The largest eigenvalue of finite rank deformation of large Wigner matrices: convergence and non-universality of the fluctuations

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Step 1: Inclusion of the spectrum of $M_N = W_N/\sqrt{N} + A_N$

$$\mathbb{P}[\mathrm{Spect}(M_N) \subset K_\sigma(\theta_1, \cdots, \theta_J) + (-\varepsilon; \varepsilon) \text{ for large } N\ ] = 1.$$

$$K_\sigma(\theta_1, \cdots, \theta_J) := \left\{\rho_{\theta_J}; \cdots ; \rho_{\theta_{J-J_{-\sigma}+1}}\right\} \cup [-2\sigma; 2\sigma] \cup \left\{\rho_{\theta_{J_{+\sigma}}}; \cdots ; \rho_{\theta_1}\right\}.$$

$\rho_{\theta_i} = \theta_i + \frac{\sigma^2}{\theta_i}$ if $|\theta_i| > \sigma$.

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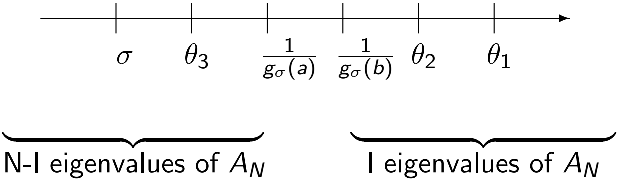
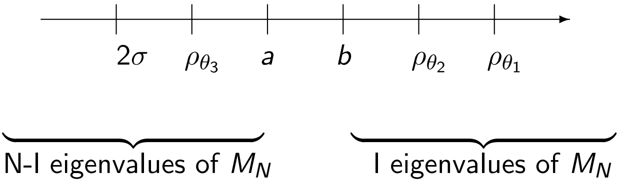
x	$-\infty$		ρ_{θ_J}		-2σ	‖	2σ		ρ_{θ_2}		ρ_{θ_1}		$+\infty$
$\frac{1}{g_\sigma(x)}$	$-\infty$	$\nearrow$	θ_J	$\nearrow$	$-\sigma$	‖	σ	$\nearrow$	θ_2	$\nearrow$	θ_1	$\nearrow$	$+\infty$

Step 2: Exact separation phenomenon

$[a, b]$ gap in $\mathrm{Spect}(M_N) \longleftrightarrow [\frac{1}{g_\sigma(a)}, \frac{1}{g_\sigma(b)}]$ gap in $\mathrm{Spect}(A_N)$





Theorem

Exact separation phenomenon

$K_\sigma(\theta_1, \cdots, \theta_J) :=$

$$\left\{ \rho_{\theta_J}; \cdots ; \rho_{\theta_{J-J_{-\sigma}+1}} \right\} \cup [-2\sigma; 2\sigma] \cup \left\{ \rho_{\theta_{J_{+\sigma}}}; \cdots ; \rho_{\theta_1} \right\}.$$

$[a, b] \subset {}^c K_\sigma(\theta_1, \ldots, \theta_J)$, $i_N \in \{0, \ldots, N\}$ *s.t*

$$\lambda_{i_N+1}(A_N) < \frac{1}{g_\sigma(a)} \quad \text{and} \quad \lambda_{i_N}(A_N) > \frac{1}{g_\sigma(b)}$$

($\lambda_0 := +\infty$ *and* $\lambda_{N+1} := -\infty$*).*
Then

$$\mathbb{P}[\lambda_{i_N+1}(M_N) < a \ \text{ and } \ \lambda_{i_N}(M_N) > b, \text{ for large } N \ ] = 1.$$

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$$\Longrightarrow \mathbb{P}[\lambda_1(M_N) < \rho_{\theta_1} + \epsilon \text{ for large } N\ ] = 1.$$

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By the exact separation phenomenon with $[a; b] = [\theta_2 + \eta; \theta_1 - \eta]$, $(\theta_1 - \eta = \frac{1}{g_\sigma(\rho_{\theta_1} - \epsilon)})$

