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Complex Analytic Methods in Free Probability Theory

Hari Bercovici

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- A_x w* closed algebra generated by $\{e_x((-\infty, t)) : t \in \mathbb{R}\}$ (σ-algebra of x)

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- Corresponding notion: *-freeness

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- same notation, different semigroup

Examples

- $\delta_r \boxplus \delta_s = \delta_{r+s}$; $c_r \boxplus c_s = c_{r+s}$ for Cauchy (arctangent)

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- There are corresponding results for $\boxtimes$

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Limit laws

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- for the circle, there are no $\boxtimes$-idempotents

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- Then $F_{\mu \boxplus \nu} \prec F_\mu$ (and $F_{\mu \boxplus \nu} \prec F_\nu$)

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- Then $F_{\mu \boxplus \nu} \prec F_\mu$ (and $F_{\mu \boxplus \nu} \prec F_\nu$)
- Note: subordination functions $F_\mu^{<-1>} \circ F_{\mu \boxplus \nu}$ obviously exist at ∞; the important point is they continue to $\mathbb{C}^+$.

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- $\mu \boxplus \nu$ has no singular continuous component
- the density of $\mu \boxplus \nu$ is locally analytic a.e.
- But: the density of $\mu \boxplus \nu$ may have points of nondifferentiability even when those of μ and ν don't

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- what about λ?

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- This is a “Fourier coefficient” of order n

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Operator-valued

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- Fully matricial analytic functions may be needed for full understanding.

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- many results extend to this operation, questions remain

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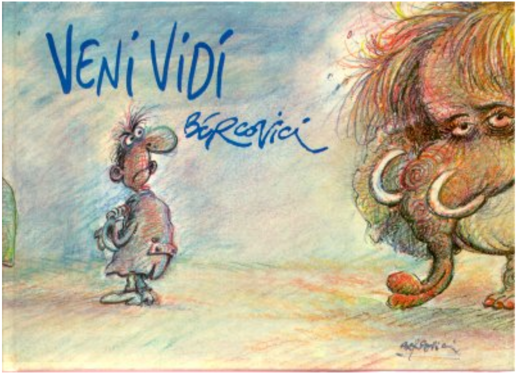
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