Asymptotics of the Weingarten function

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joint work with Roland Speicher

$\mathcal{U}_N = N \times N$ unitary group with Haar measure (normalized to be a probability measure)

$u_{ij} : \mathcal{U}_N \to \mathbb{C}$ (i,j) entry

<u>Problem:</u> $d < N$, $i_1, \dots, i_d$, $i'_1, \dots, i'_d$
$j_1, \dots, j_d$, $j'_1, \dots, j'_d$; Calculate

$$\int u_{i_1 j_1} \cdots u_{i_d j_d} \overline{u_{i'_1 j'_1}} \cdots \overline{u_{i'_d j'_d}} \, dU$$

By symmetry : $\int |u_{11}|^2 \, dU = \frac{1}{N}$

- $\int |u_{ij}|^4 \, du = \binom{N+1}{2}^{-1}$
- $\int |u_{12}|^2 |u_{11}|^2 \, du = \frac{1}{N(N+1)}$
- $\int |u_{ij}|^{2k} \, du = \binom{N+k-1}{k}^{-1}$

for these and others see Hiai & Petz

Weingarten Function $\rightsquigarrow$ systematic method

S_d = permutations of $[d] = \{1, 2, 3, \dots, d\}$

$\#(\sigma)$ = n° cycles in cycle decomposition

$|\sigma| = d - \#(\sigma)$ = n° transpositions in a minimal factorization

$\phi(\pi) = N^{\#(\pi)}$ for $\pi \in S_d$, $d < N$

$\phi \in \mathbb{C}[S_d] = \{ f : S_d \to \mathbb{C} \mid$ with convolution product $\}$

<u>THM</u> $\phi \in \mathbb{C}[S_d]^{-1}$ and central

Example: S_d $d=2$

S_2 acts on $\mathbb{C}^2$ via left reg. rep.

$$e \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (1,2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\phi \mapsto \phi(e)e + \phi((1,2))\,(1,2)$$

$$= N^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + N \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= N \begin{pmatrix} N & 1 \\ 1 & N \end{pmatrix}$$

$$\phi^{-1} = N^{-1} \begin{pmatrix} N & 1 \\ 1 & N \end{pmatrix}^{-1} = \frac{N^{-1}}{N^2-1} \begin{pmatrix} N & -1 \\ -1 & N \end{pmatrix}$$

$$= \frac{1}{N^2-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{-1}{(N-1)N(N+1)} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\therefore \phi^{-1}(e) = \frac{1}{N^2-1} \qquad \phi^{-1}((1,2)) = \frac{-1}{(N-1)N(N+1)}$$

Thm (Weingarten-Collins)

Let $Wg_N = \phi^{-1}$

- $Wg_N(\pi) = \int u_{11}\overline{u_{1\pi(1)}} \cdots u_{dd}\overline{u_{d\pi(d)}}\,du$

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- $$Wg_N(\pi) = \int u_{11} \overline{u_{1\pi(1)}} \cdots u_{dd} \overline{u_{d\pi(d)}} \, du$$

- $$\int u_{i_1 j_1} \cdots u_{i_d j_d} \overline{u_{i_1' j_1'}} \cdots u_{i_d' j_d'} \, dU_N$$

$$= \sum_{\pi, \sigma \in S_d} \delta_{i, i' \circ \pi} \, \delta_{j, j' \circ \sigma} \, Wg_N(\pi \sigma^{-1})$$

- $$Wg_N(\pi) = \frac{1}{(d!)^2} \sum_{\lambda \vdash d} \frac{\chi^\lambda(e)^2}{s_\lambda(1^d)} \chi^\lambda(\pi)$$

$$\lambda = \lambda_1, \ldots, \lambda_k \quad , \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_k = d$$

$$s_\lambda(1^d) = \dim(\pi^\lambda_{U_N})$$

When $d = 3$

$$Wg_N = \frac{1}{3!} \left\{ \frac{\chi^{(3)}}{N(N+1)(N+2)} + \frac{\chi^{(1,1,1)}}{N(N-1)(N-2)} + \frac{2\chi^{(2,1)}}{(N-1)N(N+1)} \right\}$$

$$\mathrm{Wg}(e) = \frac{N^2-2}{N(N^2-1)(N^2-4)} = N^{-3} + 3N^{-5} + \cdots$$

$$\mathrm{Wg}((1,2)(3)) = \frac{-1}{(N^2-1)(N^2-4)} = -N^{-4} - 5N^{-6} - \cdots$$

$$\mathrm{Wg}((1,2,3)) = \frac{2}{N(N^2-1)(N^2-4)} = 2N^{-5} + 10N^{-7} + \cdots$$

$$\mathrm{Wg}((1,2,3,\ldots,k))$$

$$= \frac{(-1)^{k-1}\, C_{k-1}}{\prod_{\ell=-(k-1)}^{k-1} (N-\ell)}$$

$$C_k = \frac{1}{k+1}\binom{2k}{k} = k^{th} \text{ Catalan } n^{o}$$

Cumulants of the Weingarten Function

<u>Notation:</u> Suppose we have a sequence of multi-linear functions $\{k_r\}_{r=1,2,3,\ldots}$