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$$\Longrightarrow \mathbb{P}[\rho_{\theta_1} - \epsilon < \lambda_1(M_N) < \rho_{\theta_1} + \epsilon \text{ for large } N\ ] = 1.$$

Theorem

$A_N = \mathrm{diag}(\theta, 0, \cdots, 0)$ with $\theta > \sigma$. Then

$$\sqrt{N}\Big(\lambda_1(M_N) - \rho_\theta\Big) \xrightarrow{\mathcal{D}} (1 - \frac{\sigma^2}{\theta^2})\Big\{\mu * \mathcal{N}(0, v_\theta)\Big\}.$$

$$v_\theta = \frac{t}{4}\Big(\frac{m_4 - 3\sigma^4}{\theta^2}\Big) + \frac{t}{2}\frac{\sigma^4}{\theta^2 - \sigma^2}$$

with $t = 4$ (resp. $t = 2$) if W_N is real (resp. complex) and $m_4 := \int x^4 d\mu(x)$.

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In the other particular case $(A_N)_{ij} = \frac{\theta}{N}\ \forall 1 \leq i, j \leq N$, μ symmetric with sub-gaussian moments, D. Féral and S. Péché: if $\theta > \sigma$, then $\sqrt{N}\Big(\lambda_1(M_N) - \rho_\theta\Big) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma_\theta^2)$, $\sigma_\theta = \sigma\sqrt{1 - \frac{\sigma^2}{\theta^2}}$.

$\widehat{M}_{N-1}$: the $N-1 \times N-1$ matrix obtained from M_N removing the first row and the first column. $\Rightarrow \frac{\sqrt{N}}{\sqrt{N-1}}\widehat{M}_{N-1}$ is a non-Deformed Wigner matrix associated with the measure μ.

$$\check{M}_{\cdot 1} = \ {}^t\left((M_N)_{21}, \ldots, (M_N)_{N1}\right).$$

$$M_N = \left(\begin{array}{c|c} \theta + \frac{(W_N)_{11}}{\sqrt{N}} & \check{M}_{\cdot 1}^* \\ \hline & \\ \check{M}_{\cdot 1} & \widehat{M}_{N-1} \end{array}\right)$$

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$$M_N V = \lambda_1 V \Longleftrightarrow \begin{cases} \lambda_1 v_1 = (\theta + \frac{(W_N)_{11}}{\sqrt{N}})v_1 + \check{M}^*_{\cdot 1}\widehat{V} \\ \lambda_1 \widehat{V} = v_1 \check{M}_{\cdot 1} + \widehat{M}_{N-1}\widehat{V} \end{cases}$$

$0 < \delta < \frac{\rho_\theta - 2\sigma}{4}$. $(\rho_\theta > 2\sigma)$

$$\Omega_N = \left\{ \lambda_1(\widehat{M}_{N-1}) \leq 2\sigma + \delta; \lambda_{N-1}(\widehat{M}_{N-1}) \geq -2\sigma - \delta; \lambda_1(M_N) \geq \rho_\theta - \delta \right\}$$

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$$\Leftrightarrow \left\{ \begin{array}{lcl} \lambda_1 v_1 & = & \theta v_1 + \frac{(W_N)_{11}}{\sqrt{N}} v_1 + v_1 \check{M}_{.1}^* \widehat{G}(\lambda_1) \check{M}_{.1}. \\ \widehat{V} & = & v_1 \widehat{G}(\lambda_1) \check{M}_{.1}. \end{array} \right.$$

