their neighbors (gf, fg, ff, gg)

- Thus we are left to just sum over "effective" quantities Γ, F, G

- By symmetry $\Gamma_{fg} = \Gamma_{gf}$

- Γ's involve same terms as mu but are broken into disjoint terms
  - Some go to Γ_{ff}, some go to Γ_{gf}

- If **H** is Gaussian $\Gamma_{gg} = \Gamma_{ff} = 0$



Result

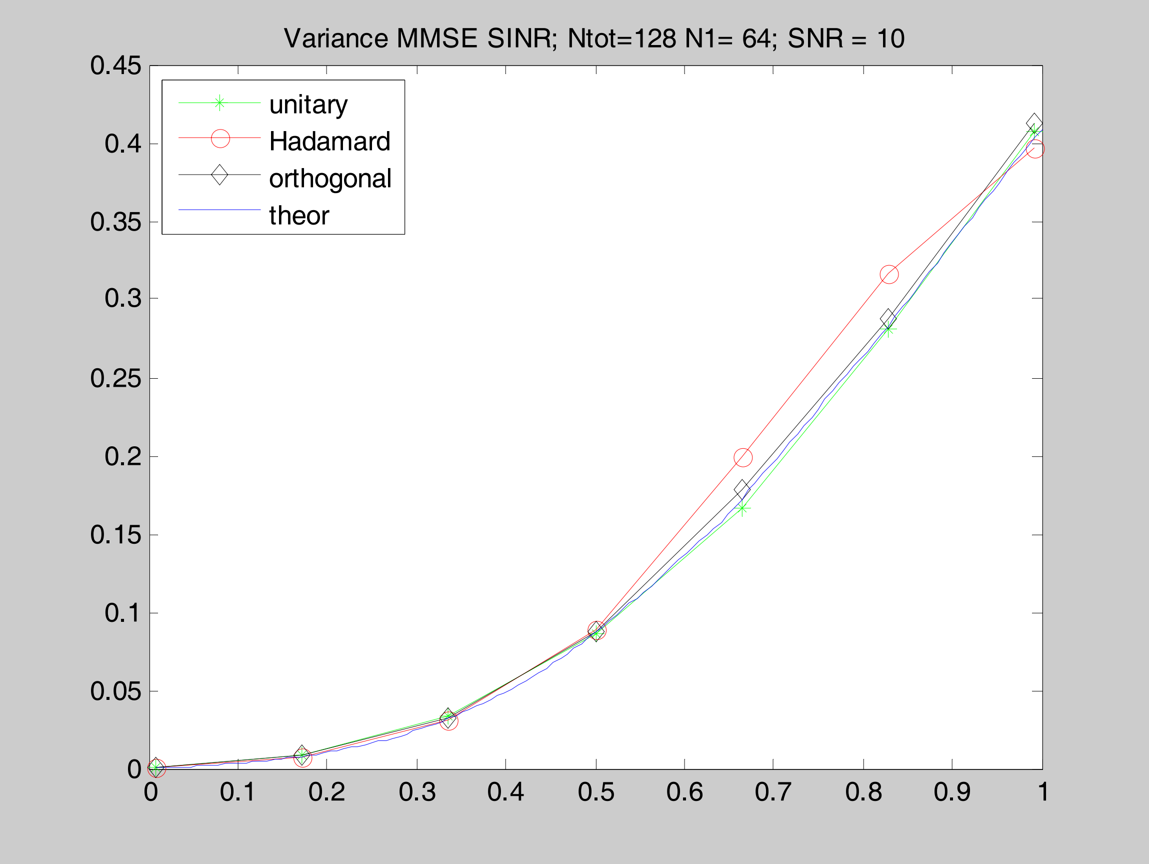
- After resumming we finally get

$$
\begin{aligned}
E[I_1 I_2]_c &= E\left[Tr\log\left[z\mathbf{I}-\mathbf{A_1HB_1H}^\dagger\right]Tr\log\left[z\mathbf{I}-\mathbf{A_2HB_2H}^\dagger\right]\right]_c \\
&= -\log\left[(1-\Gamma_{ff}t_f)(1-\Gamma_{gg}t_g)-\Gamma_{fg}^2 t_g t_f\right] \\
t_f &= tr\left[\mathbf{F_1F_2}\right] \\
t_g &= tr\left[\mathbf{G_1G_2}\right] \\
f_i &= tr\mathbf{F_i} = E\left[tr\left[\mathbf{B}_i/(\mathbf{I}-\mathbf{B_iHA_iH}^\dagger/z)\right]\right] = tr\left[\frac{\mathbf{B}_i}{\mathbf{I}-\mathbf{B_i}g_i m(f_i g_i)}\right] \\
g_i &= tr\mathbf{G_i} = E\left[tr\left[\mathbf{A}_i/(\mathbf{I}z-\mathbf{A_iH}^\dagger\mathbf{B_iH})\right]\right] = tr\left[\frac{\mathbf{A}_i}{\mathbf{I}z-\mathbf{A_i}f_i m(f_i g_i)}\right] \\
\Gamma_{ff},\Gamma_{gg},\Gamma_{fg} &= \mathcal{F}(\{f_i,g_i\})
\end{aligned}
$$