If $V = \{i_1, \ldots, i_\ell\} \subseteq [n]$ let

$$k_V(a_1, \ldots, a_n) = k_\ell(a_{i_1}, a_{i_2}, \ldots, a_{i_\ell})$$

If $\pi = \{V_1, \ldots, V_k\} \in \mathcal{P}(n)$ let

$$k_\pi(a_1, \ldots, a_n) = k_{V_1}(a_1, \ldots, a_n) \cdots k_{V_k}(a_1, \ldots, a_n)$$

e.g. $k_{\{(1,3),(2,4)\}}(a_1, a_2, a_3, a_4)$

$$= k_2(a_1, a_3)\, k_2(a_2, a_4)$$

Suppose $n_1 + n_2 + n_3 + \cdots + n_r = n$

$\sigma_1 \in S_{n_1}, \sigma_2 \in S_{n_2}, \ldots, \sigma_r \in S_{n_r}$

$\sigma = \sigma_1 \times \sigma_2 \times \cdots \times \sigma_r \in S_{n_1} \times S_{n_2} \times \cdots \times S_{n_n} \subseteq S_n$

Define $k_r(\sigma_1, \sigma_2, \ldots, \sigma_r)$ by the system

$$Wg(\sigma_1 \cdots \sigma_r) = \sum_{\pi \in \mathcal{P}(r)} k_\pi(\sigma_1, \sigma_2, \ldots, \sigma_r)$$

$$Wg(\sigma) = k_1(\sigma)$$

$$Wg(\sigma_1\sigma_2) = k_2(\sigma_1, \sigma_2) + k_1(\sigma_1)\, k_1(\sigma_2)$$

i.e. $k_2(\sigma_1, \sigma_2) = Wg(\sigma_1\sigma_2) - Wg(\sigma_1)\, Wg(\sigma_2)$

$$k_3(\sigma_1, \sigma_2, \sigma_3) = Wg(\sigma_1\sigma_2\sigma_3) - \{Wg(\sigma_1)\, Wg(\sigma_2\sigma_3)$$
$$+ Wg(\sigma_1\sigma_3)\, Wg(\sigma_2) + Wg(\sigma_1\sigma_2)\, Wg(\sigma_3)\}$$
$$+ 2\, Wg(\sigma_1)\, Wg(\sigma_2)\, Wg(\sigma_3)$$

Thm (Collins 2004) $\pi \in S_d$ $\quad d < N$

$Wg_N(\pi)$ power series in $\frac{1}{N}$ Then

- highest degree term is $d + |\pi|$
- terms decrease in steps of 2
- $$Wg_N(\pi) = \omega_1(\pi)\, N^{-(d+|\pi|)} + \omega_2(\pi)\, N^{-(d+|\pi|+2)} + \omega_3(\pi)\, N^{-(d+|\pi|+4)} + \cdots$$

- $\pi = c_1 \cdots c_k$ = decomposition into cycles

$W_1(\pi) = \omega_1(c_1) \cdots \omega_1(c_k)$

- $w_1\, c((1,\ldots,k)) = (-1)^{k-1} \frac{1}{k} \binom{2k-2}{k-1}$

$= (-1)^{k-1} C_{k-1}$ Catalan

- order of $k_r(\sigma_1, \sigma_2, \ldots, \sigma_r)$ is at most $d + |\sigma| + 2(r-1)$

$\left(d = n_1 + \cdots + n_r \quad \sigma_i \in S_{n_i} \quad \sigma = \sigma_1 \cdots \sigma_r\right)$

Definition Let $\mu_r(\sigma_1, \ldots, \sigma_r)$ be the coefficient of the highest order term of $k_r(\sigma_1, \ldots, \sigma_r)$

$$\mu_1(\sigma) = \omega_1(\sigma)$$

$$\mu_2(\sigma_1, \sigma_2) = \omega_2(\sigma_1\sigma_2) - \omega_1(\sigma_1)\,\omega_2(\sigma_2) - \omega_2(\sigma_1)\,\omega_1(\sigma_2)$$

since we know $\omega_2(\sigma)$ when σ has only one cycle, knowledge of $\mu_2(\sigma_1, \sigma_2)$ tells us $\omega_2(\sigma)$ when σ has two cycles. Thus we can inductively recover ω_2 from μ_2.

<u>Problem</u>: calculate $\mu_2(\pi_1, \pi_2)$ for arbitrary permutations π_1 and π_2.

- $$\mu_2(\pi_1 \pi_2, \pi_3) = \mu_1(\pi_1)\mu_2(\pi_2, \pi_3) + \mu_2(\pi_1, \pi_3)\mu_1(\pi_2)$$

so it suffices to know $\mu_2(\gamma_p, \gamma_q)$ where $\gamma_p = (1,2,3,\dots,p)$, $\gamma_q = (1,2,3,\dots,q)$

Let $\gamma = \gamma_{2p,2q} = (1,2,\dots,2p)(2p+1,\dots,2(p+q)) \in S_{2(p+q)}$

$$k_2(\gamma_p, \gamma_q) = k_{2p+2q}\left(u_{1\gamma(1)}, \overline{u_{\gamma(2)2}}, \dots, u_{2p+2q-1,\gamma(2p+2q-1)}, \overline{u_{\gamma(2p+2q),2p+2q}}\right)$$

- $\lim_N N^{2p+2q} k_{2p+2q}(u_{1\gamma(1)}, \overline{u_{\gamma(2)2}}, \dots)$

$= \kappa_{2p,2q}(u, u^*, u, u^*, \dots, u, u^*)$

Haar unitary

second order free cumulant

order of $k_2(\gamma_p, \gamma_q)$ is $2p+2q$

$\therefore \mu_2(\gamma_p, \gamma_2) = \kappa_{2p,2q}(u, u^*, \dots, u, u^*)$

$= (-1)^{p+q} \, c_{p,q}$

$c_{p,q}$ = n° non-crossing permutations of a (p,q)-annulus