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$$\Rightarrow \; \lambda_1 = \theta + \frac{(W_N)_{11}}{\sqrt{N}} + \check{M}^*_{.1}\widehat{G}(\lambda_1)\check{M}_{.1}$$

$$\sqrt{N}(\lambda_1 - \rho_\theta) \quad = \quad (W_N)_{11} + \sqrt{N}(\check{M}_{\cdot 1}^{*}\widehat{G}(\lambda_1)\check{M}_{\cdot 1} - \frac{\sigma^2}{\theta})$$

$$
\begin{aligned}
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&= (W_N)_{11} + \sqrt{N}(\check{M}^*_{\cdot 1}\widehat{G}(\rho_\theta)\check{M}_{\cdot 1} - \frac{\sigma^2}{\theta}) \\
&\quad + \sqrt{N}\check{M}^*_{\cdot 1}\left[\widehat{G}(\lambda_1) - \widehat{G}(\rho_\theta)\right]\check{M}_{\cdot 1}
\end{aligned}
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(using $g_\sigma(\rho_\theta) = \frac{1}{\theta}$)

$$\left\{1+\frac{\sigma^2}{\theta^2-\sigma^2}+o(1)\right\}\sqrt{N}(\lambda_1-\rho_\theta)+o(1)$$

$$=\quad (W_N)_{11}+\sqrt{N}(\check{M}_{\cdot 1}^{*}\widehat{G}(\rho_\theta)\check{M}_{\cdot 1}-\sigma^2 tr_{N-1}\widehat{G}(\rho_\theta))$$

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$$Y_{N-1}:=\frac{\sqrt{N}}{\sigma}\check{M}_{\cdot 1}.$$

Theorem

(Bai-Yao and Baik-Silverstein)
$B = (b_{ij})$: *a* $N \times N$ *random Hermitian matrix*
$Y_N = ^t (y_1, \ldots, y_N)$: *an independent vector of size* N *whith i.i.d standardized entries with bounded fourth moment and s.t.* $\mathbb{E}(y_1^2) = 0$ *if* y_1 *is complex. Assume that*

(i) $\exists\ a > 0$ *(not depending on* N*) such that* $||B|| \leq a$,
(ii) $\frac{1}{N}\mathrm{Tr}B^2$ *converges in probability to a number* a_2,
(iii) $\frac{1}{N}\sum_{i=1}^N b_{ii}^2$ *converges in probability to a number* a_1^2.

Then the random variable $\frac{1}{\sqrt{N}}(Y_N^* B Y_N - \mathrm{Tr}B)$ *converges in distribution to a Gaussian variable with mean zero and variance*

$$(\mathbb{E}|y_1|^4 - 1 - t/2)a_1^2 + (t/2)a_2$$

where $t = 4$ *when* Y_1 *is real and is* 2 *when* y_1 *is complex.*

$$\left\{\frac{\theta^2}{\theta^2-\sigma^2}+o(1)\right\}\sqrt{N}(\lambda_1-\rho_\theta)+o(1)$$

$$= \underset{\underset{\textstyle\mu}{\downarrow}}{(W_N)_{11}} + \underbrace{\sigma^2\sqrt{\frac{N-1}{N}}\,\frac{1}{\sqrt{N-1}}\left\{Y_{N-1}^*\widehat{G}(\rho_\theta)Y_{N-1}-Tr\widehat{G}(\rho_\theta)\right\}}_{\substack{\downarrow\\ \mathcal{N}(0,v_\theta)}}$$

The real setting

- Our approach dealing with $A_N = \mathrm{diag}(\theta, 0, \cdots, 0)$ with $\theta > \sigma$ is the same in the real setting as in the complex setting $\Rightarrow$ NON-UNIVERSALITY OF THE FLUCTUATIONS OF $\lambda_1(M_N)$

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- Dealing with $(A_N)_{ij} = \frac{\theta}{N}$ $\forall 1 \leq i, j \leq N$, S. Péché and D. Féral proved that If somebody is able to establish the fluctuations in the deformed GOE case then there will be universality of the fluctuations.

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Corollary of our result by the orthogonal invariance of a GOE matrix: Let A_N be an arbitrary deterministic symmetric matrix of rank one having a non-null eigenvalue θ such that $\theta > \sigma$. Then the largest eigenvalue of the Deformed GOE fluctuates as

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$\Rightarrow$ UNIVERSALITY OF THE FLUCTUATIONS OF $\lambda_1(M_N)$ with a full deformation $(A_N)_{ij} = \frac{\theta}{N}$ $\forall 1 \leq i, j \leq N$.