Results

Example: variance of MMSE SINR for synchronous downlink (no interference for simplicity)

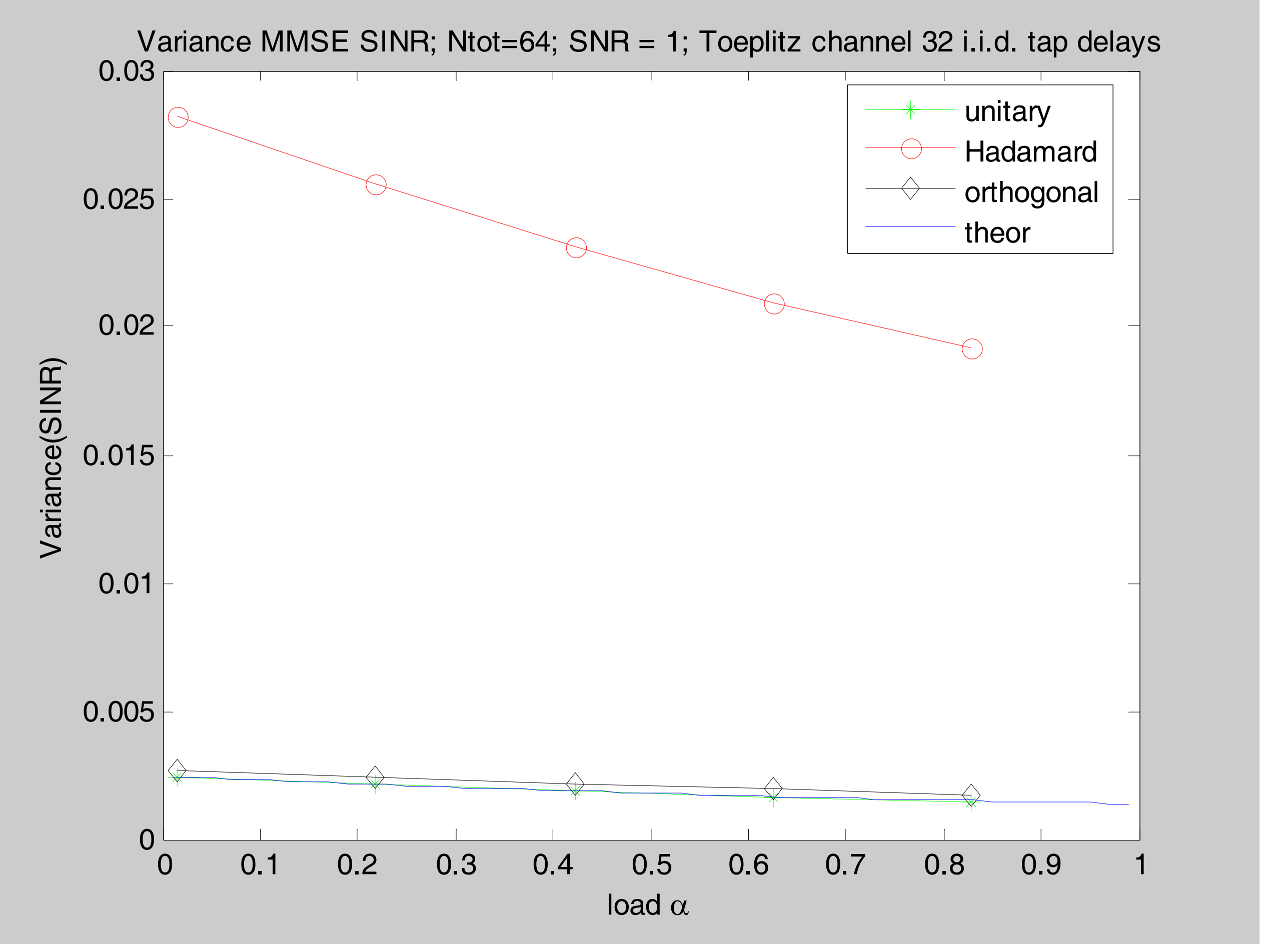
- Variance for N= 128, SNR = 10
- Cell Loading α=K/N=0.5
- Channel matrix ***H*** with iid Gaussian elements of size xaxis*N
- Good agreement with theory



Results

For channel matrix of Toeplitz form Hadamard behavior quite different

- Approach may be invalid due to lack of "randomization" of Haar eigenvalue matrix for channel
- Second order statistics no longer follow unitary asymptotic results !!!



Methods:

- Replicas
  - Originally applied to dirty magnetic systems (1970's)
  - Calculate moment generating function

$$g(\nu) = E\left[\left(\mathbf{I} + \rho\mathbf{H}\mathbf{H}^{\dagger}\right)^{\nu}\right]$$

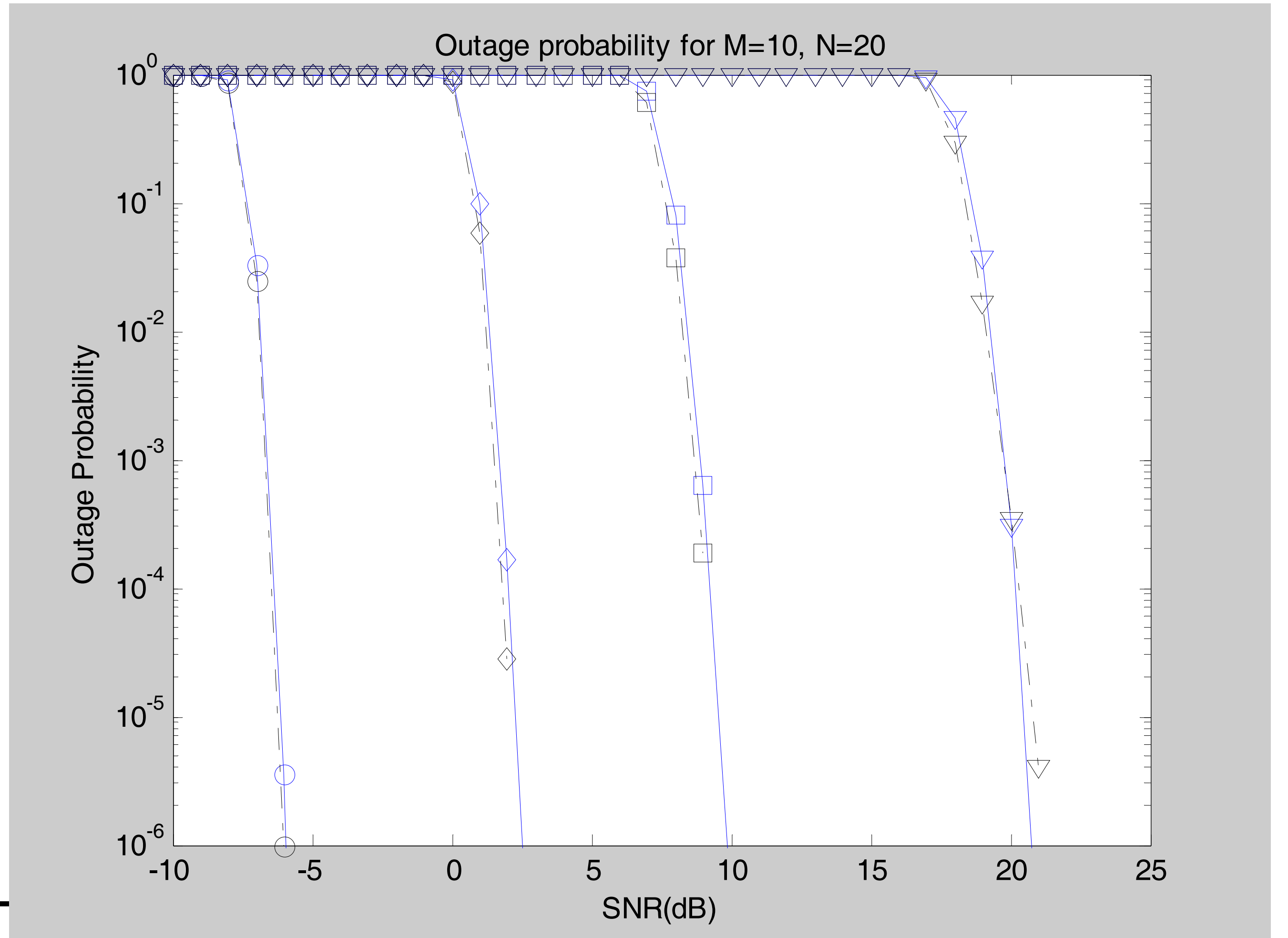
  - Replica trick: Calculate MGF for *integer* values of v
  - Analytically continue for real values of v
  - Technical Assumption: Replica Symmetry
    - Not always valid
    - Valid for random matrices with continuous symmetries (***H*** complex Gaussian w/ SU(N) rotational symmetry)
  - Applications: Calculation of mean, variance, higher order moments of trace, variance, higher moments of trlog *(and hence of MMSE SINR)*

Results

Mutual Information distribution of MMSE SINR for MIMO Gaussian channels

- Need to calculate $E[\beta_i\beta_j]$ and then calculate MI $I = \sum_{n=1}^{M} \log(1+\beta_n)$

- Outage probability well behaved down to small errors.
- For increasing N behavior better
- Better when N>M



Conclusions – Open Questions

- Applied diagrammatic approach to calculate asymptotic mean and variance of MMSE SINR with/without interference
- Applications
  - MMSE SIR for synchronous channels with (a)synchronous interference
  - MMSE SIR for synchronous downlink channels
    - Works well for orthogonal, unitary matrices
    - Hadamard matrices do not fare well WHY?
  - MMSE SIR capacity for MIMO channels
    - Reasonable waterfall curves, even for large SNRs.
    - Crossover to bad behavior is function of SNR, N
- Open Questions
  - Spectrum of AUBU' + CUDU'?
    - Known only for Gaussian U (using replicas)